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Borel-Serre Spaces and Spaces of SL(2)-Orbits

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Contents

Introduction

Let $\mathfrak{h}_g$ be the Siegel upper half space of genus g, and Γ an arithmetic subgroup of $\mathrm{Sp}(2g)$. Then the following compactifications of $\Gamma\backslash\mathfrak{h}_g$ were constructed.

(1) Satake-Baily-Borel compactification ([Sa], [BB]).

(2) Borel-Serre compactification ([BS]).

(3) Toroidal compactifications ([AMRT]).

Denote the compactification (2) by $\Gamma\backslash(\mathfrak{h}_g)_{\mathrm{BS}}$ ($\Gamma\backslash\bar{\mathfrak{h}}_g$ in their notation).

Let D be a Griffiths domain, *i.e.*, a classifying space of polarized Hodge structures (cf. §1). In the case of the Hodge numbers $h^{1,0} = h^{0,1} = g$ and $h^{p,q} = 0$ otherwise, D coincides with $\mathfrak{h}_g$. Replacing $\mathfrak{h}_g$ by D, we generalize in this paper the construction of $(\mathfrak{h}_g)_{\mathrm{BS}}$ in two directions. The one is a Borel-Serre space D_{BS} (§2), and the other is a space of

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SL(2)-orbits $D_{\mathrm{SL}(2)}$ (§3). We give remarks about the compactification (1) (3.15, 6.9). In the forthcoming paper [KU2], we will generalize (3) and also (1).

Now let $G_{\mathbf{R}}$, $G_{\mathbf{Z}}$ be the groups in NOTATION below. $G_{\mathbf{R}}$ acts on D transitively. (In the case $D = \mathfrak{h}_g$, $G_{\mathbf{R}} = \mathrm{Sp}(2g, \mathbf{R})$ and $G_{\mathbf{Z}} = \mathrm{Sp}(2g, \mathbf{Z})$.) The construction of the space D_{BS} is similar to that of $(\mathfrak{h}_g)_{\mathrm{BS}}$, and the quotient $\Gamma\backslash D_{\mathrm{BS}}$, by a subgroup Γ of $G_{\mathbf{Z}}$ of finite index, is a compact Hausdorff space (§5). On the other hand, the construction of the space $D_{\mathrm{SL}(2)}$ is based on the theory of SL(2)-orbits in [CKS]. The space $D_{\mathrm{SL}(2)}$ is Hausdorff (§3) but not always locally compact (6.12). The quotient $\Gamma\backslash D_{\mathrm{SL}(2)}$ is Hausdorff (§5), and has a nice property for period maps (3.15) which is an advantage of $\Gamma\backslash D_{\mathrm{SL}(2)}$, that $\Gamma\backslash D_{\mathrm{BS}}$ does not have.

The two spaces D_{BS} and $D_{\mathrm{SL}(2)}$ are not related directly (cf. §6). To remedy this situation, we introduce the spaces $D_{\mathrm{BS,val}}$ (resp. $D_{\mathrm{SL}(2),\mathrm{val}}$), the projective limit of the blowing-ups of D_{BS} (resp. $D_{\mathrm{SL}(2)}$) (2.6, 3.7). We then have the following diagram of topological spaces (3.1 (1)).

$$\begin{array}{ccc} D_{\mathrm{SL}(2),\mathrm{val}} & \hookrightarrow & D_{\mathrm{BS,val}} \\ \downarrow & & \downarrow \\ D_{\mathrm{SL}(2)} & & D_{\mathrm{BS}} \end{array}$$

In the classical situation, that is, when D is a Hermitian symmetric space and its horizontal tangent bundle is the whole tangent bundle (see 6.6; actually, this case is dealt with in [Sa], [BB], [BS], [AMRT]), we have $D_{\mathrm{SL}(2)} = D_{\mathrm{BS}}$ and $D_{\mathrm{SL}(2),\mathrm{val}} = D_{\mathrm{BS,val}}$ except one case (6.7). As a corollary, we derive the canonical surjection from the Borel-Serre compactification $\Gamma\backslash D_{\mathrm{BS}}$ to the Satake compactification $\Gamma\backslash D_S$ (6.9), which was defined by Zucker [Z2] in another way.

In a sequel paper [KU2], we will generalize the theory of toroidal compactifications of $\Gamma\backslash\mathfrak{h}_g$ by Mumford et al. replacing $\mathfrak{h}_g$ by general D, whose summary is in [KU1]. The results of the present paper will be also used there. The whole picture is explained in 3.16 Remark below.

After having written out this paper, the authors were informed by Steven Zucker that a recent work of Borel and Ji [BJ] independently gives generalization of Borel-Serre compactifications.

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NOTATION

Throughout this paper, we use the following notation.

Let H be a $\mathbf{Z}$-module. For $A = \mathbf{Q}, \mathbf{R}, \mathbf{C}$, we denote $H_A := A \otimes_{\mathbf{Z}} H$. We fix a 4-tuple

$$\Phi_0 = (w, (h^{p,q})_{p,q \in \mathbf{Z}}, H_0, \langle\ ,\ \rangle_0)$$

where w is an integer, $(h^{p,q})_{p,q\in\mathbf{Z}}$ is a set of non-negative integers satisfying

$$\begin{cases} h^{p,q} = 0 \text{ for almost all } p,\, q, \\ h^{p,q} = 0 \text{ if } p + q \neq w, \\ h^{p,q} = h^{q,p} \text{ for any } p,\, q, \end{cases}$$

H_0 is a free $\mathbf{Z}$-module of rank $\sum_{p,q} h^{p,q}$, and $\langle\ ,\ \rangle_0$ is a $\mathbf{Q}$-rational non-degenerate $\mathbf{C}$-bilinear form on $H_{0,\mathbf{C}}$ which is symmetric if w is even and anti-symmetric if w is odd. In the case w is even, say $w = 2t$, we assume that the signature (a, b) of $(H_{0,\mathbf{R}}, \langle\ ,\ \rangle_0)$ satisfies

a (resp. b) $= \sum_j h^{t+j,t-j}$,

where j ranges over all even (resp. odd) integers.

Let

$$G_{\mathbf{Z}} := \mathrm{Aut}(H_0, \langle\ ,\ \rangle_0),$$

and for $A = \mathbf{Q}, \mathbf{R}, \mathbf{C}$, let

$$\begin{aligned} G_A &:= \mathrm{Aut}(H_{0,A}, \langle\ ,\ \rangle_0), \\ \mathfrak{g}_A &:= \mathrm{Lie}\, G_A \\ &= \{N \in \mathrm{End}_A(H_{0,A}) \mid \langle Nx, y\rangle_0 + \langle x, Ny\rangle_0 = 0\ (\forall x, \forall y \in H_{0,A})\}. \end{aligned}$$

Following [BS], a *parabolic* subgroup of $G_{\mathbf{R}}$ means a parabolic subgroup of $(G^\circ)_{\mathbf{R}}$, where G° denotes the connected component of G in the Zariski topology containing the unity. (Note that $G^\circ = G$ if w is odd, and $G^\circ = \{g \in G \mid \det(g) = 1\}$ if w is even.)

§1. Classifying spaces of polarized Hodge structures

In this section, we recall polarized Hodge structures, the classifying space D of polarized Hodge structures, horizontal tangent bundles, polarized variations of Hodge structure and the associated period maps (cf. [G]). Let $\Phi_0 = (w, (h^{p,q})_{p,q\in\mathbf{Z}}, H_0, \langle\ ,\ \rangle_0)$ be as in NOTATION.

1.1. A *Hodge structure* of weight w and of Hodge type $(h^{p,q}) = (h^{p,q})_{p,q\in\mathbf{Z}}$ is a pair $(H_{\mathbf{Z}}, F)$ consisting of a free $\mathbf{Z}$-module $H_{\mathbf{Z}}$ of rank

$\sum_{p,q} h^{p,q}$ and of a decreasing filtration F of $H_{\mathbf{C}}$, which satisfy the following two conditions.

(i) $\dim_{\mathbf{C}} F^p = \sum_{r \geq p} h^{r,w-r}$ for all p.

(ii) $H_{\mathbf{C}} = \bigoplus_p H^{p,w-p}$, where $H^{p,w-p} := F^p \cap \overline{F}^{w-p}$.

Here $^{-}$ is the complex conjugation with respect to $H_{0,\mathbf{R}}$. Note that $\dim H^{p,w-p} = h^{p,w-p}$ for all p.

1.2. A *polarized Hodge structure* of weight w and of Hodge type $(h^{p,q})$ is a triple $(H_{\mathbf{Z}}, \langle\ ,\ \rangle, F)$ consisting of a Hodge structure $(H_{\mathbf{Z}}, F)$ and a $\mathbf{Q}$-rational non-degenerate $\mathbf{C}$-bilinear form $\langle\ ,\ \rangle$ on $H_{\mathbf{C}}$, symmmetric for even w and anti-symmetric for odd w, which satisfy the following two conditions.

(i) $\langle F^p, F^q \rangle = 0$ for $p + q > w$.

(ii) The Hermitian form $H_{0,\mathbf{C}} \times H_{0,\mathbf{C}} \to \mathbf{C}$, $(x, y) \mapsto \langle C_F(x), \overline{y} \rangle$, is positive definite.

Here C_F is the Weil operator which is defined by $C_F(x) := i^{2p-w}x$ for $x \in H^{p,w-p}$. The condition (i) (resp. (ii)) is called the Riemann-Hodge first (resp. second) bilinear relation.

1.3. A *polarized variation of Hodge structure* of weight w and of Hodge type $(h^{p,q})$ on a complex manifold X is a triple $(H_{\mathbf{Z}}, \langle\ ,\ \rangle, F)$ consisting of a local system $H_{\mathbf{Z}}$ on X, of a locally constant $\mathbf{Q}$-rational non-degenerate $\mathbf{C}$-bilinear form $\langle\ ,\ \rangle$ on $\mathbf{C} \otimes H_{\mathbf{Z}}$ and of a decreasing filtration F of $\mathcal{O}_X \otimes H_{\mathbf{Z}}$ by subbundles, which satisfy the following two conditions.

(i) $(H_{\mathbf{Z},x}, \langle\ ,\ \rangle_x, F(x))$ is a polarized Hodge structure of weight w and of Hodge type $(h^{p,q})$ $(\forall x \in X)$.

(ii) Griffiths transversality $\nabla F^p \subset \Omega^1_X \otimes F^{p-1}$ holds $(\forall p)$. ($\nabla := d \otimes \mathrm{id}_{H_{\mathbf{Z}}}$ is the connection of $\mathcal{O}_X \otimes H_{\mathbf{Z}}$.)

Definitions 1.4. The *classifying space D of polarized Hodge structures of type Φ_0* is the set of all decreasing filtrations F on $H_{0,\mathbf{C}}$ such that the triple $(H_0, \langle\ ,\ \rangle_0, F)$ is a polarized Hodge structure of weight w and of Hodge type $(h^{p,q})$.

Note that, by the condition on the signature of $(H_{0,\mathbf{R}}, \langle\ ,\ \rangle_0)$ (see NOTATION), D isnon-empty.

Definitions 1.5. The *compact dual $\check{D}$* of D is the set of all decreasing filtrations F on $H_{0,\mathbf{C}}$ such that the pair (H_0, F) is a Hodge structure of weight w and of Hodge type $(h^{p,q})$ and that the triple $(H_0, \langle\ ,\ \rangle_0, F)$ satisfies the condition 1.2 (i).

Note that D (resp. $\check{D}$) is homogeneous under $G_{\mathbf{R}}$ (resp. $G_{\mathbf{C}}$) and that D is an open subset of $\check{D}$.

Definition 1.6. Let $F \in D$ and let $T_D(F)$ be the tangent space of D at F. The *horizontal tangent space* $T_D^h(F)$ of D at F is defined as follows:

$$T_D^h(F) = F^{-1}(\mathfrak{g}_{\mathbf{C}})/F^0(\mathfrak{g}_{\mathbf{C}}) \subset T_D(F) = \mathfrak{g}_{\mathbf{C}}/F^0(\mathfrak{g}_{\mathbf{C}}),$$

where $F^r(\mathfrak{g}_{\mathbf{C}}) := \{N \in \mathfrak{g}_{\mathbf{C}} \mid N(F^p) \subset F^{p+r}\ (\forall p \in \mathbf{Z})\}$.

