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Levi-Flat Minimal Hypersurfaces in Two-dimensional Complex Space Forms

Robert L. Bryant

Abstract.

The purpose of this article is to classify the real hypersurfaces in complex space forms of dimension 2 that are both Levi-flat and minimal. The main results are as follows:

When the curvature of the complex space form is nonzero, there is a 1-parameter family of such hypersurfaces. Specifically, for each one-parameter subgroup of the isometry group of the complex space form, there is an essentially unique example that is invariant under this one-parameter subgroup.

On the other hand, when the curvature of the space form is zero, i.e., when the space form is $\mathbb{C}^2$ with its standard metric, there is an additional 'exceptional' example that has no continuous symmetries but is invariant under a lattice of translations. Up to isometry and homothety, this is the unique example with no continuous symmetries.

Introduction

A real hypersurface $\Sigma^3 \subset \mathbb{C}^2$ is *Levi-flat* [CM] if it is foliated by complex curves. (If such a foliation exists, it is necessarily unique.) Thus, a Levi-flat hypersurface in $\mathbb{C}^2$ is essentially a 1-parameter family of complex curves in $\mathbb{C}^2$. If one imposes the further condition that the hypersurface be *minimal*, there is, in addition to the obvious example of a real hyperplane, the deleted cone $C^* \subset \mathbb{C}^2 \setminus \{(0,0)\}$ defined by

$$|z_1|^2 - |z_2|^2 = 0, \quad |z_1|^2 + |z_2|^2 > 0.$$

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This article was begun during a June 1999 visit at the IHES. I would like to thank the IHES for its hospitality. The idea for this article came to me during a conversation with Mikhail Gromov and Gennadi Henkin, who asked (perhaps idly) whether any nontrivial examples of the kind mentioned in the title exist. I thank them for their stimulating conversation.

This cone is foliated by the (punctured) lines $z_1 = \lambda z_2$ with $|\lambda| = 1$ and hence is Levi-flat. Since C^* is the cone on the Clifford torus, it is also minimal as a submanifold of $\mathbb{C}^2$.

It is not obvious that there are any examples of minimal, Levi-flat hypersurfaces in $\mathbb{C}^2$ that are distinct from these up to rigid motion. The condition of being either minimal or Levi-flat constitutes a single non-linear second order PDE for the hypersurface Σ. A short calculation shows that the combined conditions form a second order system that is not involutive in Cartan's sense. In fact, by Cartan's classification [Ca] of the involutive second order systems for one function of three variables, there is no second order equation that is in involution with the minimal hypersurface equation for a hypersurface in a Riemannian 4-manifold. Thus, describing the solutions of such a system requires analysis that goes beyond an application of the Cartan-Kähler theorem.

In this article, I carry out this analysis, classify the solutions of this overdetermined system, both locally and globally, and show that there are many other examples. Since it is no harder to do the analysis for the general two-dimensional complex space form, I do the computations in this more general setting. While the calculations were guided by certain concepts from exterior differential systems, this article has been written so that no knowledge of this subject is required of the reader beyond the (elementary) Frobenius theorem on integrable plane fields. Nevertheless, the reader who wonders how some of the calculations in §2 could be motivated might want to consult [BCG, Chapter VI]. General references on calculations via the moving frame could also be helpful, in which case I recommend [Sp] or [Gr].

The results can be described as follows: Each local solution extends to a unique maximal solution and the space of maximal solutions is finite dimensional, breaking up into two or three different families.

The members of the first family are those hypersurfaces Σ whose complex leaves are totally geodesic in the ambient space form. In flat space, there are only two such examples up to isometry: the hyperplane and the Clifford cone constructed above. When the space form has positive sectional curvature and hence is $\mathbb{P}^2$ with its standard Fubini-Study metric up to a constant scale factor, there is only one example up to rigid motion. Its closure in $\mathbb{P}^2$ has one singular point, near which it resembles the Clifford cone in flat space. When the space form has negative sectional curvature and hence is the complex hyperbolic 2-ball $\mathbb{B}^2$ (i.e., the noncompact dual of $\mathbb{P}^2$) up to a scale factor, there are three distinct examples up to isometry. The closure of one of these examples has a singular point, near which it resembles the Clifford cone. The other two examples are nonsingular, complete, embedded hypersurfaces. For details, see §3.1.

The remaining two families are somewhat more difficult to describe explicitly. The structure equations for the second family show that each such example Σ^3 is invariant under a one-parameter group of isometries of the ambient space and that this one-parameter group acts on the hypersurface Σ preserving each of its complex leaves. Conversely, each one-parameter group of isometries of the ambient space preserves a family of holomorphic curves that foliates the ambient space in the complement of the fixed point set. Up to ambient isometry, there is a unique one-parameter family of these curves whose union is a minimal hypersurface. The minimal Levi-flat hypersurfaces constructed in this way that do not belong to the first family constitute the members of the second family. In §3.2, I construct these hypersurfaces explicitly for each conjugacy class of one-parameter subgroup of the isometry group of the ambient space form. The examples in this second family often have some sort of singular locus and can be either real algebraic or transcendental, see §3.2.

The third family is the most difficult to describe explicitly. It only exists when the ambient curvature is zero, i.e., in the case of $\mathbb{C}^2$ itself. Up to holomorphic isometry and homothety, there is only one such example and it is periodic with respect to a lattice $\Lambda \subset \mathbb{C}^2$ of type F_4. The quotient hypersurface $\Sigma^3 \subset \mathbb{C}^2/\Lambda$ has quite interesting properties. Its complex leaves are compact Riemann surfaces of genus 3 and the 1-parameter family of genus 3 surfaces that makes up this hypersurface is a nontrivial variation in moduli. The formula that defines the embedding of Σ into the abelian variety $\mathbb{C}^2/\Lambda$ is essentially a quotient of the Abel-Jacobi mapping on each complex leaf. There is reason to believe that this hypersurface is an open dense subset of a 'real algebraic' hypersuface in the algebraic variety $\mathbb{C}^2/\Lambda$, but I have not verified this in detail. I would like to thank Dave Morrison for a helpful conversation about the algebraic geometry of this example.

§1. Two-Dimensional Complex Space Forms

This section introduces the structure equations for complex space forms of dimension 2 and establishes the notation that will be used for the remainder of the article. For further discussion of these models, the reader might consult [He] or [KN].

1.1. The group G_R

Let R be a real number and let $G_R \subset \mathrm{SL}(3,\mathbb{C})$ be the connected subgroup whose Lie algebra $\mathfrak{g}_R$ consists of the matrices of the form

$$\begin{pmatrix} ir_1 & -R\bar{x} & -R\bar{y} \\ x & ir_2 & -\bar{z} \\ y & z & -i(r_1+r_2) \end{pmatrix}$$

where r_1 and r_2 are real and x, y, and z are complex.

When $R \neq 0$, this is the identity component of the set of unimodular matrices $\mathbf{g}$ that satisfy ${}^t\bar{\mathbf{g}} H_R \mathbf{g} = H_R$, where

$$H_R = {}^t\overline{H_R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & R \end{pmatrix}.$$

In this case, H_R defines a nondegenerate Hermitian inner product $\langle , \rangle_R$ on $\mathbb{C}^3$. Even the matrix H_0 defines a (very degenerate) Hermitian inner product $\langle , \rangle_0$ on $\mathbb{C}^3$ and G_0 preserves it.

1.2. The complex space form $\mathbb{P}^2_R$

The set $\mathbb{P}^2_R \subset \mathbb{P}^2$ consisting of the lines through $0 \in \mathbb{C}^3$ on which $\langle , \rangle_R$ is positive is a homogeneous space of G_R. Write the general element of G_R as

$$\mathbf{g} = \begin{pmatrix} \mathbf{e}_0 & \mathbf{e}_1 & \mathbf{e}_2 \end{pmatrix},$$

where the columns $\mathbf{e}_i$ of $\mathbf{g}$ are to be regarded as $\mathbb{C}^3$-valued functions on G_R. The map $\pi\colon G_R \to \mathbb{P}^2_R$ defined by $\pi(\mathbf{g}) = \mathbb{C}\cdot\mathbf{e}_0$ is a submersion. The fibers of π are the left cosets of the connected subgroup $K \subset G_R$ whose Lie algebra consists of matrices of the form

$$\begin{pmatrix} ir_1 & 0 & 0 \\ 0 & ir_2 & -\bar{z} \\ 0 & z & -i(r_1+r_2) \end{pmatrix}.$$

The group K is compact and isomorphic to the nontrivial double cover of $\mathrm{U}(2)$. In particular, $\mathbb{P}^2_R \simeq G_R/K$ as a homogeneous space.

1.3. The structure equations

Write the left invariant Maurer-Cartan form on G_R in the form

$$\gamma = \mathbf{g}^{-1}d\mathbf{g} = \begin{pmatrix} i\tau & -R\bar{\eta} & -R\bar{\omega} \\ \eta & i(\phi+\tau) & -\bar{\sigma} \\ \omega & \sigma & -i(\phi+2\tau) \end{pmatrix},$$

so that the *first structure equation* becomes

$$\begin{pmatrix} d\mathbf{e}_0 & d\mathbf{e}_1 & d\mathbf{e}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{e}_0 & \mathbf{e}_1 & \mathbf{e}_2 \end{pmatrix} \begin{pmatrix} i\tau & -R\bar\eta & -R\bar\omega \\ \eta & i(\phi+\tau) & -\bar\sigma \\ \omega & \sigma & -i(\phi+2\tau) \end{pmatrix}.$$

There exist on $\mathbb{P}^2_R$ a unique metric ds^2 and a ds^2-orthogonal complex structure J with corresponding Kähler form Ω for which

$$\pi^*\left(ds^2\right) = \eta\circ\bar\eta + \omega\circ\bar\omega \quad \text{and} \quad \pi^*(\Omega) = \tfrac{i}{2}(\eta\wedge\bar\eta + \omega\wedge\bar\omega).$$

The *second structure equation* $d\gamma = -\gamma\wedge\gamma$ shows that this Kähler structure has constant holomorphic sectional curvature $4R$. (I.e., the Gauss curvature of any totally geodesic complex curve in $\mathbb{P}^2_R$ is $4R$.)

From now on, the fibration $\pi\colon G_R \to \mathbb{P}^2_R$ will be taken as the standard unitary bundle structure for the Kähler geometry of $\mathbb{P}^2_R$. (Strictly speaking, of course, this is not quite correct since one should first divide out by the center of G_R, a cyclic subgroup of order 3, but for simplicity, I will not do this. It should not cause any confusion.)

§2. Real Hypersurfaces

Let Σ^3 be a connected, smooth, embedded[1] real hypersurface in $\mathbb{P}^2_R$. The preimage $B_0 = \pi^{-1}(\Sigma)$ is a principal K-bundle over Σ. From now on, all the forms on G_R are to be understood as pulled back to B_0.

2.1. First invariants

Since Σ is a hypersurface, there will be one linear relation among the real and imaginary parts of the two 1-forms η and ω. Let $B_1 \subset B_0$ be the subset where this relation is $\eta = \bar\eta$. Then B_1 is a union of left K_1-cosets where $K_1 \simeq S^1$ is the group of matrices of the form

$$E_\theta = \begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{i\theta} & \\ 0 & 0 & e^{-2i\theta} \end{pmatrix}.$$

From now on, all the forms on B_0 are to be understood as pulled back to B_1. In addition to the relation $\eta = \bar\eta$, there will be relations of the form

$$\begin{aligned} \phi &= H\eta - ia\omega + i\bar a\bar\omega \\ \sigma &= c\eta + iL\omega - 2s\bar\omega \end{aligned}$$

[1]All these calculations will be local, so embeddedness is not a serious restriction.

for some functions a, c, $H = \bar{H}$, L, and s on B_1. (The choice of numerical coefficients is cosmetic.) By the structure equations,

$$d\eta = -i\phi \wedge \eta + \bar{\sigma} \wedge \omega = -(a\omega - \bar{a}\bar{\omega}) \wedge \eta + \overline{(c\eta + iL\omega - 2s\bar{\omega})} \wedge \omega.$$

Since η is real, the imaginary part of the right hand expression must vanish. I.e.,

$$L = \bar{L} \quad \text{and} \quad c = -2\bar{a}.$$

Let $R_\theta \colon B_1 \to B_1$ denote right action by the matrix E_θ. Then η, H, and L are invariant under R_θ while

$$R_\theta^*\omega = e^{3i\theta}\omega, \quad R_\theta^* a = e^{-3i\theta} a, \quad R_\theta^* s = e^{6i\theta} s.$$

Note that quantities such as η, $a\omega$, $\bar{s}\omega^2$, L, H, $|a|^2$, and $|s|^2$ are π-semibasic and invariant under R_θ and so can be considered to be well defined as functions or forms on Σ.

2.1.1. Levi-flatness. The equation $\eta = 0$ defines the preimage in B_1 of the bundle of complex tangent spaces to Σ. Consequently, Σ will be Levi-flat if and only if $\eta \wedge d\eta = 0$. However, by the structure equations and the relations just derived,

$$\eta \wedge d\eta = iL\eta \wedge \omega \wedge \bar{\omega}.$$

Thus, Levi-flatness is equivalent to the condition $L = 0$. From now on, I will assume that Σ is Levi-flat.

2.1.2. Minimality. The induced metric on Σ pulls back to B_1 to be the quadratic form $\eta^2 + \omega \circ \bar{\omega}$, while the second fundamental form II satisfies

$$\pi^*(\mathrm{II}) = c_1 H\eta^2 + \mathrm{Re}(c_2 a\omega) \circ \eta + \mathrm{Re}(c_3 \bar{s}\omega^2)$$

for some nonzero constants c_1, c_2, and c_3 (the explicit values will not be important for what follows). In particular, H is the the mean curvature function of Σ (up to some universal constant multiple), i.e., Σ is minimal if and only if H vanishes identically on B_1. From now on, I will assume that Σ is minimal (as well as Levi-flat).

2.2. Differential consequences of the structure equations

At this point, the forms on B_1 satisfy the reality condition $\eta = \bar{\eta}$, the nondegeneracy condition $\eta \wedge \omega \wedge \bar{\omega} \neq 0$, and the relations

$$\begin{aligned} \phi &= -ia\omega + i\bar{a}\bar{\omega}, \\ \sigma &= -2\bar{a}\eta - 2s\bar{\omega}. \end{aligned}$$

Thus, γ pulled back to B_1 has the form

$$\gamma = \begin{pmatrix} i\tau & -R\eta & -R\bar{\omega} \\ \eta & i\tau + a\omega - \bar{a}\bar{\omega} & 2a\eta + 2\bar{s}\omega \\ \omega & -2\bar{a}\eta - 2s\bar{\omega} & -2i\tau - a\omega + \bar{a}\bar{\omega} \end{pmatrix}.$$

The structure equation $d\gamma = -\gamma \wedge \gamma$ expands to the relations

$$\begin{aligned} d\tau &= iR\omega \wedge \bar{\omega}, \\ d\eta &= (a\omega + \bar{a}\bar{\omega}) \wedge \eta, \\ d\omega &= (3i\tau - \bar{a}\bar{\omega}) \wedge \omega + 2s\bar{\omega} \wedge \eta, \end{aligned}$$

and implies the existence of complex-valued functions x and y on B_1 so that

$$\begin{aligned} da &= -3ia\tau - 6\bar{a}\bar{s}\eta + (x - 3a^2)\omega - \left(\tfrac{1}{2}R - |a|^2 + 2|s|^2\right)\bar{\omega}, \\ ds &= \quad 6is\tau + \bar{x}\eta + 3sa\omega + y\bar{\omega}. \end{aligned}$$

Remark 1. These equations imply strong conditions about the vanishing locus of s on each complex leaf $L \subset \Sigma$. In a small neighborhood U of any point $p \in L$, one can choose a complex coordinate z so that, on $B_L = \pi^{-1}(L)$, there is a nonzero function h so that $\omega = h\pi^*(dz)$ holds on B_L. Correspondingly, there will be a function f on U so that $\bar{a}\bar{\omega} = \pi^*(f_{\bar{z}}\, d\bar{z})$ and a function g on U so that $\bar{s}\omega^2 = \pi^*(g\, dz^2)$. The above structure equations then imply that the product $e^{-3f}g$ is holomorphic in z. Consequently, the quadratic form $\bar{s}\omega^2$ is a nonvanishing multiple of a holomorphic quadratic form on L and so either vanishes identically or else only vanishes at discrete points of L and then only to finite order. Note that $|s|^2$ vanishes identically on a complex leaf if and only if that leaf is totally geodesic in $\mathbb{P}^2_R$.

