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Appendix

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This is an appendix for the paper *"Infinitesimal logarithmic Torelli problem for degenerating hypersurfaces in $\mathbb{P}^n$"* by S. Saito. In Theorem (2-1), the injectivity of $d\rho_Z$ was proved for degenerating hypersurfaces. But $d\rho_Z$ is not injective in case n is odd and $\delta = 2$. We want to know the meaning of the exceptional cases. It is explained here, by using the extended period map, which is defined by K. Kato and S. Usui.

When we fix integers $\delta \geq 2$, $s \geq 1$ and $d \geq s\delta$, and general coefficients $a_\alpha \in \mathbb{C}$, a strict semistable degeneration $\tilde{X} \to B$ of hypersurfaces in $\mathbb{P}^{m+1}$ over the unit disk is constructed in Section 1. We denote the central fiber by $Z = Z_0 \cup Z_1 \cup \cdots \cup Z_s$, where $Z_0 = \tilde{X} \cap \tilde{H}_t$ and $Z_k = \tilde{X} \cap \mathbb{E}_k$.

Proposition 1. *Assume $d \geq s\delta + 1$. The mixed Hodge structure on $H^m(Z, \mathbb{Q})$ satisfies*

- $\mathrm{Gr}_l^W H^m(Z, \mathbb{Q}) = 0$ *if* $l \leq m - 2$,
- $\mathrm{Gr}_{m-1}^W H^m(Z, \mathbb{Q}) \simeq H^{m-1}_{\mathrm{prim}}(Z_0 \cap Z_s, \mathbb{Q})$.

$Z_0 \cap Z_s$ is a nonsingular hypersurface of degree δ in $\tilde{H}_t \cap \mathbb{E}_s \cong \mathbb{P}^m$.

Proof. The spectral sequence

$$E_1^{p,q} = H^q(Z^{[p]}, \mathbb{Q}) \Rightarrow H^{p+q}(Z, \mathbb{Q})$$

defines the weight filtration on $H^i(Z, \mathbb{Q})$, where $Z^{[p]} = \coprod_{0 \leq i_0 < \cdots < i_p \leq s} Z_{i_0} \cap \cdots \cap Z_{i_p}$. Let $\tilde{\mathbb{P}_o} = \tilde{H}_t \cup \mathbb{E}_1 \cup \cdots \cup \mathbb{E}_s$ be the central fiber of $\tilde{\mathbb{P}_B} \to B$. We know that $\tilde{\mathbb{P}_o}^{[p]} = Z^{[p]} = \emptyset$ for $p \geq 3$, so $\mathrm{Gr}_l^W H^m(Z, \mathbb{Q}) = 0$ for $l \leq m - 3$.

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We recall that

$$\begin{cases} \tilde{\mathbb{P}_o}^{[2]} = \coprod_{1\le k\le s-1} \tilde{H}_t \cap \mathbb{E}_k \cap \mathbb{E}_{k+1}, \\ \tilde{\mathbb{P}_o}^{[1]} = \coprod_{1\le k\le s} \tilde{H}_t \cap \mathbb{E}_k \amalg \coprod_{1\le k\le s-1} \mathbb{E}_k \cap \mathbb{E}_{k+1}, \end{cases}$$

and

$$\begin{cases} Z^{[2]} = \coprod_{1\le k\le s-1} Z_0 \cap Z_k \cap Z_{k+1}, \\ Z^{[1]} = \coprod_{1\le k\le s} Z_0 \cap Z_k \amalg \coprod_{1\le k\le s-1} Z_k \cap Z_{k+1}. \end{cases}$$

Here we remark that $Z_0 \cap Z_k \cap Z_{k+1}$, $Z_0 \cap Z_k$ and $Z_k \cap Z_{k+1}$ are not connected if $m = 2$ and $1 \le k \le s-1$.