1.7. Let $\Phi_0 = (w, (h^{p,q})_{p,q\in\mathbf{Z}}, H_0, \langle\ ,\ \rangle_0)$ be as in NOTATION. Let X be a connected complex manifold and let $H = (H_{\mathbf{Z}}, \langle\ ,\ \rangle, F)$ be a polarized variation of Hodge structure on X of weight w and of Hodge type $(h^{p,q})$ with $(H_{\mathbf{Z},x}, \langle\ ,\ \rangle_x) \simeq (H_0, \langle\ ,\ \rangle_0)$ for some (hence any) point $x \in X$. Fix such a point x and identify $(H_{\mathbf{Z},x}, \langle\ ,\ \rangle_x) = (H_0, \langle\ ,\ \rangle_0)$. Put $\Gamma := \mathrm{Im}(\pi_1(X) \to G_{\mathbf{Z}})$. Then we have the associated period map

$$\varphi : X \to \Gamma\backslash D. \tag{1}$$

The differential of the period map at $x \in X$ factors through $T_D^h(\tilde{\varphi}(x))$. Here $\tilde{\varphi} : U \to D$ is a local lifting of φ on a neighborhood U of x.

§2. Borel-Serre spaces

2.1. *Summary.* Let $\mathcal{X}$ be the set of all maximal compact subgroups of $G_{\mathbf{R}}$. Then $G_{\mathbf{R}}$ acts transitively on $\mathcal{X}$ by inner automorphisms. Since the normalizer in $G_{\mathbf{R}}$ of each $K \in \mathcal{X}$ is K itself, we have a $G_{\mathbf{R}}$-equivariant isomorphism

$$G_{\mathbf{R}}/K \xrightarrow{\sim} \mathcal{X}, \quad g \mapsto gKg^{-1},$$

for each fixed $K \in \mathcal{X}$. By using this isomorphism, we introduce a topology on $\mathcal{X}$. This topology does not depend on the choice of K. Borel and Serre constructed in [BS] a space $\mathcal{X}_{\mathrm{BS}}$ ($\overline{\mathcal{X}}$ in their notation) which contains $\mathcal{X}$ as an open dense subset. The action of $G_{\mathbf{Z}}$ on $\mathcal{X}$ extends to an action on $\mathcal{X}_{\mathrm{BS}}$. The space $\mathcal{X}_{\mathrm{BS}}$ has the following remarkable properties:

(i) If Γ is a subgroup of $G_{\mathbf{Z}}$ of finite index, then the quotient space $\Gamma\backslash\mathcal{X}_{\mathrm{BS}}$ is compact.

(ii) If Γ is a neat subgroup of $G_{\mathbf{Z}}$, then the projection $\mathcal{X}_{\mathrm{BS}} \to \Gamma\backslash\mathcal{X}_{\mathrm{BS}}$ is a local homeomorphism.

Here Γ being neat means that the subgroup of $\mathbf{C}^\times$ generated by all eigenvalues of all $\gamma \in \Gamma$ is torsion free.

In this section, we enlarge D to get a topological space D_{BS}, which contains D as a dense open subspace, in the same way as $\mathcal{X}$ was enlarged to $\mathcal{X}_{\mathrm{BS}}$. We also construct a topological space $D_{\mathrm{BS,val}}$, as the projective limit of the blowing-ups of D_{BS}, which also contains D as a dense open subspace. These spaces are related by continuous proper surjective maps in the following way:

$$D_{\mathrm{BS,val}} \to D_{\mathrm{BS}} \to \mathcal{X}_{\mathrm{BS}}.$$

2.2. *Borel-Serre action.* Let P be a parabolic subgroup of $G_{\mathbf{R}}$ and P_u be its unipotent radical. Let S_P be the maximal $\mathbf{Q}$-split torus in the center of P/P_u. Let A_P be the connected component of S_P containing the unity. ($S_P \simeq (\mathbf{R}^\times)^r$, $A_P \simeq (\mathbf{R}_{>0})^r$, where $r := \operatorname{rank} S_P$.)

For $K \in \mathcal{X}$, let $\theta_K : G_{\mathbf{R}} \to G_{\mathbf{R}}$ be the Cartan involution associated to the maximal compact subgroup K, *i.e.*, the unique automorphism of $G_{\mathbf{R}}$ characterized by $\theta_K^2 = \mathrm{id}$ and $K = \{g \in G_{\mathbf{R}} \mid \theta_K(g) = g\}$. By [BS], for each $K \in \mathcal{X}$ and $a \in S_P$, there exists a unique element $a_K \in P$ satisfying both of the following.

(i) $(a_K \mod P_u) = a$.

(ii) $\theta_K(a_K) = a_K^{-1}$.

Then the map $S_P \to P$, $a \mapsto a_K$, is a homomorphism of algebraic groups over $\mathbf{R}$. We call a_K ($a \in S_P$) the *Borel-Serre lifting of* a *at* K.

For $F \in D$, we use the following notation:

$$(1)\qquad \begin{aligned} K_F &:= \left\{ g \in G_{\mathbf{R}} \;\middle|\; \begin{matrix} g \text{ preserves the Hermitian} \\ \text{inner product } \langle C_F(\ ), \bar{\ } \rangle_0 \end{matrix} \right\}, \\ K'_F &:= \{g \in G_{\mathbf{R}} \mid gF = F\} \subset K_F, \end{aligned}$$

where C_F is the Weil operator in 1.2 (cf. [Sc, §8]). Note that K_F is a maximal compact subgroup of $G_{\mathbf{R}}$ and the Cartan involution θ_{K_F} is given by

$$(2)\qquad \theta_{K_F} = \mathrm{Int}(C_F).$$

Note also that we have the canonical $G_{\mathbf{R}}$-equivariant continuous proper map

$$\begin{array}{ccc} D & \longrightarrow & \mathcal{X} \\ \downarrow\wr & & \downarrow\wr \\ G_{\mathbf{R}}/K'_F & \longrightarrow & G_{\mathbf{R}}/K_F, \end{array} \qquad F \longmapsto K_F. \tag{3}$$

For $a \in A_P$ and $K \in \mathcal{X}$ (resp. $F \in D$), we define an action $\circ$ by

$$a \circ K := \mathrm{Int}(a_K)K \quad (\text{resp. } a \circ F := a_{K_F}F). \tag{4}$$

We call this the *Borel-Serre action.*

Lemma 2.3. *For $a \in A_P$ and $p \in P$, we have $a_{\mathrm{Int}(p)K} = \mathrm{Int}(p)a_K$.*

Proof. This follows from the fact

$$\theta_{\mathrm{Int}(p)K} = \mathrm{Int}(p)\theta_K\, \mathrm{Int}(p)^{-1} \tag{1}$$

and the definition of the Borel-Serre liftings in 2.2. Q.E.D.

Lemma 2.4. *For $a \in A_P$, $p \in P$ and $F \in D$, we have $a \circ pF = p(a \circ F)$.*

Proof. By 2.3, we have

$$\begin{aligned} a \circ pF &= a_{K_{pF}}pF = a_{\mathrm{Int}(p)K_F}pF \\ &= (\mathrm{Int}(p)a_{K_F})pF = pa_{K_F}F = p(a \circ F). \quad \text{Q.E.D.} \end{aligned}$$

By 2.4, we see that the Borel-Serre action is indeed an action of A_P on D. In fact, for $a, b \in A_P$ and $F \in D$, we have

$$a \circ (b \circ F) = a \circ (b_{K_F}F) = b_{K_F}(a \circ F) = b_{K_F}a_{K_F}F = (ba)_{K_F}F = (ab) \circ F.$$

It can be verified in a similar way that A_P acts on $\mathcal{X}$ via the Borel-Serre action.

Definition 2.5. The *generalized Borel-Serre space* D_{BS} (resp. *Borel-Serre space* $\mathcal{X}_{\mathrm{BS}}$) is defined by

$$D_{\mathrm{BS}}\ (\text{resp. } \mathcal{X}_{\mathrm{BS}}) := \left\{ (P, Z) \,\middle|\, \begin{array}{l} P \text{ is a } \mathbf{Q}\text{-parabolic subgroup of } G_{\mathbf{R}}, \\ Z \text{ is an } (A_P\circ)\text{-orbit in } D \ (\text{resp. } \mathcal{X}) \end{array} \right\}$$

For an abelian group L and a submonoid V of L, we say V is *valuative* if $V \cup V^{-1} = L$. Also, put $V^{\times} := V \cap V^{-1}$.

Definition 2.6. We define the space $D_{\mathrm{BS,val}}$ by

$$D_{\mathrm{BS,val}} := \left\{ (T,Z,V) \;\middle|\; \begin{array}{l} T \text{ is an } \mathbf{R}\text{-split torus of } G_{\mathbf{R}}, \\ Z \text{ is a } (T_{>0})\text{-orbit in } D, \\ V \text{ is a valuative submonoid of } X(T), \\ \text{which satisfy the following (i)–(iii)} \end{array} \right\}$$

Here, $T_{>0}$ is the connected component of T containing the unity, and $X(T)$ is the character group of T.

(i) $\theta_{K_F}(t) = t^{-1}$ $(\forall F \in Z, \forall t \in T)$.

(ii) $V^\times = \{1\}$.

(iii) Let $H_{0,\mathbf{R}} = \bigoplus_{\chi \in X(T)} H(\chi)$ be the decomposition into eigenspaces $H(\chi) := \{v \in H_{0,\mathbf{R}} \,|\, tv = \chi(t)v \ (\forall t \in T)\}$. Then, for any $\chi \in X(T)$, the $\mathbf{R}$-subspace

$$M_\chi := \bigoplus_{\chi' \in \chi V^{-1}} H(\chi')$$

of $H_{0,\mathbf{R}}$ is $\mathbf{Q}$-rational.

2.7. Let M be a finite set of $\mathbf{R}$-subspaces of $H_{0,\mathbf{R}}$ satisfying the following two conditions:

(i) M contains 0 and $H_{0,\mathbf{R}}$ and is totally ordered with respect to the inclusion, *i.e.*,
$M = \{M_j\}_{0 \le j \le m}$, $0 = M_0 \subset M_1 \subset \cdots \subset M_m = H_{0,\mathbf{R}}$.

(ii) $M_j^\perp = M_{m-j}$ $(0 \le j \le m)$,
where $M_j^\perp := \{v \in H_{0,\mathbf{R}} \mid \langle v, M_j \rangle_0 = 0\}$.

Let $P := \{g \in (G^\circ)_{\mathbf{R}} \mid gM_j = M_j \ (0 \le j \le m)\}$. Then P is a parabolic subgroup of $G_{\mathbf{R}}$. To see this, it is sufficient to show that $(G^\circ)_{\mathbf{R}}/P$ is a projective variety. In fact, $(G^\circ)_{\mathbf{R}}/P$ is identified with the projective variety of finite sets $M' = \{M'_j\}_{0 \le j \le m}$ of $\mathbf{R}$-subspaces of $H_{0,\mathbf{R}}$ satisfying the above conditions (i), (ii) and $\dim M'_j = \dim M_j$ $(0 \le j \le m)$.

If all M_j $(0 \le j \le m)$ are $\mathbf{Q}$-rational, then P is a $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$. (It can be shown that any $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$ is obtained in this way. See 6.6.)

2.8. For a torus T of $G_{\mathbf{R}}$ and for a valuative submonoid V of $X(T)$, let

$$P_{T,V} := \{g \in (G^\circ)_{\mathbf{R}} \mid gM_\chi = M_\chi \ (\forall \chi \in X(T))\},$$

where M_χ is defined as in 2.6 (iii).

For $(T, Z, V) \in D_{\mathrm{BS,val}}$, we prove the following two assertions:

(1) $P_{T,V}$ is a $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$.