Remark 2. It will be useful to understand the metric $\omega \circ \bar{\omega}$ induced on the complex leaves, in particular, the Gauss curvature of this induced metric. Now, the equation for $d\omega$ can be written in the form

$$d\omega = -i\rho \wedge \omega + 2s\bar{\omega} \wedge \eta$$

where $\rho = -3\tau + i(a\omega - \bar{a}\bar{\omega})$. The equation

$$d\rho \equiv -\tfrac{i}{2}(4R - 8|s|^2)\omega \wedge \bar{\omega} \bmod \eta$$

then shows that the function $K = 4(R - 2|s|^2)$ restricts to each complex leaf to be its Gauss curvature.

2.2.1. First case. Using the structure equations to expand the identity $d(da) = 0$ and then reducing the result modulo ω yields

$$sx - \bar{s}\bar{x} = 0.$$

There are now two cases to consider. First, suppose that s vanishes identically. Then so do x and y, and the remaining structure equation for a is

$$da = -3ia\tau - 3a^2\omega - (\tfrac{1}{2}R - |a|^2)\bar{\omega}.$$

Differentiating this equation just yields an identity. Thus, the system

$$\begin{aligned} d\tau &= iR\omega \wedge \bar{\omega} \\ d\eta &= (a\omega + \bar{a}\bar{\omega}) \wedge \eta \\ d\omega &= (3i\tau - \bar{a}\bar{\omega}) \wedge \omega \\ da &= -3ia\tau - 3a^2\omega - (\tfrac{1}{2}R - |a|^2)\bar{\omega} \end{aligned} \tag{1}$$

is differentially closed[2] and describes the class of solutions Σ for which the complex leaves are totally geodesic. This class will be analyzed in the next section, after all of the integrability conditions have been found for the remaining cases.

2.2.2. Second and third cases. Suppose now that s does not vanish identically. Since Σ is real analytic and connected and since $|s|^2$ is well-defined on Σ, there is a dense open set $\Sigma^* \subset \Sigma$ on which $|s|^2 > 0$. On the bundle $B_1^* = \pi^{-1}(\Sigma^*) \cap B_1$, which is a dense open subset of B_1, write $x = \bar{s}p$, where p is real. The structure equations are now

$$\begin{aligned} da &= -3ia\tau - 6\bar{a}\bar{s}\eta + (\bar{s}p - 3a^2)\omega - \left(\tfrac{1}{2}R - |a|^2 + 2|s|^2\right)\bar{\omega} \\ ds &= \quad 6is\tau + sp\eta + 3sa\omega + sy\bar{\omega} \end{aligned}$$

(where, to simplify equations to follow, I have replaced the former y by sy, which is permissible since s is nonzero).

Now, a cannot vanish identically. If it were to do so, then the above equations would imply $p = 0$ and $R = -4|s|^2 < 0$ (since s is nonzero). The equation for ds would then simplify to $ds = 6is\tau + sy\bar{\omega}$. Differentiating the relation $R + 4|s|^2 = 0$ then shows that $y = 0$, in turn implying

[2] I.e., the exterior derivatives of these equations are identities. Of course, it then follows from Cartan's generalization of Lie's Third Fundamental Theorem that there are solutions to these equations, but the explicit computations in the next section will make recourse to Cartan's theorem unnecessary. This same comment applies to the other two cases that will turn up in the next subsubsection.

that $ds = 6is\tau$, which then implies that $d\tau = 0$, contradicting the structure equation for τ since $R \neq 0$. By the real analyticity and connectedness of Σ, it follows that $|a|^2$ is nonzero on a dense open set $\Sigma^{**} \subset \Sigma^*$ and I can restrict attention to the corresponding subbundle B_1^{**}, which I will do from now on. Thus, a is nonzero on B_1^{**}.

Now, the structure equations plus the reality of p yield

$$0 = \frac{d(da) \wedge \bar{\omega}}{\bar{s}} + \frac{d(d\bar{a}) \wedge \omega}{s} = 6(\bar{a}\bar{y} - ay)\eta \wedge \omega \wedge \bar{\omega}.$$

Thus ay is real, implying that there exists a function $q = \bar{q}$ for which $y = \bar{a}(q+3)$. (Writing $q+3$ instead of q here simplifies the following formulae.) Expanding the identity $d(da) = 0$ and using the reality of p implies that p satisfies the equation

$$\begin{aligned} dp = {} & (2R - 64|a|^2 + 8|s|^2 - 6|a|^2q - p^2)\eta \\ & - (ap + 24\bar{a}\bar{s} + 2\bar{a}\bar{s}q)\omega - (\bar{a}p + 24as + 2asq)\bar{\omega}. \end{aligned}$$

By this structure equation and the reality of q,

$$0 = \frac{d(ds) \wedge a\omega}{s} + \frac{d(d\bar{s}) \wedge \bar{a}\bar{\omega}}{\bar{s}} = 4q(a^2s - \bar{a}^2\bar{s})\eta \wedge \omega \wedge \bar{\omega}.$$

Thus, either q or the imaginary part of a^2s vanishes identically. These two cases will be considered separately.

First, suppose that a^2s is real and introduce a real-valued function $t = \bar{t}$ so that $s = \bar{a}^2t$. Using the reality of t and expanding the identities $0 = d(da) = d(ds) = d(dp)$ yields

$$q = R + 4|a|^2 + 2|a|^2pt + |a|^4t^2$$

plus a formula for dt. The result is structure equations of the form

$$\begin{aligned} d\tau &= iR\omega \wedge \bar{\omega} \\ d\eta &= (a\omega + \bar{a}\bar{\omega}) \wedge \eta \\ d\omega &= (3i\tau - \bar{a}\bar{\omega}) \wedge \omega + 2\bar{a}^2t\bar{\omega} \wedge \eta \\ da &= -3ia\tau - 6|a|^2at\eta + a^2(tp-3)\omega - \left(\tfrac{1}{2}R - |a|^2 + 2|a|^4t^2\right)\bar{\omega} \\ dp &= -(4R + 16|a|^2 + 16|a|^4t^2 + 12|a|^2pt + p^2)\eta \\ &\quad - \left(p + 8|a|^2t + 2t(R + 4|a|^4t^2 + 2|a|^2pt)\right)(a\omega + \bar{a}\bar{\omega}) \\ dt &= t(p + 12|a|^2t)\eta + t(1 + 4|a|^2t^2 + R/|a|^2)(a\omega + \bar{a}\bar{\omega}). \end{aligned} \tag{2}$$

Differentiating these equations yields only identities, so this represents a set of solutions. These will be analyzed below. This system is compatible with the relation $t = 0$, in which case the structure equations specialize to (1), the first solution found. Thus, the solutions (1) can be regarded as special cases of (2).

On the other hand, if $q \equiv 0$, then the structure equations yield $d\big(d(s)\big) = 6sR\omega\wedge\bar\omega$, so this case can only occur when $R = 0$. Assuming this, the structure equations found so far are

$$
\begin{aligned}
d\tau &= 0\\
d\eta &= (a\omega + \bar a\bar\omega)\wedge\eta\\
d\omega &= (3i\tau - \bar a\bar\omega)\wedge\omega + 2s\bar\omega\wedge\eta\\
da &= -3ia\tau - 6\bar a\bar s\eta + (\bar s p - 3a^2)\omega + \big(|a|^2 - 2|s|^2\big)\bar\omega\\
ds &= s(6i\tau + p\eta + 3a\omega + 3\bar a\bar\omega)\\
dp &= \big(8|s|^2 - 64|a|^2 - p^2\big)\eta - (ap + 24\bar a\bar s)\omega - (\bar a p + 24as)\bar\omega.
\end{aligned}
\tag{3}
$$

Differentiating these equations yield only identities, so this represents a class of solutions that exist only in the case $R = 0$. These will be analyzed below. Since a^2s is not, in general, real for these solutions, they are not special cases of (2), although when $s = 0$, these solutions do specialize to the $t = 0$ solutions of (2) in the case $R = 0$. These special solutions are the only overlap between the two.

§3. Existence of Solutions

In this section, I will prove general existence results that assure that there are solutions to the equations (1), (2) and (3). In each case, this will be followed by an analysis of the equations that allows a complete description of the corresponding solutions.

3.1. Solutions of type 1

3.1.1. Existence via the Frobenius theorem. Let $M^{10} = G_R\times\mathbb{C}$ and let $\mathbf{g}\colon M \to G_R$ and $\mathbf{a}\colon M \to \mathbb{C}$ be the projections onto the factors. I will regard forms on G_R or $\mathbb{C}$ as forms on M via the pullbacks under these two maps and will not notate the pullback explicitly. Let $\mathcal{I}_1$ be the exterior ideal on M generated by the linearly independent real-valued 1-forms $\theta_1,\dots,\theta_6$ where

$$
\begin{aligned}
\theta_1 &= i(\bar\eta - \eta)\\
\theta_2 &= \phi + i\mathbf{a}\omega - i\bar{\mathbf{a}}\bar\omega\\
\theta_3 + i\theta_4 &= \sigma + 2\bar{\mathbf{a}}\eta\\
\theta_5 + i\theta_6 &= d\mathbf{a} + 3i\mathbf{a}\tau + 3\mathbf{a}^2\omega + \big(\tfrac12 R - |\mathbf{a}|^2\big)\bar\omega.
\end{aligned}
$$

The structure equations $d\gamma = -\gamma\wedge\gamma$ imply that $\mathcal{I}_1$ is differentially closed. Thus, the Frobenius theorem implies that M is foliated by 4-dimensional integral manifolds of $\mathcal{I}_1$. Each leaf $L \subset M$ is the image of a bundle $B_1 \subset G_R$ of a minimal Levi-flat hypersurface Σ satisfying equations (1) under the embedding $\mathrm{id} \times a\colon B_1 \to G_R \times \mathbb{C}$. This gives an abstract description of the solutions of type (1).

Since G_R acts by left translation on $G_R \times \mathbb{C}$ preserving the ideal $\mathcal{I}_1$, this left action permutes the integral manifolds, and two integral manifolds are equivalent under this action if and only if they correspond to congruent hypersurfaces in $\mathbb{P}^2_R$. In particular, two leaves L_1 and L_2 represent equivalent solutions if and only if they satisfy $\mathbf{a}(L_1) = \mathbf{a}(L_2)$. Note that this happens if and only if the two images $\mathbf{a}(L_1)$ and $\mathbf{a}(L_2)$ have nonempty intersection.

3.1.2. Explicit description of the solutions. On any connected solution to (1), the structure equations imply

$$
\begin{aligned}
4da \wedge d\bar{a} &= \big(R + 4|a|^2\big)\big((R - 8|a|^2)\omega \wedge \bar{\omega} + 6i\tau \wedge (a\omega + \bar{a}\bar{\omega})\big)\\
d(R + 4|a|^2) &= -2\big(R + 4|a|^2\big)(a\omega + \bar{a}\bar{\omega}).
\end{aligned}
$$

It follows that for any leaf L of $\mathcal{I}_1$, either the function $R + 4|\mathbf{a}|^2$ vanishes identically or else $\mathbf{a}\colon L \to \mathbb{C}$ is an immersion.

Now, when $R > 0$, the only possibility is that $\mathbf{a}\colon L \to \mathbb{C}$ is an immersion everywhere. Moreover, using the left action of G_R plus the existence of a leaf through any point of $G_R \times \mathbb{C}$, it follows that $\mathbf{a}\colon L \to \mathbb{C}$ is a surjective submersion for every leaf. In particular, all of the leaves are equivalent under the action of G_R. Since $|s|^2$ vanishes identically on L, it follows that the complex leaves of Σ are totally geodesic in $\mathbb{P}^2_R$, which is, up to a constant scale factor, isometric to $\mathbb{CP}^2$ endowed with the Fubini-Study metric. Thus, Σ must be a 1-parameter family of complex lines in $\mathbb{CP}^2$. In fact, Σ must be congruent to the smooth locus C_1^* of the 'cone'

$$
C_1 = \left\{ \begin{bmatrix} z \\ w \\ e^{ir}w \end{bmatrix} \in \mathbb{CP}^2 \;\middle|\; r \in \mathbb{R},\ [z, w] \in \mathbb{CP}^1 \right\}.
$$

It is evident that C_1^* is both Levi-flat and minimal. Note that C_1 has only one singular point (the intersection of the complex lines that foliate it) and is otherwise smooth.

When $R = 0$, so that $\mathbb{P}^2_0$ is isometric to $\mathbb{C}^2$ with the standard flat metric, there are two possibilities. The first possibility is that $|a|^2$ vanishes identically, in which case the corresponding Σ is congruent to a real hyperplane:

$$H_0 = \left\{ \begin{bmatrix} 1 \\ z \\ r \end{bmatrix} \in \mathbb{P}^2_0 \;\middle|\; r \in \mathbb{R},\ z \in \mathbb{C} \right\}.$$

The second possibility is that $|a|^2$ never vanishes. By the same sort of argument made for the case of positive holomorphic sectional curvature, one sees that all of these cases are equivalent to the smooth part of the cone

$$C_0 = \left\{ \begin{bmatrix} 1 \\ z \\ e^{ir} z \end{bmatrix} \in \mathbb{P}^2_0 \;\middle|\; r \in \mathbb{R},\ z \in \mathbb{C} \right\}.$$

When $R < 0$, there is no loss of generality in setting $R = -1$, so I will do so for this discussion. Then

$$\mathbb{P}^2_{-1} = \left\{ \begin{bmatrix} 1 \\ z^1 \\ z^2 \end{bmatrix} \in \mathbb{CP}^2 \;\middle|\; \left|z^1\right|^2 + \left|z^2\right|^2 < 1 \right\}$$

is the hyperbolic complex 2-ball and there are three possibilities, depending on the sign of $R + 4|a|^2 = 4|a|^2 - 1$.

The solutions with $4|a|^2 - 1 > 0$ are all congruent to the smooth part of the hyperbolic version of the cone:

$$C_{-1} = \left\{ \begin{bmatrix} 1 \\ z \\ e^{ir} z \end{bmatrix} \in \mathbb{P}^2_{-1} \;\middle|\; r \in \mathbb{R},\ \sqrt{2}\,|z| < 1 \right\}.$$

This cone has one singular point. The leaves of $dr = 0$ are the complex leaves, each one biholomorphic to a punctured disk.

All solutions with $4|a|^2 - 1 = 0$ are congruent to the 'horosphere' solution

$$S_{-1} = \left\{ \begin{bmatrix} 1 \\ z \\ ir(1-z) \end{bmatrix} \in \mathbb{P}^2_{-1} \;\middle|\; r \in \mathbb{R},\ |z|^2 + r^2|1-z|^2 < 1 \right\}.$$

The complex leaves in S_{-1} are the leaves of $dr = 0$ in the chosen parametrization. All of these complex leaves intersect at one point on the boundary of the ball. This solution can be interpreted as a limit of the cone C_{-1} as one moves the singular point of the cone out to the boundary of $\mathbb{P}^2_R$ in $\mathbb{P}^2$.

All solutions with $4|a|^2 - 1 < 0$ are congruent to the hyperbolic version of the hyperplane solution, namely

$$H_{-1} = \left\{ \begin{bmatrix} 1 \\ z \\ r \end{bmatrix} \in \mathbb{P}^2_0 \;\middle|\; r \in \mathbb{R},\ r^2 + |z|^2 < 1 \right\}.$$

This completes the list of solutions of the system (1).

3.2. Solutions of type 2

Consider the solutions of the system (2). To avoid repetition, I am going to consider only solutions for which t is non-zero, since the solutions with t vanishing identically have already been accounted for as solutions of type (1).