The semistable degeneration is constructed by

$$\begin{array}{ccccccccc} B \times \mathbb{P}^{m+1} = \mathbb{P}_B & \xleftarrow{\pi_1} & \mathbb{P}_B^{(1)} & \xleftarrow{\pi_2} & \mathbb{P}_B^{(2)} & \xleftarrow{\pi_3} \cdots \xleftarrow{\pi_s} & \mathbb{P}_B^{(s)} & = \tilde{\mathbb{P}_B} \\ \cup & & \cup & & \cup & & \cup & \\ X & \longleftarrow & X^{(1)} & \longleftarrow & X^{(2)} & \longleftarrow \cdots \longleftarrow & X^{(s)} & = \tilde{X}, \end{array}$$

where π_1 is the blowing-up of $\mathbb{P}_B$ along the singular point $p \in X$, π_k is the blowing-up of $\mathbb{P}_B^{(k-1)}$ along the singular locus $L_{k-1} \cong \mathbb{P}^1$ of $X^{(k-1)}$, and $X^{(k)}$ is the proper transform of X. $\mathbb{E}_k$ is the proper transform in $\tilde{\mathbb{P}_B}$ of the exceptional set of π_k. $\tilde{H}_t \subset \tilde{\mathbb{P}_B}$ is the proper transform of $H_t = \{t = 0\} \subset \mathbb{P}_B$.

$Z_k \cap Z_{k+1}$ is isomorphic to $\mathbb{P}^1 \times (Z_0 \cap Z_k \cap Z_{k+1})$, and contains $Z_0 \cap Z_k \cap Z_{k+1}$ as a fiber of the projection $\mathbb{P}^1 \times (Z_0 \cap Z_k \cap Z_{k+1}) \to \mathbb{P}^1$. So the restriction $H^{m-2}(Z_k \cap Z_{k+1}) \to H^{m-2}(Z_0 \cap Z_k \cap Z_{k+1})$ is surjective. Since $Z_k \cap Z_{k+1}$ meet only Z_0, this shows $\mathrm{Gr}^W_{m-2} H^m(Z, \mathbb{Q}) = 0$.

There is a commutative diagram

$$\begin{array}{ccccccc} H^{m-1}(\tilde{\mathbb{P}_o}^{[0]}) & \to & H^{m-1}(\tilde{\mathbb{P}_o}^{[1]}) & \to & H^{m-1}(\tilde{\mathbb{P}_o}^{[2]}) & \to & H^{m-1}(\tilde{\mathbb{P}_o}^{[3]}) = 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^{m-1}(Z^{[0]}) & \to & H^{m-1}(Z^{[1]}) & \to & H^{m-1}(Z^{[2]}) & \to & H^{m-1}(Z^{[3]}) = 0, \end{array}$$

where the horizontal sequences are complex, and $\mathrm{Gr}^W_{m-1} H^{m-1+i}(Z, \mathbb{Q})$ is the i-th cohomology of the second sequence. So $\mathrm{Gr}^W_{m-1} H^m(Z, \mathbb{Q}) \simeq H^{m-1}_{\mathrm{prim}}(Z_0 \cap Z_s, \mathbb{Q})$ is proved by the following:

1. $H^{m-1}(\tilde{\mathbb{P}_o}^{[2]}) \simeq H^{m-1}(Z^{[2]})$.

2. $\mathrm{Coker}(H^{m-1}(\tilde{\mathbb{P}_o}^{[1]}) \to H^{m-1}(Z^{[1]})) \simeq H^{m-1}_{\mathrm{prim}}(Z_0 \cap Z_s)$.
3. The composition $H^{m-1}(Z^{[0]}) \to H^{m-1}(Z^{[1]}) \to H^{m-1}_{\mathrm{prim}}(Z_0 \cap Z_s)$ is zero.
4. $\mathrm{Gr}^W_{m-1} H^m(\tilde{\mathbb{P}_o}) = 0$ and $\mathrm{Gr}^W_{m-1} H^{m+1}(\tilde{\mathbb{P}_o}) = 0$. (This means that the first sequence in the diagram is exact.)