(2) $T \subset P_{T,V}$, the map $T \to P_{T,V}/P_{T,V,u}$ is injective, and the image of this map is contained in S_P.

To prove (1), it is enough to show that the set $\{M_\chi \mid \chi \in X(T)\}$ has the properties (i) and (ii) in 2.7. In fact, for $\chi, \chi' \in X(X)$, we have either $\chi'\chi^{-1} \in V$ or $\chi\chi'^{-1} \in V$, and we have $M_\chi \subset M_{\chi'}$ in the former case and $M_{\chi'} \subset M_\chi$ in the latter case. This proves 2.7 (i). Furthermore, $(M_\chi)^\perp = \sum_{\chi'} M_{\chi'}$, where χ' ranges over all elements of $X(T)$ which are not contained in $\chi^{-1}V$, and hence $(M_\chi)^\perp = M_{\chi'}$ for some χ' by 2.7 (i).

Next, (2) follows from $P/P_u \subset \prod_{\chi \in X(T)} \mathrm{GL}(H(\chi))$ and the fact that T acts on each $H(\chi)$ as scalars.

Now we have maps

$$
\begin{aligned}
&D_{\mathrm{BS,val}} \xrightarrow{\alpha} D_{\mathrm{BS}} \xrightarrow{\beta} \mathcal{X}_{\mathrm{BS}}, \quad \text{where} \\
&\alpha : (T, Z, V) \mapsto (P_{T,V}, A_{P_{T,V}} \circ Z), \\
&\beta : \text{the map induced by 2.2 (3).}
\end{aligned} \tag{3}
$$

2.9. For a $\mathbf{Q}$-parabolic subgroup P of $G_{\mathbf{R}}$, we define

$$
\begin{aligned}
&D_{\mathrm{BS}}(P) := \{(Q, Z) \in D_{\mathrm{BS}} \,|\, Q \supset P\}, \\
&\mathcal{X}_{\mathrm{BS}}(P) := \{(Q, Z) \in \mathcal{X}_{\mathrm{BS}} \,|\, Q \supset P\}, \\
&D_{\mathrm{BS,val}}(P) := \{(T, Z, V) \in D_{\mathrm{BS,val}} \,|\, P_{T,V} \supset P\}.
\end{aligned} \tag{1}
$$

In 2.10–2.14 below, we give preliminaries to define topologies on the spaces D_{BS}, $\mathcal{X}_{\mathrm{BS}}$, $D_{\mathrm{BS,val}}$.

2.10. Let P be a $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$. A subset Δ_P of $X(S_P)$ is defined as follows. Let $\tilde{S}_P \subset P$ be any torus which lifts $S_P \subset P/P_u$. Let

$$
\Delta'_P := \left\{ \chi \in X(\tilde{S}_P) \;\middle|\; \begin{matrix} 0 \neq \exists v \in \mathrm{Lie}(P_u) \text{ such that} \\ \mathrm{Ad}(a)v = \chi(a)^{-1}v \; (\forall a \in \tilde{S}_P) \end{matrix} \right\}
$$

Identify $X(\tilde{S}_P)$ with $X(S_P)$ via the canonical isomorphism $\tilde{S}_P \xrightarrow{\sim} S_P$. Then Δ'_P is independent of the choice of the liftings $\tilde{S}_P$, and it is a finite subset of $X(S_P)$ which generates $\mathbf{Q} \otimes X(S_P)$ over $\mathbf{Q}$. Then there exists a unique subset Δ_P of Δ'_P satisfying the following two conditions:

(i) $\sharp(\Delta_P) = \mathrm{rank}\, S_P$.

(ii) Δ'_P is contained in the submonoid of $X(S_P)$ generated by Δ_P.

Let P be as above. Let Q be a $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$ containing P. Then we have injective maps

$$S_Q \hookrightarrow S_P, \tag{1}$$

$$\Delta_Q \hookrightarrow \Delta_P, \tag{2}$$

obtained as follows, which we regards as inclusion maps. Note $Q \supset P \supset P_u \supset Q_u$. We have that $S_Q \subset P/Q_u$ in Q/Q_u, that the canonical map $S_Q \to P/P_u$ is injective, and that the image of this map is contained in S_P. This gives the injection (1). Let $I := \{\chi \in \Delta_P \mid \chi \text{ annihilates } S_Q\}$, and let $J \subset \Delta_P$ be the complement of I in Δ_P. Then the restriction to S_Q gives a bijection $J \xrightarrow{\sim} \Delta_Q$. The injection (2) is obtained as the composite $\Delta_Q \xleftarrow{\sim} J \hookrightarrow \Delta_P$. It is known that we have a bijection

$$\left\{\begin{matrix}\mathbf{Q}\text{-parabolic subgroup} \\ \text{of } G_{\mathbf{R}} \text{ containing } P\end{matrix}\right\} \xrightarrow{\sim} \{\text{subset of } \Delta_P\}, \quad Q \mapsto \Delta_Q \tag{3}$$

(cf. [BS, §4], [BT, §5], [B, §11]).

2.11. Let $M = \{M_j\}_{0 \le j \le m}$ be as in 2.7, and let P be the associated parabolic subgroup $\{g \in (G^\circ)_{\mathbf{R}} \mid gM_j = M_j \ \ (0 \le j \le m)\}$ of $G_{\mathbf{R}}$. We describe in terms of M the unipotent radical P_u, the quotient P/P_u, and, in the case when all M_j are $\mathbf{Q}$-rational, S_P and Δ_P.

(1) $P_u = \{g \in (G^\circ)_{\mathbf{R}} \mid (g-1)M_j \subset M_{j-1} \ \ (1 \le j \le m)\}$.

(2) $$P/P_u = \left\{ (g_j)_j \in \prod_{1 \le j \le m} \mathrm{GL}(\mathrm{gr}_j^M) \;\middle|\; \begin{matrix} {}^t g_j = g_{m+1-j}^{-1} \ \ (1 \le j \le m), \\ \det(g_{(m+1)/2}) = 1 \text{ if } m \text{ is odd} \end{matrix} \right\}.$$

Here ${}^t(\ )$ is the transpose with respect to the pairing

$$\langle\ ,\ \rangle_0 : \mathrm{gr}_j^M \times \mathrm{gr}_{m+1-j}^M \to \mathbf{R}.$$

Assume now that all M_j $(0 \le j \le m)$ are $\mathbf{Q}$-rational. We say that M is *exceptional* if either one of the following two conditions is satisfied:

(a) w is even, m is even, and $\dim \mathrm{gr}^M_{(m/2)+1} = 1$.

(b) w is even, m is odd, and $\dim \mathrm{gr}^M_{(m+1)/2} = 2$, and there exist elements $e_1, e_2 \in M_{(m+1)/2} \cap H_{0,\mathbf{Q}}$ such that $\langle e_1, e_1\rangle_0 = \langle e_2, e_2\rangle_0 = 0$ and $\langle e_1, e_2\rangle_0 = 1$.

We have

$$S_P = \left\{ (g_j)_j \in P/P_u \subset \prod_{1\le j\le m} \mathrm{GL}(\mathrm{gr}_j^M) \;\middle|\; \begin{array}{l} g_j \in \mathbf{G_{m,R}} \subset \mathrm{GL}(\mathrm{gr}_j^M) \text{ unless } m \text{ is odd} \\ \text{and } j = (m+1)/2;\;\; g_{(m+1)/2} = 1 \\ \text{if } m \text{ is odd and } M \text{ is not exceptional} \end{array} \right\}. \tag{3}$$

(4) Δ_P is described in terms of M as follows.
Assume first M is not exceptional. Let $r := m/2$ if m is even, and $r := (m-1)/2$ if m is odd. Then $r = \operatorname{rank} S_P$ and Δ_P consists of

$$(g_j)_{1\le j\le m} \mapsto g_{j+1}g_j^{-1} \quad (1 \le j \le r) \quad (g_j \in \mathbf{G_{m,R}}).$$

Assume next M is exceptional and m is even. Let $r := m/2$. Then $r = \operatorname{rank} S_P$ and Δ_P consitsts of

$$(g_j)_{1\le j\le m} \mapsto g_{j+1}g_j^{-1} \quad (1 \le j \le r-1) \quad \text{and}$$
$$(g_j)_{1\le j\le m} \mapsto g_{r+1}g_{r-1}^{-1}.$$

Assume lastly M is exceptional and m is odd. Let $r := (m+1)/2$. Then $r = \operatorname{rank} S_P$ and Δ_P consitsts of

$$(g_j)_{1\le j\le m} \mapsto g_{j+1}g_j^{-1} \quad (1 \le j \le r-2),$$
$$(g_j)_{1\le j\le m} \mapsto g_{r,1}g_{r-1}^{-1}, \quad \text{and}$$
$$(g_j)_{1\le j\le m} \mapsto g_{r,2}g_{r-1}^{-1},$$

where $g_{r,k} \in \mathbf{G_m}$ $(k = 1, 2)$ are defined by

$$g_r e_k = g_{r,k} e_k \quad \text{in} \quad \mathrm{gr}^M_{(m+1)/2}\,.$$

2.12. *Identification of* $D_{\mathrm{BS}}(P)$ *with* $D \times^{A_P} \overline{A}_P$. Let P be a $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$. Let $\overline{A}_P :=$ Map$(\Delta_P, \mathbf{R}_{\ge 0})$ (= the set of maps $\Delta_P \to \mathbf{R}_{\ge 0}$). Hence $\overline{A}_P \simeq \mathbf{R}^r_{\ge 0}$, $r = \operatorname{rank} S_P$. Via $A_P \overset{\sim}{\to}$ Map$(\Delta_P, \mathbf{R}_{>0})$, A_P acts on $\overline{A}_P$ in the natural way. Denote $D \times^{A_P} \overline{A}_P := (D \times \overline{A}_P)/A_P$ under the action $a(F, b) = (a \circ F, a^{-1}b)$ $(a \in A_P, (F, b) \in D \times \overline{A}_P)$. Then we have a bijection

$$D_{\mathrm{BS}}(P) \simeq D \times^{A_P} \overline{A}_P, \quad (Q, Z) \longleftrightarrow (F, b),$$

defined as follows. For $(Q, Z) \in D_{\mathrm{BS}}(P)$, F is any element of Z and $b \in \overline{A}_P$ is defined by $b(\chi) = 0$ if $\chi \in \Delta_Q$ and $b(\chi) = 1$ if $\chi \notin \Delta_Q$. For

$(F, b) \in D \times^{A_P} \overline{A}_P$, Q is the $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$ containing P such that $\Delta_Q = \{\chi \in \Delta_P \mid b(\chi) = 0\}$, and $Z := \{a \circ F \mid a \in A_P, \chi(a) = b(\chi)$ for any $\chi \in \Delta_P - \Delta_Q\}$.