3.2.1. Existence via the Frobenius theorem. Let $M^{12} = G_R \times \mathbb{C}^* \times \mathbb{R} \times \mathbb{R}$, and let $\mathbf{g}\colon M \to G_R$, $\mathbf{a}\colon M \to \mathbb{C}^*$, $\mathbf{p}\colon M \to \mathbb{R}$, and $\mathbf{t}\colon M \to \mathbb{R}$ be the projections onto the first through fourth factors, respectively. Let $\mathcal{I}_2$ be the exterior ideal on M generated by the linearly independent real-valued 1-forms $\theta_1, \ldots, \theta_8$ where

$$\begin{aligned}
\theta_1 &= i(\bar\eta - \eta)\\
\theta_2 &= \phi + i\mathbf{a}\omega - i\bar{\mathbf{a}}\bar\omega\\
\theta_3 + i\theta_4 &= \sigma + 2\bar{\mathbf{a}}\eta + 2\bar{\mathbf{a}}^2\mathbf{t}\bar\omega\\
\theta_5 + i\theta_6 &= d\mathbf{a} + 3i\mathbf{a}\tau + 6|\mathbf{a}|^2\mathbf{a}\mathbf{t}\eta\\
&\qquad - \mathbf{a}^2(\mathbf{t}\mathbf{p} - 3)\omega + \left(\tfrac{1}{2}R - |\mathbf{a}|^2 + 2|\mathbf{a}|^4\mathbf{t}^2\right)\bar\omega\\
\theta_7 &= d\mathbf{p} + 4(R + 4|\mathbf{a}|^2 + 4|\mathbf{a}|^4\mathbf{t}^2 + 3|\mathbf{a}|^2\mathbf{p}\mathbf{t} + \mathbf{p}^2)\eta\\
&\qquad + \left(\mathbf{p} + 2\mathbf{t}(R + 4|\mathbf{a}|^2 + 4|\mathbf{a}|^4\mathbf{t}^2 + 2|\mathbf{a}|^2\mathbf{p}\mathbf{t})\right)(\mathbf{a}\omega + \bar{\mathbf{a}}\bar\omega)\\
\theta_8 &= d\mathbf{t} - \mathbf{t}\left((\mathbf{p} + 12|\mathbf{a}|^2\mathbf{t})\eta + (1 + 4|\mathbf{a}|^2\mathbf{t}^2 + R/|\mathbf{a}|^2)(\mathbf{a}\omega + \bar{\mathbf{a}}\bar\omega)\right).
\end{aligned}$$

(The reason for the restriction $\mathbf{a} \neq 0$ is the division by $|\mathbf{a}|^2$ in the last formula.) The structure equations show that the ideal $\mathcal{I}_2$ is closed under exterior differentiation, so M is foliated by 4-dimensional integral manifolds of $\mathcal{I}_2$.

By construction, each leaf $L \subset M$ is the image of the bundle $B_1^{**} \subset G_R$ over the nondegenerate part Σ^{**} of a minimal Levi-flat hypersurface Σ satisfying equations (2) under the embedding

$$\mathrm{id} \times a \times p \times t\colon B_1 \longrightarrow G_R \times \mathbb{C}^* \times \mathbb{R} \times \mathbb{R} = M.$$

Since G_R acts by left translation on $G_R \times \mathbb{C}^* \times \mathbb{R} \times \mathbb{R}$ preserving the ideal $\mathcal{I}_2$, this left action permutes its integral manifolds, and two integral manifolds are equivalent under this action if and only if they correspond to congruent hypersurfaces in $\mathbb{P}^2_R$. In particular, two leaves L_1 and L_2 represent equivalent solutions if and only if they satisfy $(\mathbf{a},\mathbf{p},\mathbf{t})(L_1) = (\mathbf{a},\mathbf{p},\mathbf{t})(L_2)$.

In fact, in order for two leaves L_1 and L_2 to be equivalent under G_R, it suffices that the two image sets $(\mathbf{a},\mathbf{p},\mathbf{t})(L_1)$ and $(\mathbf{a},\mathbf{p},\mathbf{t})(L_2)$ in $\mathbb{C}^* \times \mathbb{R} \times \mathbb{R}$ have a nonempty intersection. To see why this is so, note that if L_i contains (g_i, a, p, t), then the submanifold L described by

$$L = \big\{(g_2 {g_1}^{-1} g, b, q, u) \mid (g, b, q, u) \in L_1\big\}$$

contains $(g_2, a, p, t) \in L_2$, is evidently a maximal integral manifold of $\mathcal{I}_2$, and so must equal L_2. In particular, in order to classify the solutions up to rigid motion, it would suffice to determine the partition of $\mathbb{C}^* \times \mathbb{R} \times \mathbb{R}$ into the images $(\mathbf{a},\mathbf{p},\mathbf{t})(L)$ as L ranges over the leaves of $\mathcal{I}_2$. Moreover, this argument shows that the fibers of the map $(\mathbf{a},\mathbf{p},\mathbf{t})\colon L \to \mathbb{C}^* \times \mathbb{R} \times \mathbb{R}$ are the orbits of the action on L of the ambient symmetry group of the corresponding solution Σ^{**}.

The structure equations imply that the function $\mathbf{t}$ cannot vanish anywhere on a leaf L unless it vanishes identically on L. As mentioned at the begining of this subsection, the leaves on which $\mathbf{t}$ vanishes identically are of type (1) and so can be set aside in this discussion. For the rest of this subsection, the assumption that $\mathbf{t}$ is nonvanishing on L will be in force.

3.2.2. Symmetries of the solutions. One might expect the images $(\mathbf{a},\mathbf{p},\mathbf{t})(L)$ to have dimension 4, at least at 'generic' points, since each leaf L has dimension 4. However, equations (2) imply that $da\wedge d\bar{a}\wedge dp\wedge dt$ vanishes identically. Consequently, the rank of $(\mathbf{a},\mathbf{p},\mathbf{t})\colon L \to \mathbb{C}^* \times \mathbb{R} \times \mathbb{R}$ is strictly less than 4 at all points, implying that the fibers of this map (and hence the symmetry group of L) must have positive dimension.

It is not hard to make these fibers explicit. By the structure equations (2), the (real) nowhere vanishing vector field Y on L that satisfies

$$\begin{aligned}
\tau(Y) &= R + 2|a|^2(pt-4) + 4|a|^4t^2\\
\eta(Y) &= 0\\
\omega(Y) &= 6i\bar{a}
\end{aligned}$$

also satisfies $da(Y) = dp(Y) = dt(Y) = 0$. The structure equations also show that, for the generic value $(a_0, p_0, t_0) \in \mathbb{C}^* \times \mathbb{R} \times \mathbb{R}$, the leaf L whose $(\mathbf{a},\mathbf{p},\mathbf{t})$-image contains (a_0, p_0, t_0) has the property that $(d\mathbf{a}, d\mathbf{p}, d\mathbf{t})$ has rank 3 along the preimage of (a_0, p_0, t_0). In particular, Y spans the tangent to the fiber at such points.

Given this, it would not be surprising to find that Y can be scaled so as to become a symmetry vector field. In fact, one finds that the flow of $X = e^f Y$ preserves the coframing (τ, ρ, ω) on L if and only if f satisfies the equation

$$df = 8|a|^2 t\eta - (pt - 2)(a\omega + \bar{a}\bar{\omega}).$$

Now, by the structure equations, the right hand side of this equation is a closed 1-form on L. This shows that, at least locally (or, more precisely, on some covering space of L), a scaling factor e^f exists making $X = e^f Y$ a symmetry vector field. Moreover, this f is unique up to the addition of a constant.

This implies that any solution hypersurface $\Sigma \subset \mathbb{P}^2_R$ whose structure equations are of the form (2) must actually be invariant under a one-parameter group of isometries of $\mathbb{P}^2_R$, i.e., a one-parameter subgroup of G_R. Moreover, because $\eta(Y) = 0$, this one-parameter subgroup can be chosen (if it is not actually unique) so that it preserves each complex leaf in Σ.

3.2.3. Explicit solutions invariant under a given 1*-parameter subgroup.* A one-parameter subgroup of isometries of $\mathbb{P}^2_R$ isz of the form $\{e^{tz} \mid t \in \mathbb{R}\}$ for some $z \neq 0$ in $\mathfrak{g}_R$. There is a unique holomorphic vector field Z on $\mathbb{P}^2_R$ whose real part is the infinitesimal generator of the action of the subgroup $\{e^{tz} \mid t \in \mathbb{R}\}$. If $U_z \subset \mathbb{P}^2_R$ denotes the open set that is the complement of the fixed locus of the flow e^{tz}, then U_z is foliated by complex curves that are the 'integral curves' of the holomorphic flow generated by Z. By the above discussion, the nondegenerate part Σ^{**} of any solution Σ of type (2) will be swept out by a (real) one-parameter family of integral curves of Z for some isometric flow e^{tz}. Since, by construction, the complex leaves of a solution of type (2) are not totally geodesic, this shows that $z \in \mathfrak{g}_R$ must be chosen so that the Z-integral curves in U_z are not totally geodesic. I will refer to a $z \in \mathfrak{g}_R$ with this property as *nondegenerate.*

Conversely, starting with any one-parameter subgroup e^{tz} of isometries of $\mathbb{P}^2_R$ and considering the corresponding holomorphic foliation of $U_z \subset \mathbb{P}^2_R$ by complex curves, one can construct e^{tz}-invariant Levi-flat hypersurfaces in U_z by taking the union of any (real) one-parameter family of complex leaves of this foliation. It now suffices to show that one can choose this one-parameter family in such a way that the resulting hypersurface will be minimal. I am going to show that this can always be done, essentially in only one way up to isometry, and that, when z is nondegenerate in the sense of the previous paragraph, this always yields a solution Σ of type (2). Thus, the solutions of type (2) correspond to the conjugacy classes of nondegenerate one-parameter subgroups of isometries of $\mathbb{P}^2_R$.

First, consider the case where $R > 0$. Without essential loss of generality, I can assume that $R = 1$, so that $G_R = G_1 = \mathrm{SU}(3)$. Every one-parameter subgroup of SU(3) is semi-simple and hence conjugate to a diagonal subgroup generated by a nonzero element

$$z = \begin{pmatrix} i\lambda_0 & 0 & 0 \\ 0 & i\lambda_1 & 0 \\ 0 & 0 & i\lambda_2 \end{pmatrix} \quad \text{where} \quad \lambda_0 + \lambda_1 + \lambda_2 = 0.$$

The corresponding vector field on $\mathbb{C}^3$ (which is also well-defined on $\mathbb{P}^2_R \simeq \mathbb{P}^2$) can be written in terms of unitary holomorphic coordinates $z = (z^a)$ as the real part of the holomorphic vector field

$$Z = i\lambda_0 z^0 \frac{\partial}{\partial z^0} + i\lambda_1 z^1 \frac{\partial}{\partial z^1} + i\lambda_2 z^2 \frac{\partial}{\partial z^2}.$$

The holomorphic integral curve of Z through $c = [c^a] \in \mathbb{P}^2_R$ is of the form

$$\left\{ \begin{bmatrix} c^0 e^{i\lambda_0 w} \\ c^1 e^{i\lambda_1 w} \\ c^2 e^{i\lambda_2 w} \end{bmatrix} \;\middle|\; w \in \mathbb{C} \right\}.$$

This will be a point or a line for all such c if and only if two of the λ_i are equal. In such a case, the integral curves of Z are open subsets of lines through a fixed point in $\mathbb{P}^2$. Thus any minimal Levi-flat hypersurface whose complex leaves are integral curves of Z will be of type (1). Set this case aside and, from now on, assume that z is nondegenerate, i.e., that the λ_i are mutually distinct.

Since radial dilation has no effect on the projective space, the flow the vector field Z induces on $\mathbb{P}^2$ is the same as that of the vector field

$$Z' = i(\lambda_1 - \lambda_0) z^1 \frac{\partial}{\partial z^1} + i(\lambda_2 - \lambda_0) z^2 \frac{\partial}{\partial z^2}$$

and this, in turn, will have the same holomorphic integral curves in $\mathbb{P}^2$ as

$$Z'' = z^1 \frac{\partial}{\partial z^1} + \lambda z^2 \frac{\partial}{\partial z^2} \quad \text{where} \quad \lambda = \frac{(\lambda_2 - \lambda_0)}{(\lambda_1 - \lambda_0)} \neq 0,\ 1.$$

The nonlinear integral curves of this vector field are of the form

$$\left\{ \begin{bmatrix} 1 \\ c e^{w} \\ e^{\lambda w} \end{bmatrix} \;\middle|\; w \in \mathbb{C} \right\},$$

where c is any nonzero complex constant. Thus, a Levi-flat hypersurface whose complex leaves are integral curves of this vector field can be locally parametrized in the form

$$\Sigma = \left\{ \begin{bmatrix} 1 \\ e^{w+x(r)+iy(r)} \\ e^{\lambda w} \end{bmatrix} \,\middle|\, w \in \mathbb{C},\ r \in I \right\},$$

where $x + iy\colon I \to \mathbb{C}$ is some smooth immersion of an interval $I \subset \mathbb{R}$. Brute force calculation then yields that such a hypersurface is minimal if and only if y is a constant function. Thus, up to a holomorphic isometry, such a minimal Levi-flat hypersurface is an open subset of the hypersurface

$$\Sigma_\lambda = \left\{ \begin{bmatrix} 1 \\ e^{w+r} \\ e^{\lambda w} \end{bmatrix} \,\middle|\, w \in \mathbb{C},\ r \in \mathbb{R} \right\}.$$

Note that Σ_λ is congruent to $\Sigma_{1/\lambda}$ but that, otherwise, the Σ_λ are mutually noncongruent. When λ is irrational, this hypersurface is dense in $\mathbb{P}^2$, but when $\lambda = p/q$ where p ($\neq 0$, q) and $q > 0$ are integers without common factors, this hypersurface is dense in an algebraically defined hypersurface that is singular at the point $z^1 = z^2 = 0$ but can also be singular along the entire lines $z^1 = 0$ and $z^2 = 0$, depending on the values of p and q. A typical such hypersurface is defined by an equation of the form $\mathrm{Im}\big((\bar{z}^0)^{p+q}(z^1)^p(z^2)^q\big) = 0$.

Next, consider the case where $R = 0$, i.e., when $\mathbb{P}^2_R = \mathbb{P}^2_0$ is isometric to $\mathbb{C}^2$ with its standard flat metric. Let (z^1, z^2) be unitary holomorphic linear coordinates on $\mathbb{C}^2$. A nonzero vector field whose flow is a holomorphic isometry on $\mathbb{C}^2$ is then conjugate via an action of G_0 to a constant multiple of the real part of either

$$Z = iz^1\frac{\partial}{\partial z^1} + i\lambda z^2\frac{\partial}{\partial z^2} \quad \text{or} \quad Z = i\frac{\partial}{\partial z^1} + i\lambda z^2\frac{\partial}{\partial z^2}$$

for some real number λ. In the first case, the holomorphic integral curves of Z will be lines in $\mathbb{C}^2$ if and only if $\lambda = 0$ or 1 while, in the second case, the holomorphic integral curves of Z will be lines in $\mathbb{C}^2$ if and only if $\lambda = 0$. These are the degenerate values that will be set aside, as these degenerate cases lead to the hyperplane or Clifford cone solutions that have already been discussed in the previous subsection.