Let X be defined in $B \times \mathbb{P}^{m+1}$ by the equation

$$\sum_{\alpha_0+\cdots+\alpha_{m+1}=d} a_\alpha t^{\max\{0, s\delta - s(\alpha_1+\cdots+\alpha_m)-\alpha_{m+1}\}} X_0^{\alpha_0} \cdots X_{m+1}^{\alpha_{m+1}} = 0.$$

We define a hypersurface $Y \subset \mathbb{P}^m$ by

$$\sum_{\alpha_1+\cdots+\alpha_m=\delta, \alpha_{m+1}=0} a_\alpha X_1^{\alpha_1} \cdots X_m^{\alpha_m} = 0,$$

which is singular at $[1 : 0 : \cdots : 0]$. Let $\tilde{Y} \subset \tilde{\mathbb{P}^m}$ be the desingularization by the blowing-up at the point, and Y_0 be the hyperplane section of Y by $\{X_0 = 0\}$. Y is the projective cone over Y_0, and $\tilde{Y}$ is a $\mathbb{P}^1$-bundle over Y_0.

For $1 \leq k \leq s-1$, there are isomorphisms

$$\begin{array}{ccc} Z_0 \cap Z_k \cap Z_{k+1} & \subset & \tilde{H}_t \cap \mathbb{E}_k \cap \mathbb{E}_{k+1} \\ \downarrow\cong & & \downarrow\cong \\ Y_0 & \subset & \mathbb{P}^{m-1}, \end{array}$$

$$\begin{array}{ccc} Z_k \cap Z_{k+1} & \subset & \mathbb{E}_k \cap \mathbb{E}_{k+1} \\ \downarrow\cong & & \downarrow\cong \\ \mathbb{P}^1 \times Y_0 & \subset & \mathbb{P}^1 \times \mathbb{P}^{m-1} \end{array} \quad \textit{and} \quad \begin{array}{ccc} Z_0 \cap Z_k & \subset & \tilde{H}_t \cap \mathbb{E}_k \\ \downarrow\cong & & \downarrow\cong \\ \tilde{Y} & \subset & \tilde{\mathbb{P}^m}. \end{array}$$

So we have $H^{m-1}(\tilde{H}_t \cap \mathbb{E}_k \cap \mathbb{E}_{k+1}) \simeq H^{m-1}(Z_0 \cap Z_k \cap Z_{k+1})$, $H^{m-1}(\mathbb{E}_k \cap \mathbb{E}_{k+1}) \simeq H^{m-1}(Z_k \cap Z_{k+1})$ and $H^{m-1}(\tilde{H}_t \cap \mathbb{E}_k) \simeq H^{m-1}(Z_0 \cap Z_k)$.

$Z_0 \cap Z_s$ is isomorphic to the hypersurface in $\tilde{H}_t \cap \mathbb{E}_s \cong \mathbb{P}^m$ defined by

$$\sum_{s\delta = s(\alpha_1+\cdots+\alpha_m)+\alpha_{m+1}} a_\alpha X_0^{\delta-(\alpha_1+\cdots+\alpha_m)} X_1^{\alpha_1} \cdots X_m^{\alpha_m} = 0.$$

The map $H^{m-1}(\tilde{H}_t \cap \mathbb{E}_s) \to H^{m-1}(Z_0 \cap Z_s)$ has the cokernel $H^{m-1}_{\mathrm{prim}}(Z_0 \cap Z_s, \mathbb{Q})$. We have proved (1) and (2).