For a $\mathbf{Q}$-parabolic subgroup Q of $G_{\mathbf{R}}$ containing P, we have $A_Q \subset A_P$ and $\overline{A}_Q \subset \overline{A}_P$. The latter follows by extending a map $\Delta_Q \to \mathbf{R}$ to Δ_P by 1. Hence we have a commutative diagram

$$\begin{array}{ccc} D_{\mathrm{BS}}(Q) & \simeq & D \times^{A_Q} \overline{A}_Q \\ \cap & & \cap \\ D_{\mathrm{BS}}(P) & \simeq & D \times^{A_P} \overline{A}_P. \end{array}$$

2.13. *The topological space* $(\mathbf{R}^r_{\geq 0})_{\mathrm{val}}$. We define a topological space $(\mathbf{R}^r_{\geq 0})_{\mathrm{val}}$ $(r \geq 0)$ as follows. Let $r \geq 0$ be an integer, and define a topological space B by

$$(1) \qquad B := \varprojlim_{I \in \Psi} \mathrm{Bl}_I((\mathbf{G}_{\mathbf{a},\mathbf{R}})^r),$$

where Ψ denotes the set of all non-zero ideals I of $\mathbf{R}[t_1, \ldots, t_r] = \mathcal{O}((\mathbf{G}_{\mathbf{a},\mathbf{R}})^r)$ generated by some monomials $f_1, \ldots, f_n$, and $\mathrm{Bl}_I((\mathbf{G}_{\mathbf{a},\mathbf{R}})^r)$ means the blowing-up $\mathrm{Proj}(\bigoplus_{s \geq 0} I^s)$ of $(\mathbf{G}_{\mathbf{a},\mathbf{R}})^r$ along I. Define an order in Ψ by: $I \leq J$ if and only if $II' = J$ for some $I' \in \Psi$. For $I \leq II' = J$ with $I = (f_1, \ldots, f_n)$, $I' = (g_1, \ldots, g_m)$, define a morphism

$$\begin{array}{l} \mathrm{Bl}_I((\mathbf{G}_{\mathbf{a},\mathbf{R}})^r) = \bigcup_{1 \leq k \leq n} \mathrm{Spec}\left(\mathbf{R}\left[t_1, \ldots, t_r, \frac{f_1}{f_k}, \ldots, \frac{f_n}{f_k}\right]\right) \\ \qquad \uparrow \\ \mathrm{Bl}_J((\mathbf{G}_{\mathbf{a},\mathbf{R}})^r) = \bigcup_{1 \leq k \leq n, 1 \leq l \leq m} \mathrm{Spec}\left(\mathbf{R}\left[t_1, \ldots, t_r, \frac{f_1 g_1}{f_k g_l}, \ldots, \frac{f_n g_m}{f_k g_l}\right]\right) \end{array}$$

by the inclusions of affine rings

$$\mathbf{R}\left[t_1, \ldots, t_r, \tfrac{f_1}{f_k}, \ldots, \tfrac{f_n}{f_k}\right] \hookrightarrow \mathbf{R}\left[t_1, \ldots, t_r, \tfrac{f_1 g_1}{f_k g_l}, \ldots, \tfrac{f_n g_m}{f_k g_l}\right]$$

$(1 \leq k \leq n, 1 \leq l \leq m)$. The projective limit in (1) is taken with respect to this ordering.

Since the centers of the blow-ups are outside $(\mathbf{G}_{\mathbf{m},\mathbf{R}})^r$ $(\subset (\mathbf{G}_{\mathbf{a},\mathbf{R}})^r)$, we have a dense open immersion $(\mathbf{G}_{\mathbf{m},\mathbf{R}})^r \hookrightarrow B$. Furthermore, there is a unique action of $(\mathbf{G}_{\mathbf{m},\mathbf{R}})^r$ on B which is compatible with the standard action of $(\mathbf{G}_{\mathbf{m},\mathbf{R}})^r$ on itself. Let $(\mathbf{R}^r_{\geq 0})_{\mathrm{val}}$ be the closure of $\mathbf{R}^r_{>0}$ in B under the composite of open immersions $\mathbf{R}^r_{>0} \subset (\mathbf{G}_{\mathbf{m},\mathbf{R}})^r \hookrightarrow B$. Then the canonical map $(\mathbf{R}^r_{\geq 0})_{\mathrm{val}} \to \mathbf{R}^r_{\geq 0}$ is proper and surjective because so is $B \to (\mathbf{G}_{\mathbf{a},\mathbf{R}})^r$ (cf. [NB, Ch. 1, §10, no. 2, Corollaire 3]). Furthermore the group $\mathbf{R}^r_{>0}$ acts on $(\mathbf{R}^r_{\geq 0})_{\mathrm{val}}$.

Let $\mathbf{N} := \mathbf{Z}_{\geq 0}$. There exists a canonical bijection between $(\mathbf{R}^r_{\geq 0})_{\text{val}}$ and the set of all pairs (V, h) where V is a valuative submonoid of $\mathbf{Z}^r$ containing $\mathbf{N}^r$ and $h : V^\times \to \mathbf{R}_{>0}$ is a group homomorphism. In fact, for a point x of $(\mathbf{R}^r_{\geq 0})_{\text{val}} \subset B$, the corresponding pair (V, h) is defined by

$$(2) \qquad \begin{cases} V := \{m \in \mathbf{Z}^r \mid \prod_j t_j^{m(j)} \text{ is regular at } x\}, \\ h(m) := \big(\prod_j t_j^{m(j)}\big)(x). \end{cases}$$

The inverse map $(V, h) \mapsto x = (x_I)_{I \in \Psi}$, $x_I \in \mathrm{Bl}_I((\mathbf{G}_{\mathbf{a},\mathbf{R}})^r)$, is given as follows. Let $\tilde{h}$ be the extension by zero of h to V. Let $I = (f_1, \dots, f_n)$, and take $1 \leq k \leq n$ such that the all exponents of $\frac{f_l}{f_k}$ $(1 \leq l \leq n)$, in the expressions as the products of t_j $(1 \leq j \leq r)$, belong to V. Define a point x_I by an $\mathbf{R}$-algebra homomorphism

$$x_I : \mathbf{R}\big[t_1, \dots, t_r, \tfrac{f_1}{f_k}, \dots, \tfrac{f_n}{f_k}\big] \to \mathbf{R}, \quad x_I\big(\prod_j t_j^{m(j)}\big) := \tilde{h}(m) \quad (m \in V).$$

The action of $a = (a_j)_{1 \leq j \leq r} \in \mathbf{R}^r_{>0}$ sends (V, h) to (V, ah), where ah is defined by

$$(ah)(m) := \big(\prod_j a_j^{m(j)}\big) h(m) \quad (m \in V^\times).$$

A directed family of elements (V_λ, h_λ) of $(\mathbf{R}^r_{\geq 0})_{\text{val}}$ converges to $(V, h) \in (\mathbf{R}^r_{\geq 0})_{\text{val}}$ if and only if the following condition is satisfied for each $m \in V$:

(3) $m \in V_\lambda$ for any sufficiently large λ, and $\tilde{h}_\lambda(m)$ converges to $\tilde{h}(m)$.

(For the generality of $(\ )_{\text{val}}$, cf. [K].)

2.14. *Identification of* $D_{\mathrm{BS},\text{val}}(P)$ *with* $D \times^{A_P} (\overline{A}_P)_{\text{val}}$. Let P be a $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$. Let $(\overline{A}_P)_{\text{val}}$ be the set of all pairs (V, h) where V is a valuative submonoid of $X(S_P)$ containing Δ_P and $h : V^\times \to \mathbf{R}_{>0}$ is a homomorphism. Then, by writing $\Delta_P = \{\chi_1, \dots \chi_r\}$ $(r = \operatorname{rank} S_P)$, we have a bijection

$$(\overline{A}_P)_{\text{val}} \simeq (\mathbf{R}^r_{\geq 0})_{\text{val}}, \quad (V, h) \mapsto (V', h'),$$

where $V' := \{m \in \mathbf{Z}^r \mid \prod_j \chi_j^{m(j)} \in V\}$ and h' is the map $(V')^\times \to \mathbf{R}_{>0}$, $m \mapsto h(\prod_j \chi_j^{m(j)})$. For the set $D_{\mathrm{BS},\text{val}}(P)$ in 2.9, we have a bijection

$$(1) \qquad D_{\mathrm{BS},\text{val}}(P) \to D \times^{A_P} (\overline{A}_P)_{\text{val}}, \quad (T, Z, V') \mapsto (F, (V, h)),$$

which is given by

$$(2)\quad \begin{cases} F \in Z \ \text{(any element)}, \\ V := (\text{the inverse image of } V' \text{ under } X(S_P) \to X(T)), \\ h : V^\times \to \mathbf{R}_{>0}, \text{ the trivial homomorphism} \\ \qquad\qquad\qquad\qquad h(m) := 1 \ (\forall m \in V^\times). \end{cases}$$

Here in the definition of V, we regard T as a subtorus of S_P via the composite of the embeddings $T \hookrightarrow S_{P_{T,V'}} \hookrightarrow S_P$ (2.8 and 2.11).

The inverse map $D \times^{A_P} \overline{A}_P \to D_{\mathrm{BS,val}}(P)$, $(F,(V,h)) \mapsto (T,Z,V')$, is given by

$$\begin{cases} T := \begin{pmatrix} \text{the image of the annihilator of } V^\times \text{ in } S_P \\ \text{under the Borel-Serre lifting } S_P \hookrightarrow G_{\mathbf{R}} \text{ at} K_F \end{pmatrix}, \\ Z := \{a \circ F \mid a \in A_P,\ \chi(a) = h(\chi)\ (\forall \chi \in V^\times)\}, \\ V' := (\text{the image of } V \text{ under } X(S_P) \to X(T)) \ (\text{so that } V' \simeq V/V^\times). \end{cases}$$

We show that (T,Z,V') satisfies the conditions (i)–(iii) in 2.6 which define $D_{\mathrm{BS,val}}$. (i) and (ii) are clear. We prove (iii), that is, $M_\chi = \bigoplus_{\chi' \in \chi V'^{-1}} H(\chi')$ is $\mathbf{Q}$-rational for any $\chi \in X(T)$. The image of T in P/P_u is $\mathbf{Q}$-rational, since it is contained in S_P. This shows that uTu^{-1} is $\mathbf{Q}$-rational for some $u \in P_u$. Hence uM_χ is $\mathbf{Q}$-rational for any $\chi \in X(T)$. Hence it is enough to prove $uM_\chi = M_\chi$. For this, it is enough to prove $lM_\chi \subset M_\chi$ for $l := \log u \in \mathrm{Lie}(P_u)$. By decomposing $\mathrm{Lie}(P_u)$ as the direct sum of eigenspaces for the adjoint action of S_P, we may assume that there exists $\chi' \in \Delta'_P$ such that $\mathrm{Ad}(t)l = \chi'(t)^{-1}l$ for any $t \in T$. Since $\Delta_P \subset V$, V' contains the restrictions of elements of Δ'_P to T. Hence $\chi'|_T \in V'$. Hence $lM_\chi \subset M_{\chi(\chi'|_T^{-1})} \subset M_{\chi V'^{-1}} = M_\chi$.

2.15. *Topologies of D_{BS}, $\mathcal{X}_{\mathrm{BS}}$ and $D_{\mathrm{BS,val}}$.* Let P be a $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$. We have, with the identification of $\overline{A}_P$ and $\mathbf{R}^r_{\geq 0}$,

$$D_{\mathrm{BS}}(P) \simeq D \times^{A_P} \mathbf{R}^r_{\geq 0} \quad \text{(see 2.12)},$$
$$\mathcal{X}_{\mathrm{BS}}(P) \simeq \mathcal{X} \times^{A_P} \mathbf{R}^r_{\geq 0} \quad \text{(analoguously to the above)},$$
$$D_{\mathrm{BS,val}}(P) \simeq D \times^{A_P} (\mathbf{R}^r_{\geq 0})_{\mathrm{val}} \quad \text{(see 2.14)}.$$

By using these isomorphisms, we introduce a topology on $D_{\mathrm{BS}}(P)$ (resp. $\mathcal{X}_{\mathrm{BS}}(P)$, $D_{\mathrm{BS,val}}(P)$). We introduce the strongest topology on D_{BS} (resp. $\mathcal{X}_{\mathrm{BS}}$, $D_{\mathrm{BS,val}}$) for which the map $D_{\mathrm{BS}}(P) \hookrightarrow D_{\mathrm{BS}}$ (resp. $\mathcal{X}_{\mathrm{BS}}(P) \hookrightarrow \mathcal{X}_{\mathrm{BS}}$, $D_{\mathrm{BS,val}}(P) \hookrightarrow D_{\mathrm{BS,val}}$) is continuous for every $\mathbf{Q}$-parabolic subgroup P of $G_{\mathbf{R}}$. Then, it can be shown as in [BS] that all these maps $D_{\mathrm{BS}}(P) \hookrightarrow D_{\mathrm{BS}}$, $\mathcal{X}_{\mathrm{BS}}(P) \hookrightarrow \mathcal{X}_{\mathrm{BS}}$, and $D_{\mathrm{BS,val}}(P) \hookrightarrow D_{\mathrm{BS,val}}$ are open embeddings.