Consider the first type of vector field with $\lambda \neq 0$ or 1. Any holomorphic integral curve of Z that is not contained in a line in $\mathbb{C}^2$ is of the form

$$\left\{ \begin{pmatrix} ce^{w} \\ e^{\lambda w} \end{pmatrix} \,\middle|\, w \in \mathbb{C} \right\},$$

where $c \in \mathbb{C}$ is a nonzero constant. A smooth Levi-flat hypersurface $\Sigma^3 \subset \mathbb{C}^2$ whose complex leaves consist of such integral curves can be locally parametrized in the form

$$\Sigma^3 = \left\{ \begin{pmatrix} e^{w+x(r)+iy(r)} \\ e^{\lambda w} \end{pmatrix} \;\middle|\; w \in \mathbb{C},\ r \in I \right\},$$

where $x + iy\colon I \to \mathbb{C}$ is some smooth immersion of an interval $I \subset \mathbb{R}$. Brute force calculation then yields that such a hypersurface is minimal if and only if y is a constant function. Consequently, it follows that, up to a holomorphic isometry, the connected solutions of this kind are all equivalent to open subsets of the immersed hypersurface

$$\Sigma_\lambda = \left\{ \begin{pmatrix} e^{w+r} \\ e^{\lambda w} \end{pmatrix} \;\middle|\; w \in \mathbb{C},\ r \in \mathbb{R} \right\}.$$

If λ is irrational, then Σ_λ is dense in $\mathbb{C}^2$ and the (implicitly described) immersion given above is an embedding. On the other hand, if $\lambda = p/q$ where $p \neq 0$ and $q > 0$ are distinct integers without common factors, then this immersion is not an embedding. Moreover, $\Sigma_{p/q}$ is dense in an algebraic real hypersurface, namely

$$\begin{aligned} (z^1)^p(\bar z^2)^q - (\bar z^1)^p(z^2)^q &= 0 \quad \text{when } p > 0, \\ (\bar z^1)^{-p}(\bar z^2)^q - (z^1)^{-p}(z^2)^q &= 0 \quad \text{when } p < 0. \end{aligned}$$

Note that these hypersurfaces are cones that are singular at the origin and along the axes except when p or q equals 1.

Consider the second type of vector field with $\lambda \neq 0$. Any holomorphic integral curve of Z that is not contained in a line in $\mathbb{C}^2$ is of the form

$$\left\{ \begin{pmatrix} w + c \\ e^{\lambda w} \end{pmatrix} \;\middle|\; w \in \mathbb{C} \right\},$$

where $c \in \mathbb{C}$ is a nonzero constant. A smooth Levi-flat hypersurface $\Sigma^3 \subset \mathbb{C}^2$ whose complex leaves consist of such integral curves can be locally parametrized in the form

$$\Sigma^3 = \left\{ \begin{pmatrix} w + x(r) + iy(r) \\ e^{\lambda w} \end{pmatrix} \;\middle|\; w \in \mathbb{C},\ r \in I \right\},$$

where $x + iy\colon I \to \mathbb{C}$ is some smooth immersion of an interval $I \subset \mathbb{R}$. Brute force calculation then yields that such a hypersurface is minimal if and only if y is a constant function. Consequently, up to a holomorphic isometry followed by a homothety, the connected solutions of this kind are open subsets of the closed embedded hypersurface

$$\Sigma = \left\{ \begin{pmatrix} w \\ re^w \end{pmatrix} \;\middle|\; w \in \mathbb{C},\ r \in \mathbb{R} \right\}.$$

(In this parametrization, the complex leaf given by $r = 0$ does not belong to Σ^*, as defined in §2.2.2.) This hypersurface can be defined implicitly by the equation

$$\operatorname{Im}(z^2 e^{-z^1}) = 0$$

and is evidently transcendental.

Finally, consider the case $R < 0$, where, without essential loss of generality, it suffices to consider only the case $R = -1$. The equivalence classes of one-dimensional subspaces of $\mathfrak{su}(2,1) = \mathfrak{g}_{-1}$ under the adjoint action are more complicated in this case. The elements z that have an eigenvector that is $\langle,\rangle_{-1}$-positive (and whose associated flow, therefore, has a fixed point in $\mathbb{P}^2_{-1}$) can be diagonalized in the form

$$z = \begin{pmatrix} i\lambda_0 & 0 & 0 \\ 0 & i\lambda_1 & 0 \\ 0 & 0 & i\lambda_2 \end{pmatrix} \quad \text{where} \quad \lambda_0 + \lambda_1 + \lambda_2 = 0.$$

If z has no $\langle,\rangle_{-1}$-positive eigenvector, then it must have a null eigenvector. In this case, the most generic possibility is for z to have three distinct eigenvalues, in which case two of the eigenvalues cannot be purely imaginary and their corresponding eigenvectors must be $\langle,\rangle_{-1}$-null. Consequently, one can normalize these eigenvectors and show that, up to a (real) multiple, z is conjugate to an element of the form

$$z = \begin{pmatrix} i\lambda & 1 & 0 \\ 1 & i\lambda & 0 \\ 0 & 0 & -2i\lambda \end{pmatrix} \quad \text{where } \lambda \in \mathbb{R}.$$

If z has a double eigenvalue with a unique $\langle,\rangle_{-1}$-null corresponding eigenvector and a $\langle,\rangle_{-1}$-negative eigenvector, then, up to a (real) multiple, z is conjugate to an element of the form

$$z = \begin{pmatrix} i(\lambda+1) & -i & 0 \\ i & i(\lambda-1) & 0 \\ 0 & 0 & -2i\lambda \end{pmatrix} \quad \text{where } 0 \neq \lambda \in \mathbb{R}.$$

If z has a triple eigenvalue, i.e., is nilpotent, then either $z^2 \neq 0$, in which case it is conjugate to an element of the form

$$z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix},$$

or else $z^2 = 0$ (the most degenerate case), in which case it is conjugate to an element of the form

$$z = \begin{pmatrix} i & -i & 0 \\ i & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Among these five cases, the holomorphic flow on $\mathbb{P}^2_{-1}$ corresponding to e^{tz} will have all integral curves be totally geodesic in two cases. In the case where z is diagonalizable, this happens when $\{\lambda_0, \lambda_1, \lambda_2\}$ are not distinct. Among the nondiagonalizable cases, this happens only for the last case, i.e., when $z^2 = 0$. These cases will be set aside, as they have already been treated in the discussion of type (1) solutions.

Now, in the diagonalizable case, the analysis proceeds exactly along the lines of the elliptic case and there is no need to give details. The end result is that a connected Levi-flat minimal hypersurface whose complex leaves are invariant under a nondegenerate diagonalizable flow is congruent to an open subset of the hypersurface

$$\Sigma_\lambda = \left\{ \begin{bmatrix} 1 \\ e^{w+r} \\ e^{\lambda w} \end{bmatrix} \;\middle|\; w \in \mathbb{C},\ r \in \mathbb{R},\ \left|e^{w+r}\right|^2 + \left|e^{\lambda w}\right|^2 < 1 \right\} \subset \mathbb{P}^2_{-1},$$

where λ is a real constant not equal to 0 or 1. This hypersurface has an algebraic defining equation if and only if λ is rational.

The next case, where z has two distinct $\langle , \rangle_{-1}$-null eigenvectors, can be analyzed in a similar manner and one finds that a connected Levi-flat minimal hypersurface whose complex leaves are invariant under the associated holomorphic flow is congruent to an open subset of the hypersurface

$$\Sigma'_\lambda = \left\{ \begin{bmatrix} e^{(1+i\lambda)(w+r)} + e^{(-1+i\lambda)w} \\ e^{(1+i\lambda)(w+r)} - e^{(-1+i\lambda)w} \\ 1 \end{bmatrix} \;\middle|\; w \in \mathbb{C},\ r \in \mathbb{R} \right\} \subset \mathbb{P}^2_{-1},$$

where λ is a real constant. When $\lambda = 0$, this is a real curve in the pencil of conics that pass through two points on the boundary of $\mathbb{P}^2_{-1}$ and have given tangents there. When λ is nonzero, the curves $r = r_0$ are not algebraic. (Of course, w and r must satisfy an inequality in order that the formula given in this description represent at point in $\mathbb{P}^2_{-1}$, but it is not useful to make this inequality explicit for the purposes at hand.)

In the case where z has a double eigenvalue (and not a triple one), a similar analysis shows that a connected Levi-flat minimal hypersurface whose complex leaves are invariant under the associated holomorphic flow is congruent to an open subset of the hypersurface

$$\Sigma''_\mu = \left\{ \begin{bmatrix} w+r+1 \\ w+r-1 \\ e^{\mu w} \end{bmatrix} \,\middle|\, w \in \mathbb{C},\ r \in \mathbb{R},\ 4r > e^{\mu(w+\bar{w})} - 2(w+\bar{w}) \right\} \subset \mathbb{P}^2_{-1},$$

where μ is a nonzero real constant.

In the final nondegenerate case, where the symmetry generator $z \in \mathfrak{g}_{-1}$ satisfies $z^2 \neq 0$ but $z^3 = 0$, the nonlinear integral curves of the associated holomorphic flow are conics (i.e., rational curves of degree 2) in $\mathbb{P}^2$, all tangent at a point on the boundary of $\mathbb{P}^2_{-1} \subset \mathbb{P}^2$. Brute force calculation shows that any Levi-flat minimal hypersurface whose complex leaves are invariant under such a flow is congruent to the hypersurface

$$\Sigma = \left\{ \begin{bmatrix} w^2+r+1 \\ w^2+r-1 \\ 2w \end{bmatrix} \,\middle|\, w \in \mathbb{C},\ r \in \mathbb{R},\ 4(\operatorname{Im} w)^2 < r \right\} \subset \mathbb{P}^2_{-1}.$$

Details will be left to the reader.

3.3. Solutions of type 3

Finally, consider the solutions of the system (3). To avoid repetition, I will set aside the cases where the solution reduces to one of type (1). This means that the solution has $s \neq 0$, which, by the structure equations (3), implies that s is nowhere vanishing.

3.3.1. Existence via the Frobenius theorem. Let $M^{13} = G_0 \times \mathbb{C} \times \mathbb{C} \times \mathbb{R}$, and let $\mathbf{g}\colon M \to G_0$, $\mathbf{a}\colon M \to \mathbb{C}$, $\mathbf{s}\colon M \to \mathbb{C}$, and $\mathbf{p}\colon M \to \mathbb{R}$ be the projections onto the first through fourth factors, respectively. Let $\mathcal{I}_3$ be the exterior ideal on M generated by the linearly independent real-valued 1-forms $\theta_1, \ldots, \theta_9$ where

$$\begin{aligned}
\theta_1 &= i(\bar{\eta} - \eta) \\
\theta_2 &= \phi + i\mathbf{a}\omega - i\bar{\mathbf{a}}\bar{\omega} \\
\theta_3 + i\theta_4 &= \sigma + 2\bar{\mathbf{a}}\eta + 2\mathbf{s}\bar{\omega} \\
\theta_5 + i\theta_6 &= d\mathbf{a} + 3i\mathbf{a}\tau + 6\bar{\mathbf{a}}\bar{\mathbf{s}}\eta - (\bar{\mathbf{s}}\mathbf{p} - 3\mathbf{a}^2)\omega - (|\mathbf{a}|^2 - 2|\mathbf{s}|^2)\bar{\omega} \\
\theta_7 + i\theta_8 &= d\mathbf{s} - \mathbf{s}\big(6i\tau + \mathbf{p}\eta + \mathbf{a}\omega + \bar{\mathbf{a}}\bar{\omega}\big)
\end{aligned}$$

$$\theta_9 = d\mathbf{p} - \big(8|\mathbf{s}|^2 - 64|\mathbf{a}|^2 - \mathbf{p}^2\big)\eta + (\mathbf{a}\mathbf{p} + 24\bar{\mathbf{a}}\bar{\mathbf{s}})\omega + (\bar{\mathbf{a}}\mathbf{p} + 24\mathbf{a}\mathbf{s})\bar{\omega}.$$

By the structure equations, the ideal $\mathcal{I}_3$ is closed under exterior differentiation, so M is foliated by 4-dimensional integral manifolds of $\mathcal{I}_3$.

By construction, each leaf $L \subset M$ is the image of the bundle $B_1^{**} \subset G_0$ over the nondegenerate part Σ^{**} of a minimal Levi-flat hypersurface Σ satisfying equations (3) under the embedding

$$\mathrm{id} \times a \times s \times p \colon B_1^{**} \longrightarrow G_0 \times \mathbb{C} \times \mathbb{C} \times \mathbb{R} = M.$$

Since G_0 acts by left translation on $G_0 \times \mathbb{C}^* \times \mathbb{R} \times \mathbb{R}$ preserving the ideal $\mathcal{I}_3$, this left action permutes its integral manifolds, and two integral manifolds are equivalent under this action if and only if they correspond to congruent hypersurfaces in $\mathbb{P}^2_0 \simeq \mathbb{C}^2$. In particular, two leaves L_1 and L_2 represent equivalent solutions if and only if they satisfy $(\mathbf{a},\mathbf{s},\mathbf{p})(L_1) = (\mathbf{a},\mathbf{s},\mathbf{p})(L_2)$.

In fact, in order for two leaves L_1 and L_2 to be equivalent under G_0, it suffices that the two image sets $(\mathbf{a},\mathbf{s},\mathbf{p})(L_1)$ and $(\mathbf{a},\mathbf{s},\mathbf{p})(L_2)$ in $\mathbb{C} \times \mathbb{C} \times \mathbb{R}$ have a nonempty intersection. To see why this is so, note that if L_i contains (g_i, a, s, p), then the submanifold L described by

$$L = \big\{(g_2 {g_1}^{-1} g, b, q, u) \bigm| (g, b, q, u) \in L_1\big\}$$

contains $(g_2, a, s, p) \in L_2$, is evidently a maximal integral manifold of $\mathcal{I}_3$, and so must equal L_2. In particular, in order to classify the solutions up to rigid motion, it would suffice to determine the partition of $\mathbb{C} \times \mathbb{C} \times \mathbb{R}$ into the images $(\mathbf{a},\mathbf{s},\mathbf{p})(L)$ as L ranges over the leaves of $\mathcal{I}_3$.

3.3.2. First integrals and the symmetry of solutions. Now, it would be reasonable to expect the images $(\mathbf{a},\mathbf{s},\mathbf{p})(L)$ to have dimension 4, at least at 'generic' points, since each leaf L has dimension 4. In fact, by the argument in the previous paragraph, it is evident that the fibers of the map $(\mathbf{a},\mathbf{s},\mathbf{p})\colon L \to \mathbb{C} \times \mathbb{C} \times \mathbb{R}$ are the orbits of the action on L of the ambient symmetry group of the corresponding solution Σ^{**}.

Consider the quantities[3]

$$\mathbf{A} = \tfrac{1}{9}|\mathbf{s}|^{2/3}\big(48|\mathbf{a}|^2 + 12|\mathbf{s}|^2 + \mathbf{p}^2\big),$$
$$\mathbf{B} = \tfrac{1}{27}|\mathbf{s}|\big(216\mathbf{a}^2\mathbf{s} + 216\bar{\mathbf{a}}^2\bar{\mathbf{s}} + 72|\mathbf{a}|^2\mathbf{p} - 36|\mathbf{s}|^2\mathbf{p} + \mathbf{p}^3\big).$$

The structure equations show that the 1-form $d(\mathbf{A}^3 - \mathbf{B}^2)$ lies in $\mathcal{I}_3$, which implies that the image $(\mathbf{a},\mathbf{s},\mathbf{p})(L)$ of any $\mathcal{I}_3$-leaf L lies in a level

[3]The significance of these quantities will become clear in the analysis to be carried out below.

set of $\mathbf{F} = \mathbf{A}^3 - \mathbf{B}^2$, a homogeneous polynomial of degree 8 in the variables $\mathbf{a}$, $\bar{\mathbf{a}}$, $\mathbf{s}$, $\bar{\mathbf{s}}$, and $\mathbf{p}$.

Calculation shows that $\mathbf{F} \geq 0$, with equality exactly along the 3-dimensional cone $C_0 \subset \mathbb{C} \times \mathbb{C} \times \mathbb{R}$ defined by the equations

$$0 = \mathbf{a}^2\mathbf{s} - \bar{\mathbf{a}}^2\bar{\mathbf{s}} = 8|\mathbf{a}|^2|\mathbf{s}|^2 - 4|\mathbf{s}|^4 - 2\mathbf{a}^2\mathbf{s}\mathbf{p} - 2\bar{\mathbf{a}}^2\bar{\mathbf{s}}\mathbf{p} + \mathbf{p}^2|\mathbf{s}|^2.$$

In particular, the $\mathcal{I}_3$-leaves that lie in $G_0 \times C_0$ represent either solutions of type (1) or of type (2), and have already been analysed in the previous subsections. Moreover, 0 is the only critical value of $\mathbf{F}$ on $\mathbb{C}\times\mathbb{C}\times\mathbb{R} \simeq \mathbb{R}^5$. The remaining level sets of $\mathbf{F}$ are smooth, connected hypersurfaces. In fact, because $\mathbf{F}$ is a homogeneous polynomial of degree 8, it follows that all of the positive level sets are diffeomorphic by homothety.