$\mathbb{E}_s$ is isomorphic to the hypersurface in $\mathbb{P}^1 \times \mathbb{P}^{m+1}$ defined by $u_0^s X_0 = u_1^s X_{m+1}$, where u_i is the parameter of $\mathbb{P}^1$. In this identification, Z_s is defined by

$$\begin{cases} u_0^s X_0 = u_1^s X_{m+1}, \\ \sum_{s\delta \geq s(\alpha_1 + \cdots + \alpha_m) + \alpha_{m+1}} a_\alpha u_0^{s\delta - \alpha_{m+1}} u_1^{\alpha_{m+1}} X_1^{\alpha_1} \cdots X_m^{\alpha_m} X_{m+1}^{\delta - (\alpha_1 + \cdots + \alpha_m)} = 0. \end{cases}$$

This shows that Z_s is a very ample divisor in $\mathbb{E}_s$. So the left vertical map in the diagram

$$\begin{array}{ccc} H^{m-1}(\mathbb{E}_s) & \rightarrow & H^{m-1}(\tilde{H}_t \cap \mathbb{E}_s) \\ \downarrow \simeq & & \downarrow \\ H^{m-1}(Z_s) & \rightarrow & H^{m-1}(Z_0 \cap Z_s) \end{array}$$

is an isomorphism. Hence $H^{m-1}(Z_s) \to H^{m-1}_{\text{prim}}(Z_0 \cap Z_s)$ is the zero map.

By Lemma 2, the left vertical map in the diagram

$$\begin{array}{ccc} H^{m-1}(\tilde{H}_t) & \rightarrow & H^{m-1}(\tilde{H}_t \cap \mathbb{E}_s) \\ \downarrow \simeq & & \downarrow \\ H^{m-1}(Z_0) & \rightarrow & H^{m-1}(Z_0 \cap Z_s) \end{array}$$

is an isomorphism. Hence $H^{m-1}(Z_0) \to H^{m-1}_{\text{prim}}(Z_0 \cap Z_s)$ is the zero map, and (3) is proved.

Because the monodromy of $\tilde{\mathbb{P}}_B \to B$ is trivial, (4) is proved by same argument in the proof of Corollary 3, using Clemens-Schmid exact sequence. Q.E.D.

Lemma 2. *If $d \geq s\delta + 1$, then $H^{m-1}(\tilde{H}_t) \simeq H^{m-1}(Z_0)$.*

Proof. $\tilde{H}_t$ is obtained by

$$\mathbb{P}^{m+1} \cong H_t = H_t^{(0)} \overset{\pi_1}{\leftarrow} H_t^{(1)} \overset{\pi_2}{\leftarrow} \cdots \overset{\pi_s}{\leftarrow} H_t^{(s)} = \tilde{H}_t,$$

where π_1 is the blowing-up along the point p, and π_k is the blowing-up along the point $L_{k-1} \cap H_t^{(k-1)}$ for $2 \leq k \leq s$, and we set $\pi = \pi_1 \circ \cdots \circ \pi_s$. We denote by $E'_k \subset H_t^{(k)}$ the exceptional divisor of π_k, and by $E_k \subset \tilde{H}_t$ its proper transform in $\tilde{H}_t$. If H is a hyperplane in H_t, then $a\pi_1^*(H) + b(\pi_1^*(H) - E'_1)$ is a very ample divisor in $H_t^{(1)}$ for $a, b \in \mathbb{Z}_{>0}$. In case $s = 1$, $Z_0 \sim d\pi^*(H) - \delta E_1 = (d - \delta)\pi^*(H) + \delta(\pi^*(H) - E_1)$ is very ample in $\tilde{H}_t$, hence Lemma is proved.