2.16. We give here a preparation for the proof of Theorem 2.17. Let P be a $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$. As we shall show below, there is a closed subset $\mathcal{X}$' of $\mathcal{X}$ such that we have homeomorphisms

$$\mathcal{X}' \times A_P \xrightarrow{\sim} \mathcal{X}, \quad (x,a) \mapsto a \circ x, \tag{1}$$

$$D' \times A_P \xrightarrow{\sim} D, \quad (x,a) \mapsto a \circ x, \tag{2}$$

where D' is the inverse image of $\mathcal{X}'$ under the canonical map $D \to \mathcal{X}$ in 2.2 (3). These (1), (2) induce homeomorphisms

$$\begin{aligned} &\mathcal{X}' \times \mathbf{R}^r_{\geq 0} \simeq \mathcal{X}_{\mathrm{BS}}(P), \\ &D' \times \mathbf{R}^r_{\geq 0} \simeq D_{\mathrm{BS}}(P), \\ &D' \times (\mathbf{R}^r_{\geq 0})_{\mathrm{val}} \simeq D_{\mathrm{BS,val}}(P), \end{aligned} \tag{3}$$

where $r = \operatorname{rank} S_P$.

Now we define $\mathcal{X}'$. Let ${}^\circ P$ be the intersection of the kernel of $|\chi| : P \to \mathbf{R}_{>0}$, $a \mapsto |\chi(a)|$, for all homomorphisms χ of algebraic groups $P \to \mathbf{G}_{\mathbf{m},\mathbf{R}}$ defined over $\mathbf{Q}$. Then $P_u \subset {}^\circ P$. By [BS, 1.2], the canonical map $A_P \to P/{}^\circ P$ is an isomorphism of topological groups. Let $|\ \ | : P \to A_P$ be the composite map $P \to P/{}^\circ P \simeq A_P$. Fix a maximal compact subgroup K of $G_{\mathbf{R}}$. Since $G_{\mathbf{R}} = PK$ (cf. [B, §11]) and since $|\ \ |$ kills the compact group $P \cap K$, there exists a unique map $G_{\mathbf{R}} \to A_P$ sending pk to $|p|$ $(p \in P, k \in K)$. This map factors through $G_{\mathbf{R}}/K \simeq \mathcal{X}$. Let $\mathcal{X}'$ be the inverse image of $1 \in A_P$ under the induced map $\mathcal{X} \to A_P$. It is seen easily that we have the homeomorphisms (1), (2).

Theorem 2.17. (i) *The maps* $\alpha : D_{\mathrm{BS,val}} \to D_{\mathrm{BS}}$ *and* $\beta : D_{\mathrm{BS}} \to \mathcal{X}_{\mathrm{BS}}$ *in* 2.8 *are proper and surjective.*

(ii) *The spaces* $\mathcal{X}_{\mathrm{BS}}$, D_{BS}, $D_{\mathrm{BS,val}}$ *are Hausdorff and locally compact.*

Proof. (i) follows from the descriptions of $\mathcal{X}_{\mathrm{BS}}(P)$, $D_{\mathrm{BS}}(P)$, $D_{\mathrm{BS,val}}(P)$ in 2.16 (3) together with the fact that $(\mathbf{R}^r_{\geq 0})_{\mathrm{val}} \to \mathbf{R}^r_{\geq 0}$ and $D' \to \mathcal{X}'$ are proper and surjective.

(ii) for $\mathcal{X}_{\mathrm{BS}}$ is proved in [BS]. (ii) for D_{BS}, $D_{\mathrm{BS,val}}$ follows from this and (i). Q.E.D.

§3. Spaces of SL(2)-orbits

3.1. *Summary.* Let $\Delta : (\mathbf{G}_{\mathbf{m},\mathbf{R}})^n \to \mathrm{SL}(2,\mathbf{R})^n$ be the homomorphism defined by

$$(t_1,\ldots,t_n) \mapsto \left(\begin{pmatrix} t_1^{-1} & 0 \\ 0 & t_1 \end{pmatrix}, \ldots, \begin{pmatrix} t_n^{-1} & 0 \\ 0 & t_n \end{pmatrix} \right).$$

For a homomorphism $\rho : \mathrm{SL}(2,\mathbf{C})^n \to G_{\mathbf{C}}$ defined over $\mathbf{R}$, the following three conditions are equivalent:

(1) ρ does not factor through the projection onto any component $\mathrm{SL}(2,\mathbf{C})^m$ of $\mathrm{SL}(2,\mathbf{C})^n$ $(m < n)$.

(2) The induced Lie algebra homomorphism $\rho_* : \mathfrak{sl}(2,\mathbf{C})^{\oplus n} \to \mathfrak{g}_{\mathbf{C}}$ is injective.

(3) The induced Lie algebra homomorphism $\mathrm{Lie}(\mathbf{G}_{\mathbf{m},\mathbf{C}})^n \xrightarrow{\Delta_*} \mathfrak{sl}(2,\mathbf{C})^{\oplus n} \xrightarrow{\rho_*} \mathfrak{g}_{\mathbf{C}}$ is injective.

Let (ρ,φ) be a pair of a homomorphism $\rho : \mathrm{SL}(2,\mathbf{C})^n \to G_{\mathbf{C}}$ defined over $\mathbf{R}$ and a map $\varphi : \mathbf{P}^1(\mathbf{C})^n \to \check{D}$. Throughout this paper, such a pair (ρ,φ) is called an SL(2)-*orbit of rank* n if it satisfies the following three conditions:

(i) ρ satisfies the equivalent conditions (1)–(3).

(ii) $\varphi(gz) = \rho(g)\varphi(z)$ for all $g \in \mathrm{SL}(2,\mathbf{C})^n$ and all $z \in \mathbf{P}^1(\mathbf{C})^n$.

(iii) Let $\mathfrak{h}$ be the upper-half plane and let $\mathbf{i} := (i,\ldots,i) \in \mathfrak{h}^n \subset \mathbf{P}^1(\mathbf{C})^n$. Then $\varphi(\mathbf{i}) \in D$, and the associated Lie algebra homomorphism $\mathrm{Lie}(\rho) : \mathfrak{sl}(2,\mathbf{C})^{\oplus n} \to \mathfrak{g}_{\mathbf{C}}$ is a homomorphism of type $(0,0)$ with respect to the Hodge structures induced by the points $\mathbf{i} \in \mathfrak{h}^n$ and $\varphi(\mathbf{i}) \in D$, respectively.

Let (ρ,φ) be an SL(2)-orbit of rank n. We denote by Y_j, $N_j = N_j^-$, N_j^+ the image under the Lie algebra homomorphism $\mathrm{Lie}(\rho)$ of

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

of the j-th factor of $\mathfrak{sl}(2,\mathbf{C})^{\oplus n}$, respectively.

Define a homomorphism

$$\tilde{\rho} : (\mathbf{G}_{\mathbf{m},\mathbf{R}})^n \to G_{\mathbf{R}}, \qquad (4)$$
$$(t_1,\ldots,t_n) \mapsto \rho(\Delta(t_1\cdots t_n, t_2\cdots t_n, \ldots, t_{n-1}t_n, t_n)).$$

(The map $\tilde{\rho}$ will be related in 3.9 to Borel-Serre liftings.)

In this section, we introduce spaces of SL(2)-orbits $D_{\mathrm{SL}(2)}$ and the projective limit $D_{\mathrm{SL}(2),\mathrm{val}}$ of the blowing-ups of $D_{\mathrm{SL}(2)}$. These spaces, together with the spaces in the previous section, will form the following diagram:

$$
\begin{array}{ccccc}
D_{\mathrm{SL}(2),\mathrm{val}} & \hookrightarrow & D_{\mathrm{BS},\mathrm{val}} & & \\
\downarrow & & \downarrow & & \\
D_{\mathrm{SL}(2)} & & D_{\mathrm{BS}} & \to & \mathcal{X}_{\mathrm{BS}}.
\end{array}
\tag{5}
$$

In general, there is no direct relation between $D_{\mathrm{SL}(2)}$ and D_{BS} (see, §6), which is why we introduce $D_{\mathrm{SL}(2),\mathrm{val}}$ and $D_{\mathrm{BS},\mathrm{val}}$.

3.2. *Weight filtrations.* For a nilpotent element $N \in \mathfrak{g}_{\mathbf{R}}$, the *weight filtration associated to* N is the increasing filtration $W = W(N)$ of $H_{0,\mathbf{R}}$ characterized by the following conditions (i), (ii) ([D]).

(i) $NW_k \subset W_{k-2}$ for all $k \in \mathbf{Z}$;

(ii) $N^k : \mathrm{gr}_k^W \xrightarrow{\sim} \mathrm{gr}_{-k}^W$ for all $k \in \mathbf{Z}_{\geq 0}$.

3.3. *Cones.* We fix terminology concerning cones. Let V be an $\mathbf{R}$-vector space. A *cone* in V is a subset σ of V which is closed under addition and under multiplication by elements of $\mathbf{R}_{\geq 0}$ and satisfies $\sigma \cap (-\sigma) = 0$. A subset σ in $\mathfrak{g}_{\mathbf{R}}$ is a *nilpotent cone in* $\mathfrak{g}_{\mathbf{R}}$ if it is a finitely generated cone in $\mathfrak{g}_{\mathbf{R}}$ consisiting of mutually commutative nilpotent elements. Let σ be a nilpotent cone in $\mathfrak{g}_{\mathbf{R}}$. For $A = \mathbf{R}, \mathbf{C}$, we denote by σ_A the A-linear span of σ in $\mathfrak{g}_A$.

Definition 3.4. Let σ be a nilpotent cone in $\mathfrak{g}_{\mathbf{R}}$, and let N_j $(1 \leq j \leq r)$ be its generators over $\mathbf{R}_{\geq 0}$. A subset Z of $\check{D}$ is a *σ-nilpotent orbit* (*resp. σ-nilpotent i-orbit*) if it satisfies the following three conditions for some $F \in Z$.

(i) $Z = \exp(\sigma_{\mathbf{C}})F$ (resp. $Z = \exp(i\sigma_{\mathbf{R}})F$).

(ii) $NF^p \subset F^{p-1}$ $(\forall p, \forall N \in \sigma)$.

(iii) $\exp(\sum_{1 \leq j \leq r} iy_j N_j)F \in D$ $(\forall y_j \gg 0)$.

It is easy to see that, in 3.4, if the conditions (i)–(iii) are satisfied by one $F \in Z$ then they are satisfied by any $F \in Z$. The condition (ii) is called *Griffiths transversality* and the condition (iii) is called *positivity.*

3.5. *Weight filtrations associated to a nilpotent orbit.* We recall here a result of Cattani and Kaplan.

Theorem-Definition ([CK2]). *Let (σ, Z) be a nilpotent i-orbit. Then, for any elements N, N' of the relative interior of σ, the filtrations*

$W(N)$ and $W(N')$ of $H_{0,\mathbf{R}}$ coincide. This common filtration is denoted by $W(\sigma)$.

Note that, as in [CKS, §4], an SL(2)-orbit (ρ, φ) of rank n defines an n-tuple of nilpotent elements $(N_j)_{1\le j\le n}$ and a nested family of nilpotent i-orbits $(\sigma_j, Z_j)_{1\le j\le n}$ by

$$\sigma_j := \mathbf{R}_{\ge 0}N_1 + \cdots + \mathbf{R}_{\ge 0}N_j, \tag{1}$$
$$Z_j := \exp(i\sigma_{j,\mathbf{R}})\varphi(\overbrace{0,\ldots,0}^{j}, i, \ldots, i).$$

We have $W(N_1 + \cdots + N_j) = W(\sigma_j)$ for each $1 \le j \le n$.