A rather laborious calculation using the structure equations above shows that for any $\mathcal{I}_3$-leaf L on which $\mathbf{F} = c^2 > 0$, the rank of the map $(\mathbf{a}, \mathbf{s}, \mathbf{p})\colon L \to \mathbb{C} \times \mathbb{C} \times \mathbb{R}$ is 4, i.e., that $(\mathbf{a}, \mathbf{s}, \mathbf{p})\colon L \to \mathbf{F}^{-1}(c^2)$ is a local diffeomorphism. The existence theorem proved above via the Frobenius theorem coupled with the G_0-invariance of $\mathcal{I}_3$ shows that this map must actually be a (surjective) covering map. Thus, there is a 1-parameter family of noncongruent solutions of type (3), one for each positive level set of $\mathbf{F}$.

3.3.3. The effect of homothety. While the members of this 1-parameter family are mutually incongruent by isometries, it turns out that they are congruent via homothety. To see this, note that if X is a vector field on $\mathbb{C}^2 \simeq \mathbb{P}^2_0$ that generates dilation about a fixed point, then X lifts to a vector field Y on G_0 that satisfies

$$\mathfrak{L}_Y\tau = \mathfrak{L}_Y\phi = \mathfrak{L}_Y\sigma = 0, \quad \mathfrak{L}_Y\eta = c\eta, \quad \mathfrak{L}_Y\omega = c\omega$$

for some nonzero (real) constant c. The vector field Y can then be lifted to a vector field Z on $G_0 \times \mathbb{C} \times \mathbb{C} \times \mathbb{R}$ so that it satisfies the same equations above as Y does but also satisfies

$$\mathfrak{L}_Z\mathbf{a} = -c\mathbf{a}, \quad \mathfrak{L}_Z\mathbf{s} = -c\mathbf{s}, \quad \mathfrak{L}_Z\mathbf{p} = -c\mathbf{p}.$$

It then follows from the formulae for the generators of $\mathcal{I}_3$ that the flow of Z leaves $\mathcal{I}_3$ invariant and therefore permutes the leaves of $\mathcal{I}_3$. Since $\mathbf{F}$ is homogeneous of degree 8 in $(\mathbf{a}, \mathbf{s}, \mathbf{p})$, it follows that $\mathfrak{L}_Z\mathbf{F} = -8c\mathbf{F}$. In particular, the flow of Z acts as homothety on the level sets of $\mathbf{F}$. Thus, any two solutions on which $\mathbf{F}$ is positive are congruent via homothety in $\mathbb{C}^2 \simeq \mathbb{P}^2_0$.

In this situation, it is therefore reasonable to restrict attention to the leaves that lie in the level set $\mathbf{F} = 1$. It is the solution corresponding to such a leaf that I am now going to describe. Note that the isometry group preserving such a solution is necessarily discrete.

3.3.4. Local integration of the equations. Suppose that one has a minimal Levi-flat hypersurface $\Sigma \subset \mathbb{C}^2$ for which the bundle B_1^{**} satisfies equations (3). Since s is nonzero on B_1^{**}, the structure equations show that there is a submanifold $B_2 \subset B_1^{**}$ defined as the set on which s is real and positive and that B_2 is a 6-fold cover of Σ^{**}. From now on, all functions and forms are to be regarded as pulled back to B_2.

The reality of s and the structure equation

$$s^{-1}ds = 6i\tau + p\eta + 3a\omega + 3\bar{a}\bar{\omega}$$

imply that $\tau = 0$. Combining this with the structure equation $d\eta = (a\omega + \bar{a}\bar{\omega}) \wedge \eta$ yields

$$d\big(s^{-1/3}\eta\big) = 0.$$

The structure equations also imply that the quantities

$$A = \tfrac{1}{9}s^{2/3}\big(48|a|^2 + 12s^2 + p^2\big),$$
$$B = \tfrac{1}{27}s\big(216a^2s + 216\bar{a}^2s + 72|a|^2p - 36s^2p + p^3\big)$$

introduced earlier satisfy equations of the form

$$dA = -4B\ s^{-1/3}\eta$$
$$dB = -6A^2s^{-1/3}\eta.$$

The assumption that the hypersurface Σ correspond to an $\mathcal{I}_3$-leaf on which $\mathbf{F}$ is identically equal to 1 is equivalent to the equation $A^3 - B^2 = 1$, so there is a unique function θ on B_2 with values in the open interval $(-\pi/2, \pi/2)$ for which

$$A = \sec^{2/3}\theta > 0 \quad \text{and} \quad B = -\tan\theta.$$

The above differential equations for A and B now imply that

$$s^{-1/3}\eta = \tfrac{1}{6}\sec^{2/3}\theta\, d\theta.$$

In particular, $d\theta$ never vanishes on B_2 and is a nonzero multiple of η.

The structure equations now imply that

$$d(s^{1/3}\omega) = \big(2s^{5/3}\bar{\omega} - \tfrac{1}{3}ps^{2/3}\omega\big) \wedge \tfrac{1}{6}\sec^{2/3}\theta\, d\theta.$$

It follows that any point $q \in B_2$ has a neighborhood U_0 on which there exists a complex valued function z, uniquely defined up to the addition of a (complex) function of t, and a complex function L, uniquely defined once z is chosen, so that

$$s^{1/3}\omega = dz + \tfrac{1}{6}L\sec^{2/3}\theta\, d\theta.$$

(Introducing such a coefficient in the $L\,d\theta$ term simplifies later calculations.) Because $dz \wedge d\bar{z} \wedge d\theta = s^{2/3}\omega \wedge \bar{\omega} \wedge d\theta \neq 0$, it follows that $(z,\theta)\colon U_0 \to \mathbb{C} \times \mathbb{R}$ is a local diffeomorphism. By restricting to an appropriate neighborhood $U_1 \subset U_0$ of q, I can assume that $(z,\theta)\colon U_1 \to \mathbb{R} \times \mathbb{C}$ defines a rectangular coordinate system (not necessarily centered on q). Write $z = x + iy$ where x and y are real-valued. Given the ambiguities in the choice of the coordinate system, partial differentiation with respect to z (or x or y) is coordinate independent although partial differentiation with respect to θ is not.

In these coordinates, the above structure equation for $d\bigl(s^{1/3}\omega\bigr)$ now becomes

$$dL \wedge \sec^{2/3}\theta\,d\theta = \bigl(2s^{4/3}d\bar{z} - \tfrac{1}{3}ps^{1/3}dz\bigr) \wedge \sec^{2/3}\theta\,d\theta,$$

so

$$L_z = -\tfrac{1}{3}ps^{1/3} \quad \text{and} \quad L_{\bar{z}} = 2s^{4/3}.$$

Set $u = s^{1/3}$. Then

$$u^{-1}du = \tfrac{1}{3}s^{-1}ds \equiv a\omega + \bar{a}\bar{\omega} \equiv u^{-1}\bigl(a\,dz + \bar{a}\,d\bar{z}\bigr) \bmod d\theta,$$

so $a = u_z$. The structure equation for da now gives

$$da \equiv (u^3p - 3a^2)u^{-1}dz + (|a|^2 - 2u^6)u^{-1}d\bar{z} \bmod d\theta,$$

so it follows that $u_{zz} = a_z = (u^3p - 3{u_z}^2)u^{-1}$, which can be written in the form

$$(u^4)_{zz} = 4u^5p.$$

Since p is real and since $v_{zz} = \frac{1}{4}(v_{xx} - v_{yy}) + \frac{i}{2}v_{xy}$ for any function v on U_1, it follows that $(u^4)_{xy} = 0$. Consequently, there exist functions f and g defined on the rectangles $(x,\theta)(U_1)$ and $(y,\theta)(U_1)$ in $\mathbb{R}^2$ so that

$$u^4 = f(x,\theta) - g(y,\theta) > 0.$$

These functions are unique up to the addition of a function of θ, i.e., one could replace $\bigl(f(x,\theta), g(y,\theta)\bigr)$ by $\bigl(f(x,\theta){+}h(\theta), g(y,\theta){+}h(\theta)\bigr)$ for some h defined on the interval $\theta(U_1)$, but this is the only ambiguity in the choice of these two functions.

Now, the equation for da also implies the equation $u_{z\bar{z}} = a_{\bar{z}} = (|u_z|^2 - 2u^6)u^{-1}$, which can be written in the form

$$(u^4)_{z\bar{z}} = u^{-4}\bigl|(u^4)_z\bigr|^2 - 8(u^4)^2.$$

Using the expression already found for u^4 plus the formulae $v_{z\bar{z}} = \frac{1}{4}(v_{xx} + v_{yy})$ and $|v_z|^2 = \frac{1}{4}\left(v_x{}^2 + v_y{}^2\right)$, this equation can be written in the form

$$f_{xx}(x,\theta) - g_{yy}(y,\theta) = \frac{f_x(x,\theta)^2 + g_y(y,\theta)^2}{f(x,\theta) - g(y,\theta)} - 32\big(f(x,\theta) - g(y,\theta)\big)^2.$$

Now, setting $v = f - g$, this can be written in the form

$$v_{xx} + v_{yy} = \frac{v_x{}^2 + v_y{}^2}{v} - 32v^2$$

and rearranged to give

$$\left(\frac{v_x}{v}\right)_x = \frac{v_y{}^2}{v^2} + \frac{v_{yy}}{v} - 32v.$$

Since $v_{xy} = 0$, both v_y and v_{yy} are constant in x. Thus, multiplying this equation by $2v_x/v$ and integrating with respect to x yields

$$\left(\frac{v_x}{v}\right)^2 = C(y,\theta) - \frac{v_y{}^2}{v^2} - \frac{2v_{yy}}{v} - 64v.$$

for some function C on $(y,\theta)(U_1)$. Now, multiplying by v^2 and substituting $v = f - g$, this can be written in the form

$$f_x(x,\theta)^2 = 2a_0(y,\theta) + 12a_1(y,\theta)f(x,\theta) + 48a_2(y,\theta)f(x,\theta)^2 - 64f(x,\theta)^3$$

for some functions a_0, a_1, and a_2 on $(y,\theta)(U_1)$. (The choice of numerical coefficients is cosmetic.)

Now, if the functions a_i really did depend on y, differentiating this equation with respect to y would then force $f(x,\theta)$ to be constant in x, making f_x vanish identically. This would, in turn, imply that $a = u_z = -\frac{1}{2}iu_y$ is purely imaginary, so that the quantity a^2s would be real. However, going back to the analysis in §2.2.2, this can only happen for solutions of type (2). Since the goal of this section is analyse the solutions of type (3) that have not already been accounted for by those of type (1) or (2), this case can therefore be set aside.

Thus, f satisfies an equation of the form

$$f_x(x,\theta)^2 = 2a_0(\theta) + 12a_1(\theta)f(x,\theta) + 48a_2(\theta)f(x,\theta)^2 - 64f(x,\theta)^3$$

for some functions a_0, a_1, and a_2 on $\theta(U_1)$. A similar analysis shows that there are functions b_0, b_1, and b_2 on $\theta(U_1)$ for which

$$g_y(y,\theta)^2 = -2b_0(\theta) - 12b_1(\theta)g(y,\theta) - 48b_2(\theta)g(y,\theta)^2 + 64g(y,\theta)^3.$$

Moreover, substituting these relations and their derivatives back into the original equation for v, it follows that $b_0 = a_0$, $b_1 = a_1$, and $b_2 = a_2$. Thus,

$$\begin{aligned} f_x(x,\theta)^2 &= 2a_0(\theta) + 12a_1(\theta)f(x,\theta) + 48a_2(\theta)f(x,\theta)^2 - 64f(x,\theta)^3, \\ g_y(y,\theta)^2 &= -2a_0(\theta) - 12a_1(\theta)g(y,\theta) - 48a_2(\theta)g(y,\theta)^2 + 64g(y,\theta)^3. \end{aligned}$$

By replacing $\big(f(x,\theta), g(y,\theta)\big)$ with $\big(f(x,\theta) - \frac{1}{4}a_2(\theta), g(y,\theta) - \frac{1}{4}a_2(\theta)\big)$, it can be arranged that $a_2 \equiv 0$. This removes the ambiguity in the choice of f and g.

At this point, f and g satisfy the equations

$$\begin{aligned} f_x(x,\theta)^2 &= 2a_0(\theta) + 12a_1(\theta)f(x,\theta) - 64f(x,\theta)^3 \\ g_y(y,\theta)^2 &= -2a_0(\theta) - 12a_1(\theta)g(y,\theta) + 64g(y,\theta)^3 \end{aligned}$$

as well as equations

$$\begin{aligned} f_{xx}(x,\theta) &= 6a_1(\theta) - 96f(x,\theta)^2, \\ g_{yy}(y,\theta) &= -6a_1(\theta) + 96g(y,\theta)^2. \end{aligned}$$

This information can now be substituted back into the previous formulae, yielding

$$\begin{aligned} s &= \big(f(x,\theta) - g(y,\theta)\big)^{3/4} \\ a &= \tfrac{1}{8}\big(f(x,\theta) - g(y,\theta)\big)^{-3/4}\big(f_x(x,\theta) + i g_y(y,\theta)\big) \\ p &= -6\big(f(x,\theta) - g(y,\theta)\big)^{-1/4}\big(f(x,\theta) + g(y,\theta)\big). \end{aligned}$$

Using these formulae, the definitions of A and B, and the equations satisfied by f and g, it now follows that

$$a_1(\theta) = A = \sec^{2/3}\theta \quad \text{and} \quad a_0(\theta) = B = -\tan\theta.$$

The previous formula for dL now simplifies to

$$dL \equiv 4f(x,\theta)dx + 4g(y,\theta)dy \bmod d\theta,$$

so that $L = F(x,\theta) + iG(y,\theta)$ for functions F and G satisfying $F_x = 4f$ and $G_y = 4g$. All this information combines to yield the formulae

$$\begin{aligned} \tau &= 0 \\ \eta &= \tfrac{1}{6}\big(f(x,\theta) - g(y,\theta)\big)^{1/4}\sec^{2/3}\theta\, d\theta \\ \omega &= \big(f(x,\theta) - g(y,\theta)\big)^{-1/4}\big(dz + \tfrac{1}{6}\sec^{2/3}\theta\big(F(x,\theta) + iG(y,\theta)\big)d\theta\big). \end{aligned}$$

Now, the cubic polynomial

$$p(\lambda,\theta) = -2\tan\theta + 12\sec^{2/3}\theta\,\lambda - 64\lambda^3$$

has three real, distinct roots in λ. In fact, defining

$$\begin{aligned}
r_1(\theta) &= \tfrac{1}{2}\sin\bigl(\tfrac{1}{3}\theta - \tfrac{2}{3}\pi\bigr)\sec^{1/3}\theta,\\
r_2(\theta) &= \tfrac{1}{2}\sin\bigl(\tfrac{1}{3}\theta\bigr)\sec^{1/3}\theta,\\
r_3(\theta) &= \tfrac{1}{2}\sin\bigl(\tfrac{1}{3}\theta + \tfrac{2}{3}\pi\bigr)\sec^{1/3}\theta,
\end{aligned}$$

one has $r_1(\theta) < r_2(\theta) < r_3(\theta)$ when $-\pi/2 < \theta < \pi/2$ and

$$p(\lambda,\theta) = -64\bigl(\lambda - r_1(\theta)\bigr)\bigl(\lambda - r_2(\theta)\bigr)\bigl(\lambda - r_3(\theta)\bigr).$$

Now, the differential equations on $f(x,\theta)$ and $g(y,\theta)$ coupled with the inequality $g(x,\theta) < f(y,\theta)$ imply the inequalities

$$r_1(\theta) < g(y,\theta) < r_2(\theta) < f(x,\theta) < r_3(\theta).$$

Moreover the differential equation for f (resp. g) can now be used to extend its range of definition from $(x,\theta)(U_1)$ (resp. $(y,\theta)(U_1)$) to all of $\mathbb{R}\times(-\pi/2,\pi/2)$. The extended functions satisfy

$$r_1(\theta) \le g(y,\theta) \le r_2(\theta) \le f(x,\theta) \le r_3(\theta)$$

and the periodicity relations

$$\begin{aligned}
f\bigl(x + 2\rho_+(\theta),\theta\bigr) &= f(x,\theta)\\
g\bigl(y + 2\rho_-(\theta),\theta\bigr) &= g(y,\theta)
\end{aligned}$$

where the functions $\rho_\pm$ are defined by the elliptic integrals

$$\begin{aligned}
\rho_+(\theta) &= \frac{1}{8}\int_{r_2(\theta)}^{r_3(\theta)} \frac{da}{\sqrt{\bigl(r_3(\theta)-a\bigr)\bigl(a-r_2(\theta)\bigr)\bigl(a-r_1(\theta)\bigr)}},\\
\rho_-(\theta) &= \frac{1}{8}\int_{r_1(\theta)}^{r_2(\theta)} \frac{da}{\sqrt{\bigl(r_3(\theta)-a\bigr)\bigl(r_2(\theta)-a\bigr)\bigl(a-r_1(\theta)\bigr)}}.
\end{aligned}$$

Note, by the way, that $\rho_+(-\theta) = \rho_-(\theta) > 0$ for θ in $(-\pi/2,\pi/2)$.