We assume $s \geq 2$. In this case, Z_0 is not ample in $\tilde{H}_t$. By the exact sequence

$$H_c^{m-1}(\tilde{H}_t \setminus Z_0) \to H^{m-1}(\tilde{H}_t) \to H^{m-1}(Z_0) \to H_c^{m-1}(\tilde{H}_t \setminus Z_0),$$

we want to show $H_c^{m-1}(\tilde{H}_t \setminus Z_0) = 0$ and $H_c^{m-1}(\tilde{H}_t \setminus Z_0) = 0$. Let $Z_0^{(k)}$ be the proper transform of $Z_0^{(k-1)}$ by π_k, where $Z_0^{(0)} = H_t \cap X$. Since $H_t \cong \mathbb{P}^{m+1}$, $E'_k \cong \mathbb{P}^m$ and

$$H_t^{(k)} \setminus (Z_0^{(k)} \cup E'_k) \cong H_t^{(k-1)} \setminus Z_0^{(k-1)},$$

$H_c^{m-1}(\tilde{H}_t \setminus Z_0) = 0$ is proved by the exact sequence

$$\begin{aligned} &H_c^{m-2}(E'_k \setminus E'_k \cap Z_0^{(k)}) \to H_c^{m-1}(H_t^{(k)} \setminus (Z_0^{(k)} \cup E'_k)) \\ &\qquad \to H_c^{m-1}(H_t^{(k)} \setminus Z_0^{(k)}) \to H_c^{m-1}(E'_k \setminus E'_k \cap Z_0^{(k)}), \end{aligned}$$

inductively.

By $E_k = \tilde{H}_t \cap \mathbb{E}_k \cong \tilde{\mathbb{P}}^m$ and $E_k \cap Z_0 \cong \tilde{Y}$, we have

$$H_c^m(E_k \setminus (E_k \cap Z_0)) = 0,$$

for $1 \leq k \leq s-1$, and these contain $E_{k-1} \cap E_k \cong \mathbb{P}^{m-1}$ and $E_{k-1} \cap E_k \cap Z_0 \cong Y_0$ as a section of the $\mathbb{P}^1$-bundle structure, so we can see

$$\begin{aligned} H_c^{m-1}(E_k \setminus (E_k \cap Z_0)) &\simeq H_c^{m-1}((E_{k-1} \cap E_k) \setminus (E_{k-1} \cap E_k \cap Z_0)) \\ &\simeq H_{\mathrm{prim}}^{m-2}(Y_0), \end{aligned}$$

$$H_c^{m-1}(E_k \setminus ((E_k \cap Z_0) \cup (E_{k-1} \cap E_k))) = 0,$$

$$H_c^{m}(E_k \setminus ((E_k \cap Z_0) \cup (E_{k-1} \cap E_k))) = 0,$$

for $2 \leq k \leq s-1$. By the exact sequence

$$\begin{aligned} &H_c^{m-1}(E_k \setminus ((E_k \cap Z_0) \cup (E_{k-1} \cap E_k))) \\ &\to H_c^m(\tilde{H}_t \setminus (Z_0 \cup E_1 \cup \cdots \cup E_k)) \to H_c^m(\tilde{H}_t \setminus (Z_0 \cup E_1 \cup \cdots \cup E_{k-1})) \\ &\qquad \to H_c^m(E_k \setminus ((E_k \cap Z_0) \cup (E_{k-1} \cap E_k))), \end{aligned}$$

we have

$$H_c^m(\tilde{H}_t \setminus (Z_0 \cup E_1 \cup \cdots \cup E_{s-1})) \simeq H_c^m(\tilde{H}_t \setminus (Z_0 \cup E_1)),$$

inductively, and by the exact sequence

$$0 = H_c^{m-1}(\tilde{H}_t \setminus Z_0) \to H_c^{m-1}(E_1 \setminus (E_1 \cap Z_0)) \to H_c^m(\tilde{H}_t \setminus (Z_0 \cup E_1)) \\ \to H_c^m(\tilde{H}_t \setminus Z_0) \to H_c^m(E_1 \setminus (E_1 \cap Z_0)) = 0,$$

we have the exact sequence

$$0 \to H_{\mathrm{prim}}^{m-2}(Y_0) \to H_c^m(\tilde{H}_t \setminus (Z_0 \cup E_1 \cup \cdots \cup E_{s-1})) \\ \to H_c^m(\tilde{H}_t \setminus Z_0) \to 0.$$