Definition 3.6. We define $D_{\mathrm{SL}(2),0} := D$ and, for a positive integer n, we define

$$D_{\mathrm{SL}(2),n} := \left\{ (\rho,\varphi) \;\middle|\; \begin{array}{l} (\rho,\varphi) \text{ is an SL(2)-orbit of rank } n, \\ W(\sigma_j) \text{ is } \mathbf{Q}\text{-rational } (1 \le j \le n) \end{array} \right\} \Big/ \sim ,$$

where $(\rho, \varphi) \sim (\rho', \varphi')$ if and only if there exists $t \in \mathbf{R}^n_{>0}$ such that $\rho' = \mathrm{Int}(\rho(\Delta(t)))(\rho)$ and $\varphi' = \rho(\Delta(t)) \cdot \varphi$. We define

$$D_{\mathrm{SL}(2)} := \textstyle\bigsqcup_{n\ge 0} D_{\mathrm{SL}(2),n}, \quad D_{\mathrm{SL}(2),\le r} := \bigsqcup_{0\le n\le r} D_{\mathrm{SL}(2),n}.$$

We denote by $[\rho, \varphi]$ the point of $D_{\mathrm{SL}(2)}$ represented by (ρ, φ).

Definition 3.7. For a non-negative integer n, we define

$$D_{\mathrm{SL}(2),\mathrm{val},n} := \left\{ ([\rho,\varphi], Z, V) \;\middle|\; \begin{array}{l} [\rho,\varphi] \in D_{\mathrm{SL}(2),n},\ Z \subset \varphi((\mathbf{R}_{>0}i)^n), \\ V : \text{a valuative submonoid of } X(\mathbf{G}^n_m), \\ \text{which satisfy (i) and (ii) below} \end{array} \right\}.$$

(i) Let $X((\mathbf{G_m})^n)_+$ be the image of $\mathbf{N}^n$ via the canonical isomorphism $\mathbf{Z}^n \simeq X((\mathbf{G_m})^n)$. Then,
$$X((\mathbf{G_m})^n)_+ \subset V \text{ and } X((\mathbf{G_m})^n)_+ \cap V^\times = \{1\}.$$

(ii) Let
$$T := \{t \in (\mathbf{G}_{\mathbf{m},\mathbf{R}})^n \mid \chi(t) = 1\ (\forall \chi \in V^\times)\}.$$

Then Z is a $\tilde{\rho}(T_{>0})$-orbit in D. Here $\tilde{\rho}$ is as in 3.1, and we denote again by $T_{>0}$ the connected component of T containing the unity.

We define

$$D_{\mathrm{SL}(2),\mathrm{val}} := \textstyle\bigsqcup_{n\ge 0} D_{\mathrm{SL}(2),\mathrm{val},n}.$$

We have the canonical surjection

$$D_{\mathrm{SL}(2),\mathrm{val}} \to D_{\mathrm{SL}(2)}, ([\rho,\varphi], Z, V) \mapsto [\rho,\varphi].$$

After preliminaries in Lemmas 3.8–3.10, we will relate $D_{\mathrm{SL}(2),\mathrm{val}}$ with $D_{\mathrm{BS},\mathrm{val}}$ in 3.11 below.

Lemma 3.8. *Let (ρ,φ) be an* SL(2)*-orbit of rank n and put $\mathbf{r} := \varphi(\mathbf{i})$. Then*

$$\theta_{K_{\mathbf{r}}}(\rho(\Delta(t))) = \rho(\Delta(t))^{-1} \quad (\forall t \in (\mathbf{G}_{\mathbf{m},\mathbf{R}})^n).$$

Proof. Let Y_j $(1 \le j \le n)$ be as in 3.1. It is enough to show $\theta_{K_{\mathbf{r}}}(Y_j) = -Y_j$ for all j. We prove this. Here $\theta_{K_{\mathbf{r}}}$ is regarded as the involution of $\mathfrak{g}_{\mathbf{C}}$ induced by the Cartan involution $\theta_{K_{\mathbf{r}}}$ of $G_{\mathbf{R}}$ at $K_{\mathbf{r}}$ by abuse of the notation. Let

$$\mathfrak{g}_{\mathbf{C}} = \textstyle\bigoplus_s \mathfrak{g}_{\mathbf{r}}^{s,-s}, \quad \mathfrak{g}_{\mathbf{r}}^{s,-s} := \{X \in \mathfrak{g}_{\mathbf{C}} \mid XH_{\mathbf{r}}^{p,w-p} \subset H_{\mathbf{r}}^{p+s,w-p-s} \ (\forall p)\}$$

be the Hodge structure on $\mathfrak{g}_{\mathbf{C}}$ induced by $\mathbf{r}$. Then, by 2.2 (2) and the definition of the Weil operator $C_{\mathbf{r}}$ in 1.2, $\theta_{K_{\mathbf{r}}}$ is given by

(1) $\theta_{K_{\mathbf{r}}}(X) = \sum_s (-1)^s X^{s,-s}$ for $X = \sum_s X^{s,-s} \in \mathfrak{g}_{\mathbf{C}} = \bigoplus_s \mathfrak{g}_{\mathbf{r}}^{s,-s}$.

On the other hand, the Hodge decomposition of $H_{\mathfrak{h},\mathbf{C}} = \mathbf{C}^2 = \mathbf{C}e_1 + \mathbf{C}e_2$ corresponding to $i \in \mathfrak{h}$ is

$$H_{\mathfrak{h},\mathbf{C}} = H_{\mathfrak{h}}^{1,0} \oplus H_{\mathfrak{h}}^{0,1} = \mathbf{C}(ie_1 + e_2) \oplus \mathbf{C}(-ie_1 + e_2) \ \text{(cf. 6.2 below)},$$

and this induces the Hodge decomposition

$$\begin{aligned}\mathfrak{sl}(2,\mathbf{C}) &= \mathfrak{sl}(2)^{1,-1} \oplus \mathfrak{sl}(2)^{0,0} \oplus \mathfrak{sl}(2)^{-1,1} \\ &= \mathbf{C}\begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix} \oplus \mathbf{C}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \oplus \mathbf{C}\begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}\end{aligned}$$

of $\mathfrak{sl}(2,\mathbf{C})$. Since

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{i}{2}\left(\begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix} - \begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix}\right),$$

$Y_j \in \mathfrak{g}_{\mathbf{r}}^{1,-1} \oplus \mathfrak{g}_{\mathbf{r}}^{-1,1}$ by 3.1 (ii) for all j. Hence, by (1), $\theta_{K_{\mathbf{r}}}(Y_j) = -Y_j$ for all j. Q.E.D.

Lemma 3.9. *Let (ρ,φ) and $\mathbf{r}$ be as in* 3.8. *For $1 \le j \le n$, let $W^{(j)} = W(\sigma_j)$ be as in* 3.5 *and let P_j be the $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}}$ defined by $W^{(j)}$. Then the j-th factor of $\tilde\rho$ (cf.* 3.1),

$$t_j \mapsto \tilde\rho(1,\ldots,1,t_j,1,\ldots,1),$$

coincides with the Borel-Serre lifting at $K_{\mathbf{r}}$ of the j-th weight map

$$\mathbf{G}_{\mathbf{m},\mathbf{R}} \to P_j/P_{j,u}, \quad t_j \mapsto (t_j^k \text{ on } \mathrm{gr}_k^{W^{(j)}})_k.$$

Proof. This follows from 3.8 and the observation

$$\begin{aligned}\tilde\rho(1,\ldots,1,t_j,1,\ldots,1) &= \rho(\Delta(\overbrace{t_j,\ldots,t_j}^{j},1,\ldots,1)) \\ &= \exp(\log(t_j)(Y_1+\cdots+Y_j)). \quad \text{Q.E.D.}\end{aligned}$$

Lemma 3.10. *Let (ρ,φ), $\mathbf{r}=\varphi(\mathbf{i})$ be as in 3.8, and W be the family of weight filtrations $(W(\sigma_j))_{1\le j\le n}$ as just before 3.6. Then, an $\mathrm{SL}(2)$-orbit (ρ,φ) of rank n is completely determined by $(W,\mathbf{r})$.*

Proof. By 3.9, $\tilde\rho$ is determined by $(W,\mathbf{r})$, and Y_j $(1\le j\le n)$ are determined by $\tilde\rho$. Let $T_D(\mathbf{r})$ (resp. $T_{\mathfrak{h}}(i)$) be the tangent space of D at $\mathbf{r}$ (resp. $\mathfrak{h}$ at i). Then we have a commutative diagram

$$(1)\qquad \begin{array}{ccc} \mathfrak{g}_{\mathbf{C}} & \xrightarrow{\alpha_{\mathbf{r}}} & T_D(\mathbf{r}) \\ \uparrow{\scriptstyle \mathrm{Lie}(\rho)} & & \uparrow{\scriptstyle d\varphi} \\ \mathfrak{sl}(2,\mathbf{C})^n & \xrightarrow{\alpha_i^n} & T_{\mathfrak{h}}(i)^n, \end{array}$$

where $\alpha_{\mathbf{r}}$ (resp. α_i) is the differential of the morphism $G_{\mathbf{C}}\to\check{D}$, $g\mapsto g\mathbf{r}$, (resp. $\mathrm{SL}(2,\mathbf{C})\to\mathbf{P}^1(\mathbf{C})$, $g\mapsto gi$,) at 1. Since $-2i\alpha_i\begin{pmatrix}0&1\\0&0\end{pmatrix} = \alpha_i\begin{pmatrix}-1&0\\0&1\end{pmatrix}$, we have $-2i\alpha_{\mathbf{r}}(N_j)=\alpha_{\mathbf{r}}(Y_j)$ $(1\le j\le n)$. Since the restriction of $\alpha_{\mathbf{r}}$ on $\mathrm{Lie}(P_{j,u})$ is injective, N_j $(1\le j\le n)$ are determined. The N_j^+ are determined by the Y_j and the N_j. Q.E.D.

Theorem 3.11. *There is an injective map*

$$(1)\qquad D_{\mathrm{SL}(2),\mathrm{val}} \to D_{\mathrm{BS},\mathrm{val}}, \quad ([\rho,\varphi],Z,V)\mapsto(\tilde\rho(T),Z,V').$$

Here T is the subtorus of $(\mathbf{G}_{\mathbf{m},\mathbf{R}})^n$ in 3.7 (ii) and $V' := \tilde\rho_*(V/V^\times)$, which is regarded as a subset of the character group of $\tilde\rho(T)$.

Proof. Let $([\rho,\varphi],Z,V)\in D_{\mathrm{SL}(2),\mathrm{val}}$ and let (T',Z,V') be its image under (1).

We check first the conditions (i)–(iii) in 2.6 hold for (T',Z,V'). (i) follows from 3.8 and 2.3. (ii) is evident. We prove (iii), that is, $M_\chi =$

$\bigoplus_{\chi' \in \chi V'^{-1}} H(\chi')$ is $\mathbf{Q}$-rational for any $\chi \in X(T')$. For $\chi \in X((\mathbf{G_m})^n)$, let

$$\tilde{H}(\chi) := \{v \in H_{0,\mathbf{R}} \mid \tilde{\rho}(t)v = \chi(t)v \quad (\forall t \in (G_{m,\mathbf{R}})^n)\},$$
$$W_\chi := \bigoplus_{\chi' \in \chi(X((\mathbf{G_m})^n)_+)^{-1}} \tilde{H}(\chi').$$

Let $\chi \in X(T')$ and let $\tilde{\chi} \in X((\mathbf{G_m})^n)$ be an element such that

$$\chi(\tilde{\rho}(t)) = \tilde{\chi}(t) \quad (\forall t \in T).$$

Then

$$M_\chi = \bigoplus_{\chi' \in \tilde{\chi} V^{-1}} \tilde{H}(\chi').$$

Since $X((\mathbf{G_m})^n)_+ \subset V$, we have

$$M_\chi = \sum_{\chi' \in \tilde{\chi} V^{-1}} W_{\chi'}.$$

Hence it is enough to show that W_χ is $\mathbf{Q}$-rational for any $\chi \in X((\mathbf{G_m})^n)$. Write $\chi = (l_1, \ldots, l_n)$ $(l_j \in \mathbf{Z})$ via $X((\mathbf{G_m})^n) = \mathbf{Z}^n$. Then $W_\chi = \bigcap_{1 \le j \le n} W(\sigma_j)_{l_j}$, where $W(\sigma_j)$ is as in 3.5. Since the $W(\sigma_j)_{l_j}$ are $\mathbf{Q}$-rational, so is W_χ. This proves that the map (1) is well-defined.