Using these extended functions, I can now modify x and y by adding functions of θ so as to arrange that

$$g(0,\theta) = r_2(\theta) = f(0,\theta).$$

This makes the coordinates (x, y, θ) unique up to replacement by coordinates of the form

$$(x^*, y^*, \theta^*) = \bigl(x + 2m\rho_+(\theta), y + 2n\rho_-(\theta), \theta\bigr)$$

for some integers m and n. These formulae will be important in the discussion of discrete symmetries that will be undertaken below.

The functions f and g are now uniquely defined on the entire strip $\mathbb{R} \times (-\pi/2, \pi/2)$ by the requirement that they satisfy the second order equations with initial conditions

$$\begin{aligned} f_{xx}(x,\theta) &= 6\sec^{2/3}\theta - 96f(x,\theta)^2, \quad f(0,\theta) = r_2(\theta), \quad f_x(0,\theta) = 0, \\ g_{yy}(y,\theta) &= -6\sec^{2/3}\theta + 96g(y,\theta)^2, \quad g(0,\theta) = r_2(\theta), \quad g_y(0,\theta) = 0. \end{aligned}$$

Then $u(x, y, \theta)^4 = f(x, \theta) - g(y, \theta) \geq 0$ is doubly periodic on $\mathbb{R} \times \mathbb{R} \times (-\pi/2, \pi/2)$ in the obvious sense and is strictly positive except along the curves $C_{m,n}$ of the form $(x, y, \theta) = \bigl(2m\rho_+(\theta), 2n\rho_-(\theta), \theta\bigr)$ for any integers m and n. The vanishing near these lines is very simple: Along $C_{0,0}$, i.e., the line $(x, y, \theta) = (0, 0, \theta)$, there are convergent Taylor expansions

$$f(x,\theta) = r_2(\theta) + \sum_{k=1}^{\infty} c_k(\theta)x^{2k}, \quad g(y,\theta) = r_2(\theta) + \sum_{k=1}^{\infty}(-1)^k c_k(\theta)y^{2k},$$

implying that there is a smooth function $\tilde{u}$ on $\mathbb{R} \times \mathbb{R} \times (-\pi/2, \pi/2)$ satisfying $\tilde{u}(0, 0, \theta) = c_1(\theta) = 3\sec^{2/3}\theta\bigl(1 - 4\sin^2(\frac{1}{3}\theta)\bigr) > 0$ for which

$$u(x,y,\theta)^4 = (x^2 + y^2)\tilde{u}(x,y,\theta).$$

By the periodicity relations, the description of the vanishing of u near the other curves $C_{m,n}$ follows from this one.

Now, examining the coefficient of $d\theta$ in the formula for ds yields the relation

$$g_y G - 6\cos^{2/3}\theta\, g_\theta - 8g^2 = f_x F - 6\cos^{2/3}\theta\, f_\theta - 8f^2.$$

The left hand side of this relation is independent of x while the right hand side is indepdendent of y, so that each side is a function of θ only. Evaluating either side at $x = y = 0$ then yields

$$\begin{aligned} g_y G - 6\cos^{2/3}\theta\, g_\theta - 8g^2 &= f_x F - 6\cos^{2/3}\theta\, f_\theta - 8f^2 \\ &= -6\cos^{2/3}\theta\, r_2'(\theta) - 8r_2(\theta)^2. \end{aligned}$$

Of course, this allows one to solve for F and G away from the places where f_x and g_y vanish, yielding formulae of the form

$$F = \left[6\cos^{2/3}\theta\,(f_\theta - r_2'(\theta)) + 8(f^2 - r_2(\theta)^2)\right]/f_x$$
$$G = \left[6\cos^{2/3}\theta\,(g_\theta - r_2'(\theta)) + 8(g^2 - r_2(\theta)^2)\right]/g_y.$$

Since $f_x(x,\theta) = 0$ if and only if x is an integer multiple of $\rho_+(\theta)$ and $g_y(y,\theta) = 0$ if and only if y is an integer multiple of $\rho_-(\theta)$, this gives integration-free formulae for F and G that are valid over a dense open set. Moreover, differentiating the relations above with respect to x or y and using the identities $F_x = 4f$ and $G_y = 4g$ yields

$$g_{yy}G - 6\cos^{2/3}\theta\,g_{y\theta} - 12gg_y = f_{xx}F - 6\cos^{2/3}\theta\,f_{x\theta} - 12ff_y = 0.$$

Since f_x and f_{xx} do not vanish simultaneously, and since g_y and g_{yy} do not vanish simultaneously, these relations together with the relations above yield explicit smooth formulae for F and G over all of $\mathbb{R}\times(-\pi/2,\pi/2)$. In particular, these formulae imply that $F(0,\theta) = G(0,\theta) = 0$, so that F and G can also be described by

$$F(x,\theta) = 4\int_0^x f(\xi,\theta)d\xi, \quad G(y,\theta) = 4\int_0^y f(\xi,\theta)d\xi.$$

The integration-free formulae yield pseudo-periodicity relations for F and G: Differentiating

$$f\big(x + 2\rho_+(\theta),\theta\big) = f(x,\theta)$$

with respect to θ shows that f_θ satisfies the pseudo-periodicity relation

$$f_\theta\big(x + 2\rho_+(\theta),\theta\big) - f_\theta(x,\theta) = -2f_x(x,\theta)\rho_+'(\theta).$$

Consequently, F satisfies the pseudo-periodicity relation

$$F\big(x + 2\rho_+(\theta),\theta\big) - F(x,\theta) = -12\rho_+'(\theta)\cos^{2/3}\theta.$$

Similarly,

$$G\big(y + 2\rho_-(\theta),\theta\big) - G(y,\theta) = -12\rho_-'(\theta)\cos^{2/3}\theta.$$

At this point, all the structure equations in (3) are identities.

3.3.5. Global structure of the solution. The local information derived in the previous subsubsection can now be used to give a global description of the corresponding minimal Levi-flat hypersurface in $\mathbb{C}^2$. To begin, define r_i for $i = 1$, 2, and 3 and $\rho_\pm$ as functions on $(-\pi/2, \pi/2)$ by the already listed formulae. Then, define functions f and g on $\mathbb{R} \times (-\pi/2, \pi/2)$ by the differential equations with initial conditions:

$$\begin{aligned} f_{xx}(x,\theta) &= 6\sec^{2/3}\theta - 96f(x,\theta)^2, \quad f(0,\theta) = r_2(\theta), \quad f_x(0,\theta) = 0, \\ g_{yy}(y,\theta) &= -6\sec^{2/3}\theta + 96g(y,\theta)^2, \quad g(0,\theta) = r_2(\theta), \quad g_y(0,\theta) = 0. \end{aligned}$$

Note that f is even and periodic of period $2\rho_+(\theta)$ in its first argument while g is even and periodic of period $2\rho_-(\theta)$ in its first argument. Moreover, these functions automatically satisfy the first order equations

$$\begin{aligned} f_x(x,\theta)^2 &= 2\tan\theta + 12\sec^{2/3}\theta\, f(x,\theta) - 64f(x,\theta)^3, \\ g_y(y,\theta)^2 &= -2\tan\theta - 12\sec^{2/3}\theta\, g(y,\theta) + 64g(y,\theta)^3. \end{aligned}$$

Define F and G on the same domain by

$$F(x,\theta) = \int_0^x 4f(\xi,\theta)d\xi, \quad G(y,\theta) = \int_0^y 4g(\xi,\theta)d\xi.$$

Let $D = \mathbb{R} \times \mathbb{R} \times (-\pi/2, \pi/2)$ and let $D^* \subset D$ be the complement of the curves

$$C_{m,n} = \left\{ \left(2m\rho_+(\theta), 2n\rho_-(\theta), \theta\right) \,\middle|\, \theta \in \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right) \right\}.$$

Finally, define functions and 1-forms on D^* by the formulae

$$\begin{aligned} s &= \big(f(x,\theta) - g(y,\theta)\big)^{3/4}, \\ a &= \tfrac{1}{8}\big(f(x,\theta) - g(y,\theta)\big)^{-3/4}\big(f_x(x,\theta) + i g_y(y,\theta)\big), \\ p &= -6\big(f(x,\theta) - g(y,\theta)\big)^{-1/4}\big(f(x,\theta) + g(y,\theta)\big), \\ \eta &= \tfrac{1}{6}\big(f(x,\theta) - g(y,\theta)\big)^{1/4}\sec^{2/3}\theta\, d\theta, \\ \omega &= \big(f(x,\theta) - g(y,\theta)\big)^{-1/4}\big(dz + \tfrac{1}{6}\sec^{2/3}\theta\,\big(F(x,\theta) + iG(y,\theta)\big)d\theta\big), \\ \tau &= 0. \end{aligned}$$

Then the structure equations (3) are satisfied on D^*. In particular, setting $\sigma = -2\bar{a}\eta - 2s\bar{\omega}$ and $\phi = -ia\omega + i\bar{a}\bar{\omega}$, the $\mathfrak{g}_0$-valued 1-form

$$\gamma = \begin{pmatrix} 0 & 0 & 0 \\ \eta & i\phi & -\bar{\sigma} \\ \omega & \sigma & -i\phi \end{pmatrix}$$

satisfies $d\gamma = -\gamma \wedge \gamma$.

By the usual moving frame argument [Gr], it follows that, if $U \subset D^*$ is any simply connected domain in D^*, then there is a map $\mathbf{g}\colon U \to G_0$, unique up to left translation by a constant, so that $\mathbf{g}^{-1}d\mathbf{g} = \gamma_{|U}$. The projection $\mathbf{g}K\colon U \to G_0/K = \mathbb{C}^2$ is then an immersion of U into $\mathbb{C}^2$ as a minimal Levi-flat hypersurface of type (3). However, this argument does not provide a description of the topology or global properties of the solution. It is to this description that I now turn.

The group $\mathbb{Z}^2$ acts on D preserving D^* via the maps

$$\Phi_{m,n}(x, y, \theta) = \big(x + 2m\rho_+(\theta), y + 2n\rho_-(\theta), \theta\big).$$

Denote the $\mathbb{Z}^2$-orbit of (x, y, θ) by $[x, y, \theta] \in N$. The periodicity relations on f and g combined with the pseudo-periodicity relations on F and G imply $\Phi^*_{m,n}\gamma = \gamma$. (In fact, all the quantitites s, a, p, η, ω, and τ $(= 0)$ are invariant under this $\mathbb{Z}^2$-action.) Thus γ is well-defined on the quotient space $N^* = D^*/\mathbb{Z}^2$, which is diffeomorphic to a punctured torus cross an open interval.

On $N^* \times G_0$, thought of as a trivialized principal left G_0-bundle over N^*, consider the $\mathfrak{g}_0$-valued connection 1-form

$$\psi = dg\, g^{-1} - g\gamma g^{-1} = g\big(g^{-1}dg - \gamma\big)g^{-1}.$$

Since $d\gamma = -\gamma \wedge \gamma$, it follows that $d\psi = \psi \wedge \psi$, i.e., that ψ is flat. Consequently, $N^* \times G_0$ is foliated by ψ-leaves, each of which is a smooth submanifold $L \subset N^* \times G_0$ such that projection onto the first factor is a covering map and such that any two leaves differ by left action in the G_0-factor by a constant element of G_0. For any such leaf L, we can regard the functions s, a, p and 1-forms η and ω as being well defined on L via pullback from the projection $L \to N^*$.

The map $(g, a, s, p)\colon L \to M^{13} = G_0 \times \mathbb{C} \times \mathbb{C} \times \mathbb{R}$ then immerses L as an $\mathcal{I}_3$-leaf lying in the locus $\mathbf{F} = 1$. By the construction of γ and the development that led up to it, the image of L is a complete $\mathcal{I}_3$-leaf. Thus, the topology of the leaves will be known once the covering map $L \to D^*$ and the projection $g\colon L \to G_0$ are understood.

The projection $g\colon L \to G_0$ is simply a diffeomorphism. This follows because, on L, the $\mathfrak{g}_0$-valued 1-form $g^{-1}dg$ is simply γ, which determines the forms ω and η and the functions s, a, and p. The construction of the coordinate system (x, y, θ) from (η, ω, s, a, p) shows that this suffices to recover the map $(x, y, \theta)\colon L \to D^*$ up to the action of $\mathbb{Z}^2$, which is the same as recovering $[x, y, \theta]\colon L \to N^*$ and hence the full embedding of L into $N^* \times G_0$. In particular, this implies that (g, a, s, p) is an embedding.

Now, a leaf L is just the holonomy bundle of ψ through each of its points. For the sake of concreteness, choose $n_0 = [\rho_0, \rho_0, 0] \in N^*$ as basepoint, where $\rho_0 = \rho_+(0) = \rho_-(0)$ and let $L \subset N^* \times G_0$ be the leaf of ψ that passes through (n_0, I_3). The intersection $L \cap (\{n_0\} \times G_0)$ is then of the form $\{n_0\} \times \Gamma$ where $\Gamma \subset G_0$ is the holonomy subgroup of ψ and this is what must be computed. The calculations below will actually determine the ψ-monodromy homomorphism $\pi_1(N^*, n_0) \to G_0$, whose image is Γ.

Since N^* is an interval cross a punctured torus, $\pi_1(N^*, n_0)$ is generated by the loops $X\colon [0, 2\rho_0] \to N^*$ and $Y\colon [0, 2\rho_0] \to N^*$ defined by

$$X(x) = [x + \rho_0, \rho_0, 0], \quad Y(y) = [\rho_0, y + \rho_0, 0].$$

To compute the ψ-monodromy around these two loops, information about the behavior of the functions f and g when $\theta = 0$ will be used. To begin, note that $r_1(0) = -\sqrt{3}/4$, $r_2(0) = 0$, and $r_3(0) = \sqrt{3}/4$ and observe that, by the symmetry properties of f and g, there is a $2\rho_0$-periodic function v on $\mathbb{R}$ that satisfies

$$v(t) = f(t + \rho_0, 0) + \sqrt{3}/4 = \sqrt{3}/4 - g(t + \rho_0, 0)$$

for all t. In fact, v is defined by the conditions that it satisfy both the initial condition $v(0) = \sqrt{3}/2$ and the Weierstraß-type differential equation

$$\big(v'(t)\big)^2 = 64 v(t)\big(\sqrt{3}/2 - v(t)\big)\big(v(t) - \sqrt{3}/4\big).$$

Note that v is positive, satisfying $\sqrt{3}/4 \le v(t) \le \sqrt{3}/2$, and that v is an even function on $\mathbb{R}$. In particular, satisfies $v(2\rho_0 - t) = v(t)$, a fact that will be used below.