To see $H_c^m(\tilde{H}_t \setminus Z_0) = 0$,

$$H_{\mathrm{prim}}^{m-2}(Y_0) \simeq H_c^m(\tilde{H}_t \setminus (Z_0 \cup E_1 \cup \cdots \cup E_{s-1}))$$

is proved in the following. We consider the rational map

$$\begin{array}{ccc} H_t \cong \mathbb{P}^{m+1} & \cdots \to & \mathbb{P}^{2m} \\ [X_0 : \cdots : X_{m+1}] & \mapsto & [X_1^s : \cdots : X_{m+1}^s : X_0^{s-1}X_1 : \cdots : X_0^{s-1}X_m] \\ & & = [y_1 : \cdots : y_{m+1} : z_1 : \cdots : z_m], \end{array}$$

which has the elimination of indeterminacy $\phi : \tilde{H}_t \to \mathbb{P}^{2m}$. Let H_t' be the image of $\pi \times \phi : \tilde{H}_t \to H_t \times \mathbb{P}^{2m}$, and $W \subset H_t \times \mathbb{P}^{2m}$ be the subvariety defined by $X_1 = \cdots = X_{m+1} = y_1 = \cdots = y_{m+1} = 0$, which is contained in H_t'. The birational morphism $\pi \times \phi : \tilde{H}_t \to H_t'$ has the exceptional set $E_1 \cup \cdots \cup E_{s-1}$, and $(\pi \times \phi)(E_1 \cup \cdots \cup E_{s-1}) = W$. We can see

$$\begin{cases} \pi^* \mathcal{O}_{H_t}(1) \simeq \mathcal{O}_{\tilde{H}_t}(\pi^* H), \\ \phi^* \mathcal{O}_{\mathbb{P}^{2m}}(1) \simeq \mathcal{O}_{\tilde{H}_t}(s(\pi^* H) - E_1 - 2E_2 - \cdots - sE_s). \end{cases}$$

Because Z_0 is linearly equivalent to $d\pi^*(H) - \delta E_1 - 2\delta E_2 - \cdots - s\delta E_s$ in $\tilde{H}_t$, $\mathcal{O}_{H_t'}(Z_0') \simeq (\mathcal{O}_{H_t}(d - s\delta) \boxtimes \mathcal{O}_{\mathbb{P}^{2m}}(\delta)) \otimes \mathcal{O}_{H_t'}$, where $Z_0' = (\pi \times \phi)(Z_0)$. By the assumption $d \geq s\delta + 1$, Z_0' is a very ample divisor in H_t', so we have

$$H_c^{m-1}(W \setminus (W \cap Z_0')) \simeq H_c^m(H_t' \setminus (Z_0' \cup W)).$$

Since $W \cap Z_0'$ is isomorphic to Y_0 in $\mathbb{P}^{m-1} \cong W$, we have $H_c^{m-1}(W \setminus (W \cap Z_0')) \simeq H_{\mathrm{prim}}^{m-2}(Y_0)$. $\pi \times \phi$ induces the isomorphism

$$\tilde{H}_t \setminus (Z_0 \cup E_1 \cup \cdots \cup E_{s-1}) \cong H_t' \setminus (Z_0' \cup W),$$

so $H_c^m(\tilde{H}_t \setminus (Z_0 \cup E_1 \cup \cdots \cup E_{s-1})) \simeq H_{\mathrm{prim}}^{m-2}(Y_0)$ is proved. Q.E.D.

The degeneration $\tilde{X} \to B$ defines the limit Hodge structure H^i_{lim}.