We prove that (1) is injective. $V \subset X((\mathbf{G_m})^n)$ is the inverse image of V' under the map $X((\mathbf{G_m})^n) \to X(T) \simeq X(T')$. Hence it is sufficient to show that $[\rho, \varphi] \in D_{\mathrm{SL}(2)}$ is determined by (T', Z, V'). Let $\mathbf{r} \in Z$ and take a representative (ρ, φ) with $\varphi(\mathbf{i}) = \mathbf{r}$. To prove that (ρ, φ) is determined by (T', Z, V') and $\mathbf{r}$, it is sufficient to show, by 3.10, that the rank n of (ρ, φ) and the family of weight filtrations $\big(W(N_1 + \cdots + N_j)\big)_{1 \le j \le n}$ associated to (ρ, φ) are determined by (T', Z, V'). Hence it is sufficient to prove that the family $(N_j)_{1 \le j \le n}$ is the unique family of elements of $\mathrm{Lie}(P_{T',V',u})$ having the following properties (i)–(iv).

(i) For $1 \le j \le n$, N_j is a non-zero eigenvector for the adjoint action of T'.

(ii) For $1 \le j \le n$, let $\chi_j : T' \to \mathbf{G}_{\mathbf{m},\mathbf{R}}$ be a character defined by $\mathrm{Ad}(t)N_j = \chi_j(t)N_j$ for $t \in T'$. Then the χ_j are non-trivial and different from each other.

(iii) In the notation (ii) above, $\chi_j \chi_{j+1}^{-1} \in V'$ $(1 \le j \le n-1)$.

(iv) Let $\alpha_\mathbf{r} : \mathfrak{g}_\mathbf{C} \to T_D(\mathbf{r})$ be the canonical $\mathbf{C}$-linear map 3.10 (1). For $1 \le j \le n$, let $\mathrm{Lie}(\chi_j) : \mathrm{Lie}(T') \to \mathbf{R}$ be the map induced by χ_j. Then

$$-i\alpha_\mathbf{r}(A) = \sum_{1 \le j \le n} \mathrm{Lie}(\chi_j)(A)\alpha_\mathbf{r}(N_j) \quad \text{for any } A \in \mathrm{Lie}(T').$$

We first show that the family $(N_j)_{1\le j\le n}$ associated to (ρ,φ) satisfies these (i)–(iv). We have

$$\mathrm{Ad}(\rho(\Delta(t_1,\dots,t_n)))(N_j)=t_j^{-2}N_j$$

and hence

$$\mathrm{Ad}(\tilde\rho(t_1,\dots,t_n))(N_j)=(t_j\cdots t_n)^{-2}N_j. \tag{2}$$

Hence (i) and (iii) are satisfied. Denote by χ_{n+1} the trivial character of T'. Suppose $\chi_j=\chi_k$ for some j, k with $1\le j\le k\le n+1$. Then the character

$$(t_1,\dots,t_n)\mapsto \textstyle\prod_{j\le l<k} t_l^2$$

of $(\mathbf{G}_{\mathbf{m},\mathbf{R}})^n$ is trivial on T and hence belongs to $V^\times$. Since it also belongs to $X((\mathbf{G}_\mathbf{m})^n)_+$ and since $X((\mathbf{G}_\mathbf{m})^n)_+\cap V^\times=\{1\}$, we have $j=k$. Thus we have proved (ii). Note that

$$\alpha_\mathbf{r}(Y_j)=-2i\alpha_\mathbf{r}(N_j)\quad(1\le j\le n)\quad(\text{cf. Proof of 3.10}). \tag{3}$$

By (2) and (3), we have, for an element $A=\sum_{1\le j\le n}b_jY_j$ of $\mathrm{Lie}(T')$,

$$-i\alpha_\mathbf{r}(A)=-2\textstyle\sum_{1\le j\le n}b_j\alpha_\mathbf{r}(N_j)=\sum_{1\le j\le n}\mathrm{Lie}(\chi_j)(A)\alpha_\mathbf{r}(N_j).$$

Next we prove that a family $(N_j)_{1\le j\le n}$ of elements of $\mathrm{Lie}(P_{T',V',u})$ satisfying (i)–(iv) is unique. Since the restriction of $\alpha_\mathbf{r}$ on $\mathrm{Lie}(P_{T',V',u})$ is injective, we have:

(4) $\sum_{1\le j\le n}\mathrm{Lie}(\chi_j)(A)N_j$ for $A\in\mathrm{Lie}(T')$ is the unique element of $\mathrm{Lie}(P_{T',V',u})$ whose image under $\alpha_\mathbf{r}$ coincides with $-i\alpha_\mathbf{r}(A)$.

As sets, we have

$$\{\chi_j\}_{1\le j\le n}=\left\{\chi\in X(T')\ \middle|\ \begin{array}{l}\text{under the action of }\mathrm{Ad}(T')\\ \text{on }\mathfrak{g}_\mathbf{C}\text{, the }\chi\text{-component}\\ \text{of }\sum_{1\le j\le n}\mathrm{Lie}(\chi_j)(A)N_j\\ (A\in\mathrm{Lie}(T'))\text{ is not zero}\end{array}\right\} \tag{5}$$

Since $(V')^\times=1$, (iii) determines the order, hence the family $(\chi_j)_{1\le j\le n}$. For each $1\le j\le n$, $\mathrm{Lie}(\chi_j)(A)N_j$ $(A\in\mathrm{Lie}(T'))$ is determined as the χ_j-component of $\sum_{1\le j\le n}\mathrm{Lie}(\chi_j)(A)N_j$ $(A\in\mathrm{Lie}(T'))$ under the action of $\mathrm{Ad}(T')$. Since $\mathrm{Lie}(\chi_j)\ne0$, N_j is determined. Q.E.D.

3.12. *Topologies on $D_{\mathrm{SL}(2)}$, $D_{\mathrm{SL}(2),\mathrm{val}}$.* A family $(W^{(j)})_{1\le j\le n}$ of increasing filtrations $W^{(j)}$ of $H_{0,\mathbf{R}}$ is called a *compatible family* if there exists a direct sum decomposition $H_{0,\mathbf{R}}=\bigoplus_{m\in\mathbf{Z}^n}H(m)$ such that $W_k^{(j)}=$

$\bigoplus_{m\in\mathbf{Z}^n,\, m_j\le k} H(m)$ for any j and k and that $\langle H(m), H(m')\rangle_0 = 0$ if $m+m' \neq 0$. Note that, for $[\rho,\varphi] \in D_{SL(2),n}$, the family of weight filtrations $(W(\sigma_j))_{1\le j\le n}$ associated to $[\rho,\varphi]$ in 3.5 is a compatible family.

Let $W = (W^{(j)})_{1\le j\le n}$ be a compatible family of $\mathbf{Q}$-rational increasing filtrations $W^{(j)}$ of $H_{0,\mathbf{R}}$. We define the subset $D_{\mathrm{SL}(2)}(W)$ of $D_{\mathrm{SL}(2)}$ by

$$D_{\mathrm{SL}(2)}(W) := \bigcup_{0\le m\le n} \left\{ x \in D_{\mathrm{SL}(2),m} \;\middle|\; \begin{array}{l} \exists s_j \in \mathbf{Z}\ (1\le j\le m) \text{ such that} \\ 1 \le s_1 < \cdots < s_m \le n \text{ and} \\ W(\sigma_j) = W^{(s_j)} (\forall j) \end{array} \right\}$$

Here $(W(\sigma_j))_{1\le j\le m}$ is the family of weight filtrations associated to $x \in D_{\mathrm{SL}(2),m}$.

We define the subset $D_{\mathrm{SL}(2),\mathrm{val}}(W)$ of $D_{\mathrm{SL}(2),\mathrm{val}}$ by the pull-back of $D_{\mathrm{SL}(2)}(W)$.

Definition 3.13. We define the *topology on* $D_{\mathrm{SL}(2),\mathrm{val}}$ as the weakest one in which the following two families of subsets are open:

(i) The pull-backs on $D_{\mathrm{SL}(2),\mathrm{val}}$ of open subsets of $D_{\mathrm{BS},\mathrm{val}}$.

(ii) The subset $D_{\mathrm{SL}(2),\mathrm{val}}(W)$ for any n and any compatible family of $\mathbf{Q}$-rational increasing filtrations $W = (W^{(j)})_{1\le j\le n}$.

Note that the injective map $D_{\mathrm{SL}(2),\mathrm{val}} \to D_{\mathrm{BS},\mathrm{val}}$ is continuous by (i). We induce the quotient topology on $D_{\mathrm{SL}(2)}$ of the above one under the projection $D_{\mathrm{SL}(2),\mathrm{val}} \to D_{\mathrm{SL}(2)}$.

This topology of $D_{\mathrm{SL}(2)}$ has the following property (see 4.19 below). For an SL(2)-orbit of rank n (ρ,φ), $[\rho,\varphi] \in D_{\mathrm{SL}(2)}$ is the limit of

$$\varphi(iy_1,\ldots,iy_n) \in D, \quad \text{as } y_j > 0 \text{ and } \frac{y_j}{y_{j+1}} \to \infty \text{ for } 1 \le \forall j \le n,$$

where y_{n+1} denotes 1.

Note that the space $D_{\mathrm{SL}(2),\mathrm{val}}$ is Hausdorff by 3.11 and 3.13.

Theorem 3.14. (i) *The canonical map* $D_{\mathrm{SL}(2),\mathrm{val}} \to D_{\mathrm{SL}(2)}$ *is proper and surjective.*

(ii) *The space* $D_{\mathrm{SL}(2)}$ *is Hausdorff.*

The proof of this theorem will be given in §4.

In Remarks 3.15, 3.16 below, we give a rough sketch of the relationship between the present results and the results in [KU1]. Precise descriptions and their proofs will be found in the forthcoming paper [KU2].

3.15 *Remark* (*Relation with period maps*). Let $\overline{X}$ be a connected complex manifold, let Y be a reduced divisor with normal crossings on $\overline{X}$, and let $X := \overline{X} - Y$. Let $H = (H_{\mathbf{Z}}, \langle\ ,\ \rangle, F)$ be a polarized variation of Hodge structure on X of weight w and of Hodge type $(h^{p,q})$. Then, as in 1.7, we have the associated period map

$$\varphi : X \to \Gamma\backslash D. \tag{1}$$

In the forthcoming paper [KU2], by using SL(2)-orbit theorem in [CKS], we will show that φ extends continuously to $\varphi_{\mathrm{SL}(2)}$ and also to $\varphi^{\flat}_{\mathrm{SL}(2)}$ in the following diagram.

$$\begin{array}{ccc} \varprojlim_{I\in\Psi} \mathrm{Bl}_I(\overline{X})^{\log} & \xrightarrow{\varphi_{\mathrm{SL}(2)}} & \Gamma\backslash D_{\mathrm{SL}(2)} \\ \downarrow & & \downarrow \\ \varprojlim_{I\in\Psi} \mathrm{Bl}_I(\overline{X}) & \xrightarrow{\varphi^{\flat}_{\mathrm{SL}(2)}} & \Gamma\backslash D^{\flat}_{\mathrm{SL}(2)}. \end{array} \tag{2}$$

Here, in analogy with 2.13, Ψ denotes the set of all non-zero $\mathcal{O}_{\overline{X}}$-ideals which are locally of forms $I = (f_1, \dots, f_n)$ generated by some local sections $f_1, \dots, f_n$ of $\mathcal{O}_{\overline{X}}$ whose zeros are contained in Y, $\mathrm{Bl}_I(\overline{X})$ means the blowing-up of $\overline{X}$ along I, $\mathrm{Bl}_I(\overline{X})^{\log}$ is the topological space defined by the method of [KN], and the projective limit is taken with respect to the ordering of the set Ψ as in 2.13. $D^{\flat}_{\mathrm{SL}(2)}$ is a space of Satake(-Baily-Borel)-Cattani-Kaplan type ([Sa], [BB], [CK]) which is defined as a quotient space of $D_{\mathrm{SL}(2)}$ under the following equivalence relation $\sim$. For $x \in D_{\mathrm{SL}(2),m}$, $y \in D_{\mathrm{SL}(2),n}$,

$$x \sim y \iff \begin{cases} m = n, \text{ and the associated families of weight,} \\ \text{filtrations coincide, say } W, \text{ and } y \in G_{W,\mathbf{R},u}x. \end{cases}$$

Here $G_{W,\mathbf{R}}$ is the subgroup of $G_{\mathbf{R}}$ preserving all the filtrations in W and $G_{W,\mathbf{R},u}$ is its unipotent radical. (When D is in the classical situation (6.6) except one case (i) in Theorem 6.7, this space $D^{\flat}_{\mathrm{SL}(2)}$ is exactly the same as the 'reductive Borel-Serre space' which was constructed by Zucker in [Z1], [Z3].) Since the centers of the blowing-ups are contained in Y, we have open immersions $X \hookrightarrow \varprojlim_{I\in\Psi} \mathrm{Bl}_I(\overline{X})$, $X \hookrightarrow \varprojlim_{I\in\Psi} \mathrm{Bl}_I(\overline{X})^{\log}$. Note that, when $\overline{X}$ is a unit disc and Y is the origin, we have $\varprojlim_{I\in\Psi} \mathrm{Bl}_I(\overline{X}) = \overline{X}$.