Now, from the definition of X it follows that

$$X^*(\gamma) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2\big(v(x)\big)^{1/2} dx \\ \big(v(x)\big)^{-1/4} dx & -2\big(v(x)\big)^{1/2} dx & 0 \end{pmatrix}.$$

Consider the $g_X\colon [0, 2\rho_0] \to G_0$ that satisfies ${g_X}^{-1} dg_X = X^*\gamma$ and $g_X(0) = \mathrm{I}_3$. Because $X^*\gamma$ takes values in $\mathfrak{g}_0 \cap \mathfrak{sl}(3, \mathbb{R})$, the map g_X has values in $G_0 \cap \mathrm{SL}(3, \mathbb{R})$ and so can be written in the form

$$g_X(x) = \begin{pmatrix} 1 & 0 & 0 \\ u_1(x) & \cos\varphi(x) & \sin\varphi(x) \\ u_2(x) & -\sin\varphi(x) & \cos\varphi(x) \end{pmatrix},$$

where the functions u_1, u_2, and φ on $[0, 2\rho_0]$ are defined by the ODE system

$$\begin{aligned} u_1'(x) &= \sin\varphi(x)\,\bigl(v(x)\bigr)^{-1/4}, & u_1(0) &= 0,\\ u_2'(x) &= \cos\varphi(x)\,\bigl(v(x)\bigr)^{-1/4}, & u_2(0) &= 0,\\ \varphi'(x) &= \qquad 2\,\bigl(v(x)\bigr)^{1/2}\;, & \varphi(0) &= 0. \end{aligned}$$

The ODE that v satisfies suggests a change of variables eliminating the explicit x-dependence, yielding

$$\begin{aligned} \varphi(2\rho_0) &= \int_0^{2\rho_0} 2\bigl(v(x)\bigr)^{1/2}dx\\ &= 4\cdot\frac{1}{8}\int_{\sqrt{3}/4}^{\sqrt{3}/2} \frac{v^{1/2}dv}{\sqrt{v\bigl(\sqrt{3}/2-v\bigr)\bigl(v-\sqrt{3}/4\bigr)}}\\ &= \frac{\pi}{2}. \end{aligned}$$

Thus φ defines a diffeomorphism $\varphi\colon [0, 2\rho_0] \to [0, \pi/2]$ that, because of the symmetries of v, has the symmetry $\varphi(2\rho_0 - x) = \pi/2 - \varphi(x)$. In turn, this implies that $u_i'(x) > 0$ for all $x \in (0, 2\rho_0)$ and, by a straightforward change of variables, that $u_1(2\rho_0) = u_2(2\rho_0) = r$ for some[4] $r > 0$.

This implies that $g_X(2\rho_0) = h_X$ where

$$h_X = \begin{pmatrix} 1 & 0 & 0\\ r & 0 & 1\\ r & -1 & 0 \end{pmatrix}.$$

This h_X represents the holonomy of ψ around the loop X. (Note that it is possible to compute the map g_X and hence the holonomy h_X by quadratures in this manner because $X^*\gamma$ takes values in a solvable subalgebra of $\mathfrak{g}_0$.)

A similar argument for Y gives

$$Y^*(\gamma) = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & -2i\bigl(v(y)\bigr)^{1/2}dy\\ \bigl(v(y)\bigr)^{-1/4}dy & -2i\bigl(v(y)\bigr)^{1/2}dy & 0 \end{pmatrix}.$$

Carrying out the same sort of analysis as was applied to X leads to the conclusion that if $g_Y\colon [0, 2\rho_0] \to G_0$ is the map that satisfies

[4]For the curious: Numerical calculation yields $\rho_0 \approx 0.498083225$ and $r \approx .565201447$.

${g_Y}^{-1}dg_Y = Y^*\gamma$ and $g_Y(0) = \mathrm{I}_3$, then $g_Y(2\rho_0) = h_Y$ where

$$h_Y = \begin{pmatrix} 1 & 0 & 0 \\ -ir & 0 & -i \\ r & -i & 0 \end{pmatrix}.$$

Thus h_Y represents the holonomy of ψ around the loop Y.

Now, setting

$$\mathbf{v} = \begin{pmatrix} 0 \\ r \end{pmatrix} \neq 0,$$

and

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{i} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \mathbf{k} = \mathbf{ij} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix},$$

it follows that

$$h_X = \begin{pmatrix} 1 & 0 \\ (\mathbf{1}+\mathbf{i})\mathbf{v} & \mathbf{i} \end{pmatrix}, \quad h_Y = \begin{pmatrix} 1 & 0 \\ (\mathbf{1}+\mathbf{j})\mathbf{v} & \mathbf{j} \end{pmatrix}.$$

Noting that $\mathbf{i}^2 = \mathbf{j}^2 = -\mathbf{1}$ while $\mathbf{k} = \mathbf{ij} = -\mathbf{ji}$, it is evident that ${h_X}^4 = {h_Y}^4 = \mathrm{I}_3$ and that any iterated product of the matrices h_X and h_Y is of the form

$$h = \begin{pmatrix} 1 & 0 \\ (a_0\mathbf{1} + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})\mathbf{v} & \mathbf{q} \end{pmatrix}$$

where $\mathbf{q}$ lies in $\{\pm\mathbf{1}, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$ and the a_i are integers whose sum is even. In particular, the subgroup $\Gamma \subset G_0$ generated by h_X and h_Y is discrete. Moreover, the homomorphism $\Gamma \to \{\pm\mathbf{1}, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$ defined by $h \mapsto \mathbf{q}$ in the above notation is surjective. It is not difficult to establish that the kernel $\hat{\Lambda}$ of this homomorphism consists exactly of the matrices of the form

$$\begin{pmatrix} 1 & 0 \\ 2(a_0\mathbf{1} + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})\mathbf{v} & \mathbf{1} \end{pmatrix}$$

where the a_i are integers whose sum is even. Since $\mathbf{v} \neq 0$, the set $\Lambda \subset \mathbb{C}^2$ consisting of the vectors $2(a_0\mathbf{1}+a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\mathbf{v}$ where the a_i are integers whose sum is even is a lattice in $\mathbb{C}^2$, i.e., a discrete abelian subgroup of rank 4. Up to rotation and scaling, Λ is a lattice of type F_4. In what follows, it will be useful to identify Λ with $\hat{\Lambda} \subset G_0$ via the identification

$$2(a_0\mathbf{1} + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})\mathbf{v} \longmapsto \begin{pmatrix} 1 & 0 \\ 2(a_0\mathbf{1} + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})\mathbf{v} & \mathbf{1} \end{pmatrix},$$

so I will do this henceforth without explicit comment.

Let $\hat{K} \subset \pi_1(N^*, n_0)$ denote the normal subgroup of index 8 consisting of those homotopy classes of loops whose ψ-holonomy lies in $\hat{\Lambda}$ and let $\hat{N}^* \to N^*$ denote the 8-fold covering space corresponding to $\hat{K}$. I am going to show that there is a way of 'completing' $\hat{N}^*$ in a natural way so that each of the complex leaves of $\hat{N}^*$ (i.e., the leaves of $\eta = 0$) is realized as a compact Riemann surface of genus 3 punctured at four points. I will then examine to what extent the functions and forms s, a, p, η, and ω extend smoothly across these punctures.

Ultimately, the goal is to show that $\mathbb{C}^2/\Lambda$ contains a minimal Levi-flat hypersurface whose complex leaves are (compact) Riemann surfaces of genus 3.

Let $\tilde{N}$ be the quotient of D by the action of $(2\mathbb{Z})^2$, i.e., the index 4 subgroup of $\mathbb{Z}^2$ generated by the transformations $\Phi_{2m,2n}$, and let $\tilde{N}^* \subset \tilde{N}$ be the image of $D^* \subset D$ under this quotient action. Let $\langle x, y, \theta\rangle \in \tilde{N}$ denote the equivalence class of $(x, y, \theta) \in D$ under the action of $(2\mathbb{Z})^2$. Any product of a finite sequence drawn from $\{h_X, h_Y\}$ that contains an odd number of copies of either h_X or h_Y will be an $h \in \Gamma$ whose corresponding $\mathbf{q}$ lies in $\{\pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$. Consequently, the quotient map $\tilde{N}^* \to N^*$ defines a 4-fold cover of N^* that is, itself, a 2-fold quotient of $\hat{N}^*$. I.e., there is a sequence of coverings

$$\hat{N}^* \xrightarrow{2-1} \tilde{N}^* \xrightarrow{4-1} N^*$$

corresponding to the inclusion of subgroups $\{\mathbf{1}\} \subset \{\pm\mathbf{1}\} \subset \{\pm\mathbf{1}, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$. The commutator loop $Y^{-1} * X^{-1} * Y * X$ is closed in $\tilde{N}^*$ and this is a loop over which the cover $\hat{N}^* \to \tilde{N}^*$ is non-trivial since this loop does not lie in $\hat{K}$.

It will be useful to construct a embedding of $\tilde{N}^*$ into $\mathbb{CP}^2 \times (-\pi/2, \pi/2)$. Consider the meromorphic solution $\mathfrak{p}$ on $\mathbb{C} \times (-\pi/2, \pi/2)$ to the second order holomorphic differential equation with initial conditions

$$\mathfrak{p}_{zz}(z,\theta) = 6\sec^{2/3}\theta - 96\mathfrak{p}(z,\theta)^2, \quad \mathfrak{p}(0,\theta) = r_2(\theta), \quad \mathfrak{p}_z(0,\theta) = 0.$$

Of course, $\mathfrak{p}$ is a version of the Weierstrass $\mathfrak{p}$-function. It satisfies the first order differential equation

$$\mathfrak{p}_z(z,\theta)^2 = -2\tan\theta + 12\sec^{2/3}\theta\,\mathfrak{p}(z,\theta) - 64\mathfrak{p}(z,\theta)^3.$$

Moreover $\mathfrak{p}(x,\theta) = f(x,\theta)$ when x is real and $\mathfrak{p}(iy,\theta) = g(y,\theta)$ when y is real, as follows easily from the Chain Rule. Now, $\mathfrak{p}$ is doubly periodic and even:

$$\mathfrak{p}\big(z+2\rho_+(\theta)\big)=\mathfrak{p}\big(z+2i\rho_-(\theta)\big)=\mathfrak{p}(-z)=\mathfrak{p}(z).$$

and also assumes the special values

$$\begin{aligned}\mathfrak{p}\big(i\rho_-(\theta)\big)&=r_1(\theta), & \mathfrak{p}(0)&=r_2(\theta),\\ \mathfrak{p}\big(\rho_+(\theta)\big)&=r_3(\theta), & \mathfrak{p}\big(\rho_+(\theta)+i\rho_-(\theta)\big)&=\infty.\end{aligned}$$

In fact, $\mathfrak{p}$ has a double pole at $\rho_+(\theta)+i\rho_-(\theta)$ and no other singularities in the fundamental rectangle. Moreover, $\mathfrak{p}_z$ has simple zeros at 0, $\rho_+(\theta)$, $i\rho_-(\theta)$ and a triple pole at $\rho_+(\theta)+i\rho_-(\theta)$.

Now consider, for each fixed θ in the interval $(-\pi/2,\pi/2)$, the quadratic form

$$ds^2_\theta=\big(f(x,\theta)-g(y,\theta)\big)(dx^2+dy^2).$$

By the earlier analysis of the vanishing locus of $u\colon D\to\mathbb{R}$, this quadratic form defines a conformal pseudo-metric on $\mathbb{C}$ that branches to order 1 at the points of the lattice

$$\Lambda_\theta=\big\{2m\rho_+(\theta)+i\,2n\rho_-(\theta)\mid m,\,n\in\mathbb{Z}\big\}\subset\mathbb{C}$$

and is periodic with respect to this lattice. Since ds^2_θ is invariant under reflection in the x-axis and the y-axis, the lines $x=m\rho_+(\theta)$ and $y=n\rho_-(\theta)$ for integer m and n are geodesics in this metric.

The structure equations show that ds^2_θ has constant Gauss curvature $K=16$, and so must be induced by pullback from the standard metric on the Riemann sphere with this curvature. In particular, there is a meromorphic function w on the z-plane so that

$$f(x,\theta)-g(y,\theta)=\frac{|w'(z)|^2}{4\big(1+|w(z)|^2\big)^2}.$$

The function w must ramify to order 1 at each of the points of Λ_θ and must carry the geodesics $x=2m\rho_+(\theta)$ and $y=2n\rho_-(\theta)$ onto a single geodesic on the Riemann sphere. (Since they intersect at right angles in the z-plane and the intersection point is a branch point of w of order 2, the image geodesics must meet at an angle of π and hence must lie along the same geodesic on the sphere.) This information is not enough to make the function w unique; it only determines w up to composition with an isometric rotation of the Riemann sphere. However, adding the requirements that $w(0)=0$ and that $w''(0)$ be real and positive do make w unique, so this will be assumed from now on.

Because the geodesic segment $t\rho_+(\theta)$ for $0 \le t \le 2$ is congruent to the geodesic segment $t\rho_+(\theta) + 2i\rho_-(\theta)$ for $0 \le t \le 2$, and because the geodesic segment $it\rho_-(\theta)$ for $0 \le t \le 2$ is congruent to the geodesic segment $2\rho_+(\theta) + it\rho_-(\theta)$ for $0 \le t \le 2$, and because there are no ramification points of w in the interior of the fundamental rectangle, it follows that the normalized w must satisfy

$$w\big(2\rho_+(\theta)\big)w\big(2i\rho_-(\theta)\big) = -1$$

with $w\big(2\rho_+(\theta)\big)$ real and positive and $w\big(2\rho_+(\theta) + 2i\rho_-(\theta)\big) = \infty$. Pursuing this analysis, it follows without much difficulty that w must be doubly periodic with periods $4\rho_+(\theta)$ and $4i\rho_-(\theta)$ and have one double pole at $2\rho_+(\theta) + 2i\rho_-(\theta)$ in the fundamental rectangle of $2\Lambda_\theta$.

By the usual properties of doubly periodic meromorphic functions on the plane, only one function w with all these properties exists. It can be written in terms of $\mathfrak{p}$ as

$$w(z,\theta) = \frac{\mathfrak{p}(\frac{1}{2}z,\theta) - r_2(\theta)}{\sqrt{\big(r_3(\theta) - r_2(\theta)\big)\big(r_2(\theta) - r_1(\theta)\big)}}.$$

The Weierstraß-type equation for $\mathfrak{p}$ shows that w itself satisfies the Weierstraß-type equation

$$(w_z)^2 = 16b(\theta)w - 48r_2(\theta)w^2 - 16b(\theta)w^3$$

where

$$b(\theta) = \sqrt{\big(r_3(\theta) - r_2(\theta)\big)\big(r_2(\theta) - r_1(\theta)\big)} > 0.$$

By symmetry considerations, w must map the boundary of the rectangle $\mathcal{R}$ with vertices 0, $2\rho_+(\theta)$, $2\rho_+(\theta) + 2i\rho_-(\theta)$, and $2i\rho_-(\theta)$ to the real line plus ∞ on the Riemann sphere and do so in a one-to-one and onto manner. Consequently w establishes a biholomorphism between the interior of $\mathcal{R}$ and the upper half plane. Because of the symmetry of the boundary values, particularly the identity $w\big(2\rho_+(\theta)\big)w\big(2i\rho_-(\theta)\big) = -1$, it follows that w must map $\rho_+(\theta) + i\rho_-(\theta)$, the center of $\mathcal{R}$, to the center of the upper half plane (endowed with its usual metric of constant *positive* curvature), i.e., that

$$w\big(\rho_+(\theta) + i\rho_-(\theta), \theta\big) = i.$$

Using this information, it is not difficult to deduce that

$$w_z\big(\rho_+(\theta) + i\rho_-(\theta), \theta\big) = 4\sqrt{3r_2(\theta) + 2b(\theta)i}.$$

(In view of the Weierstraß equation, the only problem is to fix the ambiguity of the sign of this square root, but this is not difficult. I mean the one with positive imaginary part.)