Corollary 3. *Assume $d \geq s\delta + 1$. The mixed Hodge structure on H^m_{lim} satisfies*

- $\mathrm{Gr}^W_l H^m_{\mathrm{lim}} = 0$ *if* $l \leq m-2$.
- $\mathrm{Gr}^W_{m-1} H^m_{\mathrm{lim}} \simeq H^{m-1}_{\mathrm{prim}}(Z_0 \cap Z_s, \mathbb{Q})$.

If we denote the logarithm of the monodromy by $N : H^m_{\mathrm{lim}} \to H^m_{\mathrm{lim}}$, then $N^2 = 0$, and we have $N = 0$ if and only if m is even and $\delta = 2$.

Proof. We use the Clemens-Schmid exact sequence

$$\mathrm{Gr}^W_{l-2m-2} H_{m+2}(Z, \mathbb{Q}) \to \mathrm{Gr}^W_l H^m(Z, \mathbb{Q}) \to \mathrm{Gr}^W_l H^m_{\mathrm{lim}} \xrightarrow{[N]} \mathrm{Gr}^W_{l-2} H^m_{\mathrm{lim}}.$$

Since $W_{-m-3}H_{m+2}(Z, \mathbb{Q}) = 0$, $\mathrm{Gr}^W_l H^m(Z, \mathbb{Q}) \simeq \mathrm{Gr}^W_l H^m_{\mathrm{lim}}$ is proved by the sequence and Proposition 1, inductively for $l \leq m-1$. By the property of the weight filtration on H^m_{lim},

$$\mathrm{Gr}^W_{m+i} H^m_{\mathrm{lim}} \overset{[N^i]}{\tilde{\to}} \mathrm{Gr}^W_{m-i} H^m_{\mathrm{lim}},$$

the condition $N^i = 0$ is equivalent to $W_{m-i}H^m_{\mathrm{lim}} = 0$, hence the monodromy statement is proved. Since $Z_0 \cap Z_s$ is a hypersurface of degree δ in $\mathbb{P}^m$, $H^{m-1}_{\mathrm{prim}}(Z_0 \cap Z_s, \mathbb{Q}) = 0$ if and only if m is even and $\delta = 2$. Q.E.D.

Corollary 4. *If m is even and $\delta = 2$, then $d\rho_Z$ is not injective.*

Proof. We have the extended period map $\phi : B \to \Gamma\backslash D_\sigma$, where D is the period domain, σ is the nilpotent cone $\mathbb{Q}_{\geq 0} \cdot N$, and Γ is the subgroup of $\mathrm{Aut}(H^m_{\mathrm{lim}}, <, >)$ generated by the monodromy. The log differential of the extended period map satisfies the commutative diagram

$$\begin{array}{ccc} T^{\log}_B(o) = \mathbb{C} \cdot t\frac{\partial}{\partial t} & \xrightarrow{d\phi} & T^h_{\Gamma\backslash D_\sigma}(\phi(o)) \\ \downarrow & & \downarrow \\ H^1(Z, \theta_{Z/S_o}) & \xrightarrow{d\rho_Z} & \bigoplus_{1 \leq p \leq m} \mathrm{Hom}(H^{m-p}(Z, \omega^p_{Z/S_o}), H^{m-p+1}(Z, \omega^{p-1}_{Z/S_o})). \end{array}$$

If $N = 0$, then the extended period domain $\Gamma\backslash D_\sigma$ is equal to D. Because the log structure of D is trivial, $d\phi(t\frac{\partial}{\partial t})$ must be zero. On the other hand, the image of $t\frac{\partial}{\partial t}$ by the log Kodaira-Spencer map is not zero in $H^1(Z, \theta_{Z/S_o})$. So $d\rho_Z$ is not injective. Q.E.D.

Remark 5. If $m \geq 3$ or $d \neq 3$, then the period map $\phi : B \to \Gamma \backslash D_\sigma$ is injective, by the local Torelli for smooth hypersurfaces. But the log differential $d\phi$ is zero if m is even and $\delta = 2$. The injectivity of ϕ does not necessary imply the injectivity of $d\phi$.

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