It follows that there is a well-defined map $\Psi\colon \tilde{N} \to \mathbb{CP}^2 \times (-\pi/2, \pi/2)$ satisfying

$$\Psi\big(\langle x, y, \theta\rangle\big) = \big([1, w(x+iy, \theta), w_z(x+iy, \theta)], \theta\big).$$

Note, in particular, that $\Psi\big(\langle\rho_+(\theta), \rho_-(\theta), \theta\rangle\big) = \big([1, i, 4\sqrt{3r_2(\theta)+2b(\theta)i}\,], \theta\big)$. The image of Ψ is the locus $\tilde{E} \subset \mathbb{CP}^2 \times (-\pi/2, \pi/2)$ consisting of points $\big([Z_0, Z_1, Z_2], \theta\big)$ that satisfy the equation

$$Z_0 {Z_2}^2 = 16b(\theta) {Z_0}^2 Z_1 - 48r_2(\theta) Z_0 {Z_1}^2 - 16b(\theta) {Z_1}^3.$$

Let $\tilde{E}_\theta \subset \mathbb{CP}^2$ be the smooth plane cubic curve so that $\tilde{E}_\theta \times \{\theta\} = \tilde{E} \cap \big(\mathbb{CP}^2 \times \{\theta\}\big)$. This is an elliptic curve and will be referred to as the θ-slice of $\tilde{E}$. By the discussion already given plus elementary properties of elliptic curves, Ψ is a diffeomorphism from $\tilde{N}$ to $\tilde{E}$. Moreover, $\Psi(\tilde{N}^*) = \tilde{E}^*$, which is defined as the complement in $\tilde{E}$ of the three points on each $\tilde{E}_\theta$ that lie on the line $Z_2 = 0$ together with the point at infinity (i.e., the flex tangent on the line $Z_0 = 0$) on each $\tilde{E}_\theta$.

Now, the double cover $\hat{N}^* \to \tilde{N}^* \simeq \tilde{E}^*$ is nontrivial around each of these missing points in each θ-slice. Consider the smooth plane quartic family $\hat{E} \subset \mathbb{CP}^2 \times (-\pi/2, \pi/2)$ consisting of points $\big([W_0, W_1, W_2], \theta\big)$ that satisfy the equation

$$ {W_2}^4 = 16b(\theta) {W_0}^3 W_1 - 48r_2(\theta) {W_0}^2 {W_1}^2 - 16b(\theta) W_0 {W_1}^3.$$

The map that takes $\big([W_0, W_1, W_2], \theta\big) \in \hat{E}$ to $\big([(W_0)^2, W_0W_1, (W_2)^2], \theta\big) \in \tilde{E}$ is a branched double cover over each $\tilde{E}_\theta$. The branch locus over each $\tilde{E}_\theta$ consists of the four points on $\tilde{E}_\theta$ that do not belong to $\tilde{E}^*$. Let $\hat{E}^* \subset \hat{E}$ be the inverse image of $\tilde{E}^*$ under this smooth mapping.

Now the double cover $\hat{E}^* \to \tilde{E}^* \simeq \tilde{N}^*$ is nontrivial exactly along the same curves as the double cover $\hat{N}^* \to \tilde{N}^*$. Thus, there is a diffeomorphism $\hat{\Psi}\colon \hat{N}^* \to \hat{E}^*$ that identifies the two double covers and this $\hat{\Psi}$ is unique up to composition with the deck transformation $\big([W_0, W_1, W_2], \theta\big) \to \big([W_0, W_1, -W_2], \theta\big)$ of the covering $\hat{E}^* \to \tilde{E}^*$. From now on, I will fix a choice of $\hat{\Psi}$ and use it to identify $\hat{N}^*$ with $\hat{E}^*$.

Each of the θ-slices $\hat{E}_\theta \subset \hat{E}$ is a nonsingular plane quartic and hence is a nonhyperelliptic Riemann surface of genus 3 [GH, Chapter 2]. In fact, the functions

$$w = \frac{W_1}{W_0}, \quad v = \frac{W_2}{W_0}$$

are smooth and well-defined on $\hat{E}^*$, restricting to each $\hat{E}_\theta$ to become meromorphic functions with poles located at the point at 'infinity' given by the intersection of $\hat{E}_\theta$ with the line $W_0 = 0$. The 1-forms

$$\alpha_1 = \frac{w\,dw}{v^3}, \qquad \alpha_2 = \frac{dw}{v^3}, \qquad \alpha_3 = \frac{v\,dw}{v^3} = \frac{dw}{v^2}$$

restrict to each $\hat{E}_\theta$ to be a basis for the holomorphic 1-forms on $\hat{E}_\theta$. Note that α_3 is actually invariant under the deck transformation $(w,v,\theta) \mapsto (w,-v,\theta)$ of the covering $\hat{E}^* \to \tilde{E}^*$ and hence is well-defined as a 1-form on $\tilde{E}^*$. This 1-form restricts to each $\tilde{E}_\theta$ to become the nontrivial holomorphic differential on that elliptic curve. Note that α_1 and α_2 have no common zeroes: In fact, α_2 has only one zero, which is of order 4, and this occurs at the common pole of w and v. Since w has a pole of order exactly 4 at this point, it follows that α_1 does not vanish there.

Let $\hat{n}(\theta) = \big(\big[1, i, 2\sqrt[4]{3r_2(\theta) + 2b(\theta)i}\,\big], \theta\big)$ and consider the multivalued 'function' on $\hat{E}$ 'defined' by the abelian integral

$$\begin{aligned}\vartheta([1,w,v],\theta) = \begin{pmatrix}\vartheta_1([1,w,v],\theta)\\ \vartheta_2([1,w,v],\theta)\end{pmatrix} &= \int_{\hat{n}(\theta)}^{([1,w,v],\theta)} \sqrt{2}\begin{pmatrix}w\\1\end{pmatrix}\frac{dw}{v^3}\\ &= \int_{\hat{n}(\theta)}^{([1,w,v],\theta)} \begin{pmatrix}\sqrt{2}\,\alpha_1\\ \sqrt{2}\,\alpha_2\end{pmatrix},\end{aligned}$$

where the integral is to be computed along a path joining $\hat{n}(\theta)$ to $([1,w,v],\theta) \in \hat{E}$ that lies entirely in $\hat{E}_\theta$. Of course, the value of this integral depends on the homology class of the path joining the two endpoints, so this is not well-defined as a function on $\hat{E}$. The ambiguity in the definition of ϑ will be determined below. For the time being, consider ϑ as being defined on a suitable cover $\check{E} \to \hat{E}$. Since α_1 and α_2 do not have any common zeroes, this map is an immersion on each $\hat{E}_\theta$.

Now consider the functions

$$A = \frac{\bar{v}}{v\sqrt{1+|w|^2}}, \qquad B = \frac{-\bar{v}w}{v\sqrt{1+|w|^2}}$$

defined on $\hat{E}^*$. They satisfy $|A|^2 + |B|^2 = 1$, so the function

$$h = \begin{pmatrix}1 & 0 & 0\\ 0 & \bar{A}\big(\hat{n}(\theta)\big) & -B\big(\hat{n}(\theta)\big)\\ 0 & \bar{B}\big(\hat{n}(\theta)\big) & A\big(\hat{n}(\theta)\big)\end{pmatrix}^{-1} \begin{pmatrix}1 & 0 & 0\\ \vartheta_1 & \bar{A} & -B\\ \vartheta_2 & \bar{B} & A\end{pmatrix}$$

takes values in G_0 and is well-defined on the open set $\check{E}^* \subset \check{E}$ that is the inverse image of $\hat{E}^*$ under the cover $\check{E} \to \hat{E}$. (The purpose of the first matrix is to arrange that $h(\hat{n}(\theta)) = \mathrm{I}_3$ for all θ, which will be needed below.)

Since the first factor in h depends only on θ, computation yields

$$h^{-1}dh \equiv \begin{pmatrix} 0 & 0 & 0 \\ A\,d\vartheta_1 + B\,d\vartheta_2 & A\,d\bar{A} + B\,d\bar{B} & B\,dA - A\,dB \\ \bar{A}\,d\vartheta_2 - \bar{B}\,d\vartheta_1 & \bar{A}\,d\bar{B} - \bar{A}\,d\bar{A} & \bar{A}\,dA + \bar{B}\,dB \end{pmatrix} \bmod d\theta.$$

Since $d\vartheta_i \equiv \sqrt{2}\,\alpha_i \bmod d\theta$ for $i = 1, 2$, it follows that

$$\left.\begin{aligned} A\,d\vartheta_1 + B\,d\vartheta_2 &\equiv 0 \\ \bar{A}\,d\vartheta_2 - \bar{B}\,d\vartheta_1 &\equiv \omega \end{aligned}\right\} \quad \bmod d\theta.$$

(This last follows from the identities

$$\omega \equiv (f-g)^{-1/4}dz \equiv \frac{\sqrt{2(1+|w|^2)}}{|w'(z)|^{1/2}}\frac{dw}{w'(z)} \equiv \frac{\sqrt{2(1+|w|^2)}}{|v|}\frac{dw}{v^2} \quad \bmod d\theta,$$

together with the definitions of A and B. The reader can now probably see why the factor of $\sqrt{2}$ was introduced into the definition of ϑ.) Moreover,

$$\begin{aligned} \bar{A}\,d\bar{B} - \bar{B}\,d\bar{A} &= -\bar{A}^2\,d(\bar{B}/\bar{A}) = \frac{-v^2}{|v|^2(1+|w|^2)}\,d\bar{w} \\ &= \frac{-|v|^2}{(1+|w|^2)}\frac{d\bar{w}}{\bar{v}^2} \equiv -2s\omega \equiv \sigma \quad \bmod d\theta. \end{aligned}$$

By these results, there exists a real-valued 1-form ϕ^* so that

$$h^{-1}dh \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & i\phi^* & -\bar{\sigma} \\ \omega & \sigma & -i\phi^* \end{pmatrix} \bmod d\theta.$$

The matrix on the right is almost γ. In fact, I claim that it is congruent to γ modulo $d\theta$. Since η is a multiple of $d\theta$ by definition, the only thing to check is whether $\phi^* \equiv \phi$ modulo $d\theta$. However, this follows immediately from the structure equations, which show that $d\sigma \equiv 2i\phi \wedge \sigma \bmod d\theta$ while the very fact that ϕ^* appears where it does in $h^{-1}dh$ shows that $d\sigma \equiv 2i\phi^* \wedge \sigma \bmod d\theta$. Comparing these two relations and using the fact that ϕ and ϕ^* are real then yields $\phi^* \equiv \phi \bmod d\theta$, as desired. (Alternatively, one can simply carry out the computations and compare the results.)

It has now been shown that $h^{-1}dh \equiv \gamma \bmod d\theta$. Now, γ is well-defined on $\hat{E}^*$, not just on $\check{E}^*$, so it follows that $h^{-1}dh$ is well defined on each $\hat{E}_\theta$ and has the same holonomy as γ on each $\hat{E}_\theta$. Now, it has already been shown that the holonomy of γ on $\hat{E}^*$ lies in the discrete subgroup $\hat{\Lambda} \subset G_0$ and the inclusion $\hat{E}^*_\theta \hookrightarrow \hat{E}^*$ induces and isomorphism on fundamental groups. Consequently, there is a well-defined mapping

$$\hat{\Lambda}h\colon \hat{E} \to \hat{\Lambda}\backslash G_0.$$

Note that the quotient is via the left action and not the right action. In particular, the canonical left-invariant form on G_0 is well-defined on $\hat{\Lambda}\backslash G_0$.

Now, consider the $\mathfrak{g}_0$-valued 1-form κ that is well-defined on $\check{E}^*$ by the formula $\kappa = h\gamma h^{-1} - dh\, h^{-1}$. Since

$$\kappa = h(\gamma - h^{-1}dh)h^{-1} \equiv 0 \bmod d\theta$$

since $d\kappa = -\kappa \wedge \kappa = 0$, and since κ vanishes when restricted to each $\check{E}^*_\theta$, it must be a 1-form in θ alone. In fact, a computation using the properties of f and g shows that

$$\hat{n}^*(\kappa) = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{6}\big(r_3(\theta) - r_1(\theta)\big)^{1/4} \sec^{2/3}\theta\, d\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In particular, $\kappa = k^{-1}dk$ where

$$k = \begin{pmatrix} 1 & 0 & 0 \\ m(\theta) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and where m satisfies $m(0) = 0$ and $m'(\theta) = \frac{1}{6}\big(r_3(\theta) - r_1(\theta)\big)^{1/4} \sec^{2/3}\theta$.

Since the elements of the form $k(\theta)$ commute with all of the elements of $\hat{\Lambda}$, it now follows that $\gamma = g^{-1}dg$ where $[g] = \hat{\Lambda}kh$ is well defined on $\hat{E}^*$ as a map into $\hat{\Lambda}\backslash G_0$. Since $\hat{\Lambda}\backslash G_0/K \simeq \mathbb{C}^2/\Lambda$, the map

$$\begin{aligned} &\Phi([1,w,v],\theta) \\ &\equiv \begin{pmatrix} A\big(\hat{n}(\theta)\big) & B\big(\hat{n}(\theta)\big) \\ -\bar{B}\big(\hat{n}(\theta)\big) & \bar{A}\big(\hat{n}(\theta)\big) \end{pmatrix} \begin{pmatrix} \vartheta_1([1,w,v],\theta) \\ \vartheta_2([1,w,v],\theta) \end{pmatrix} + \begin{pmatrix} m(\theta) \\ 0 \end{pmatrix} \quad \bmod \Lambda \end{aligned}$$

is well-defined as a map $\Phi\colon \hat{E} \to \mathbb{C}^2/\Lambda$.

From the formulae that went into its definition, Φ is an immerison on $\hat{E}^*$ whose image is a Levi-flat minimal hypersurface in $\mathbb{C}^2/\Lambda$ of type (3). Moreover $\Phi(\hat{E}_\theta) \subset \mathbb{C}^2/\Lambda$ is a complex leaf in this hypersurface and is immersed as a compact Riemann surface of genus 3.

Now, Φ is not an immersion near the four curves $v = 0$ in $\hat{E}$. (These are the curves that intersect each $\hat{E}_\theta$ in the four branch points.) In fact, it collapses each of these curves to a point, as can be seen by doing a local computation. Let these points be labeled $P_i \in \mathbb{C}^2/\Lambda$ for $i = 1$ to 4.

A possible 'algebraic' structure. Now $\mathbb{C}^2/\Lambda$ is a complex torus that has nontrivial divisors, for example, the genus 3 Riemann surfaces $\Phi(\hat{E}_\theta)$. It follows that $\mathbb{C}^2/\Lambda$ is an Abelian variety (actually, this also follows from the explicit description of Λ as a lattice of type F_4 that has already been given). In particular, $\mathbb{C}^2/\Lambda$ is an algebraic surface. By a standard Riemann-Roch calculation [GH, Chapter 4], one can show that the curves in the connected family of $C_\theta = \Phi(\hat{E}_\theta)$ that pass through the points P_i form a pencil, i.e., the moduli M of such curves is a $\mathbb{CP}^1$. In fact, regarding θ as a complex parameter in the formula for Φ gives a local real parameter on M near $\theta = 0$. Evidently, the curve M admits an antiholomorphic involution for which the curves C_θ are fixed points. Of course, any antiholomorphic involution of $\mathbb{CP}^1$ that has fixed points is conjugate via an automorphism of $\mathbb{CP}^1$ to the standard conjugation fixing an $\mathbb{RP}^1 \subset \mathbb{CP}^1$. Thus, it would appear that the image $\Sigma = \Phi(\hat{E}) \subset \mathbb{C}^2/\Lambda$ is a dense open set in an 'algebraic' real hypersurface $\bar{\Sigma} \subset \mathbb{C}^2/\Lambda$ that is the union of the curves in M that are fixed under the antiholomorphic involution. Presumably, the singular curves in the pencil M are unions of elliptic curves embedded in $\mathbb{C}^2/\Lambda$ linearly and are therefore the totally geodesic complex leaves in $\bar{\Sigma}$. It would be interesting to know whether or not the only singularities of $\bar{\Sigma}$ are the four points P_i and whether or not these singular points really do resemble cones on the Clifford torus, as they appear to.

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Department of Mathematics
Duke University
Box 90320, Durham
NC 27708-0320
bryant@math.duke.edu