3.16 *Remark* (*Relation with moduli of polarized logarithmic Hodge structures*). In the forthcoming paper [KU2], the diagram 3.1 (5) is enlarged as

$$
\begin{array}{ccccccccc}
 & & & & D_{\mathrm{SL}(2),\mathrm{val}} & \hookrightarrow & D_{\mathrm{BS},\mathrm{val}} & & \\
 & & & & \downarrow & & \downarrow & & \\
D^{\sharp}_{\Sigma} & \leftarrow & D^{\sharp}_{\Sigma,\mathrm{val}} & \rightarrow & D_{\mathrm{SL}(2)} & & D_{\mathrm{BS}} & \rightarrow & \mathcal{X}_{\mathrm{BS}}. \\
\downarrow & & \downarrow & & \downarrow & & & & \\
\Gamma\backslash D_{\Sigma} & \leftarrow & D_{\Sigma,\mathrm{val}} & \rightarrow & D^{\flat}_{\mathrm{SL}(2)} & & & &
\end{array}
\tag{1}
$$

We explain roughly the spaces and the morphisms appeared in this diagram (1).

Replacing $\mathfrak{g}_{\mathbf{R}}$ by $\mathfrak{g}_{\mathbf{Q}}$ in 3.3, we can define a *nilpotent cone σ in $\mathfrak{g}_{\mathbf{Q}}$* analogously. In this remark, we always consider nilpotent cones in $\mathfrak{g}_{\mathbf{Q}}$. Let Σ be a *fan* of nilpotent cones in $\mathfrak{g}_{\mathbf{Q}}$, *i.e.*, a set of nilpotent cones in $\mathfrak{g}_{\mathbf{Q}}$ satisfying

$$
\begin{cases}
\sigma \in \Sigma,\ \tau \text{ is a face of } \sigma \Longrightarrow \tau \in \Sigma, \\
\sigma, \sigma' \in \Sigma \Longrightarrow \sigma \cap \sigma' \text{ is a face of } \sigma \text{ and of } \sigma'.
\end{cases}
$$

By using the notions in Definition 3.4 (replacing $\sigma = \sum_j \mathbf{R}_{\geq 0} N_j$ by $\sigma = \sum_j \mathbf{Q}_{\geq 0} N_j$), we define sets

$$
\begin{aligned}
(2) \qquad & D_{\Sigma} \ (\text{resp. } D^{\sharp}_{\Sigma}) \\
& := \{(\sigma, Z) \text{ nilpotent orbit (resp. } i\text{-orbit)} \mid \sigma \in \Sigma,\ Z \subset \check{D}\}.
\end{aligned}
$$

There is a natural map $D^{\sharp}_{\Sigma} \to D_{\Sigma}$, $(\sigma, Z) \mapsto (\sigma, \exp(\sigma_{\mathbf{C}})Z)$. Let Γ be a neat subgroup of $G_{\mathbf{Z}}$. We assume Γ is *strongly compatible* with Σ, *i.e.*, they satisfy

$$
\begin{cases}
\Gamma \in \Gamma, \sigma \in \Sigma \Longrightarrow \Gamma^{-1}\sigma\Gamma \in \Sigma, \\
\Gamma(\sigma) := \Gamma \cap \exp(\sigma) \Longrightarrow \log \Gamma(\sigma) \text{ generates the cone } \sigma.
\end{cases}
$$

Theorem 6.2 in [KU1] says that $\Gamma\backslash D_{\Sigma}$ with a suitable structure of 'generalized fs logarithmic analytic space' is a fine moduli space of 'polarized logarithmic Hodge structures of type (Φ_0, Σ, Γ)', and that $\Gamma\backslash D^{\sharp}_{\Sigma}$, with a suitable structure of ringed space, is isomorphic to 'the ringed space $(\Gamma\backslash D_{\Sigma})^{\log}$ associated to $\Gamma\backslash D_{\Sigma}$'. This space $\Gamma\backslash D_{\Sigma}$ is our generalization of a toroidal compactification (3) in Introduction.

The space $D_{\Sigma,\mathrm{val}}$ and $D^{\sharp}_{\Sigma,\mathrm{val}}$ are certain projective limits of blowing-ups of D_{Σ} and $D^{\sharp}_{\Sigma}$, respectively. The maps $D_{\Sigma,\mathrm{val}} \to D^{\flat}_{\mathrm{SL}(2)}$ and

$D^{\sharp}_{\Sigma,\mathrm{val}} \to D_{\mathrm{SL}(2)}$ are the unique continuous maps extending the identity of D, which are constructed by using SL(2)-orbit theorem in [CKS].

We can derive good properties, such as Hausdorffness, on $\Gamma\backslash D_{\Sigma}$ from good properties 2.1 (i), (ii) on $\mathcal{X}_{\mathrm{BS}}$ along the fundamental diagram (1).

Let $H = (H_{\mathbf{Z}}, \langle\ ,\ \rangle, F)$ be a polarized variation of Hodge structure on X in 3.15. If there is a suitable fan Σ for $\overline{X}$, then the period map 3.15 (1) extends to

$$\begin{aligned} &\overline{X} \to \Gamma\backslash D_{\Sigma}, \\ &\overline{X}^{\log} \to (\Gamma\backslash D_{\Sigma})^{\log} = \Gamma\backslash D^{\sharp}_{\Sigma}, \end{aligned} \tag{3}$$

and to

$$\begin{aligned} &\varprojlim_{I\in\Psi} \mathrm{Bl}_I(\overline{X}) \to \Gamma\backslash D_{\Sigma,\mathrm{val}} \to \Gamma\backslash D^{\flat}_{\mathrm{SL}(2)}, \\ &\varprojlim_{I\in\Psi} \mathrm{Bl}_I(\overline{X})^{\log} \to \Gamma\backslash D^{\sharp}_{\Sigma,\mathrm{val}} \to \Gamma\backslash D_{\mathrm{SL}(2)}. \end{aligned} \tag{4}$$

The composite maps (4) are nothing but $\varphi^{\flat}_{\mathrm{SL}(2)}$ and $\varphi_{\mathrm{SL}(2)}$, respectively, in 3.15 (2).

§4. Proof of Theorem 3.14

4.1. *Summary.* This section is devoted to proving Theorem 3.14. To do so, we will introduce a new topology $\mathcal{T}$ on the set $D_{\mathrm{SL}(2)}(W)$ in 3.12 in terms of filters on D associated to points of $D_{\mathrm{SL}(2)}(W)$ by using Cartan decompositions (4.6, 4.8 below). Denote this new topological space by $D_{\mathrm{SL}(2)}(W)_{\mathcal{T}}$, *i.e.*, the underlying set coincides with the one of $D_{\mathrm{SL}(2)}(W)$ but whose topology is $\mathcal{T}$. We will show that the topological space $D_{\mathrm{SL}(2),\mathrm{val}}(W)$ in 3.13 is homeomorphic to $D_{\mathrm{SL}(2)}(W)_{\mathcal{T}} \times_{\mathbf{R}^n_{\geq 0}} (\mathbf{R}^n_{\geq 0})_{\mathrm{val}}$ (4.14 below). From this, we get the homeomorphism $D_{\mathrm{SL}(2)}(W) \xrightarrow{\sim} D_{\mathrm{SL}(2)}(W)_{\mathcal{T}}$ (4.15 below) and the proof of Theorem 3.14 (4.17 below).

4.2. First, we prove two Lemmas 4.3, 4.4 below. Let X_j $(1 \leq j \leq d)$ be indeterminates. A *convergent Lie power series in the* X_j is a power series with respect to the bracket product $[\ ,\ ]$ in the X_j with coefficients in $\mathbf{C}$ which converges if the X_j are elements of a finite-dimensional Lie algebra over $\mathbf{C}$ and sufficiently near 0. The *order* of a convergent Lie power series $f(X_1, \ldots, X_d)$ is the minimum of the degrees of the monomials in $f(X_1, \ldots, X_d)$ whose coefficients are not zero.

Lemma 4.3. *Let X, Y be two indeterminates. Then there exist convergent Lie power serieses $f_-(X,Y)$ and $f_+(X,Y)$ which satisfy the following two conditions.*

(i) $\exp(X+Y) = \exp(f_-(X,Y))\exp(f_+(X,Y))$.

(ii) *Each monomial in* $f_-(X,Y)$ (*resp.* $f_+(X,Y)$) *is of odd* (*resp. even*) *degree in* X.

Proof. Let A be the convergent Lie power series of order ≥ 2 defined by

$$\exp(X+Y) = \exp(X+A)\exp(Y).$$

Devide $A = A_- + A_+$ so that each monomial in A_- (resp. A_+) is of odd (resp. even) degree in X. Let A' be the convergent Lie power series of order ≥ 3 defined by

$$\exp(X+Y) = \exp(X + A_- + A')\exp(A_+)\exp(Y).$$

Then $\exp(A_+)\exp(Y) = \exp(B_+)$ for some convergent Lie power series B_+ whose monomials are of even degree in X. Devide $A' = A'_- + A'_+$ so that each monomial in A'_- (resp. A'_+) is of odd (resp. even) degree in X. Continuing this process, we obtain Lie power series $f_-(X,Y)$, $f_+(X,Y)$ which can be checked to be convergent Lie power series. Q.E.D.

Lemma 4.4. *Let* $y \in D_{\mathrm{SL}(2),\mathrm{val},n}$ *and let* $[\rho,\varphi] \in D_{\mathrm{SL}(2),n}$ *be the image of* y. *Let* $P = P_{\tilde\rho(T),V'}$ *be the* $\mathbf{Q}$*-parabolic subgroup of* $G_{\mathbf{R}}$ *associated to* y *in* 3.7 *and* 2.8. *Let* $K = K_{\mathbf{r}}$ *be the maximal compact subgroup of* $G_{\mathbf{R}}$ *associated to the point* $\mathbf{r} := \varphi(\mathbf{i})$, *and let* $\theta = \theta_K$ *be the associated Cartan involution. Let* $\mathfrak{g}_{\mathbf{R}} = \bigoplus_l \mathfrak{g}(l)$ *be the decomposition into the eigenspaces under the action of* $(\mathbf{G}_{\mathbf{m},\mathbf{R}})^n$ *through* $\mathrm{Ad}(\tilde\rho(\ ))$. *Let* X_l, Y_l *be indeterminates, where the index* l *runs over all* $l \in \mathbf{Z}^n$ *with* $\mathfrak{g}(l) \neq 0$.

Then, there exist convergent Lie power series f_P *and* f_K *in the* X_l *and the* Y_l *which satisfy the following conditions. Let* U *be a sufficiently small neighborhood of* 0 *in* $\mathfrak{g}_{\mathbf{R}}$. *For any* $x \in U$, *writing* $x = \sum_l x_l$, $x_l \in \mathfrak{g}(l)$, *we have*

(i) $f_P\big((x_l)_l, (\theta(x_l))_l\big) \in \mathrm{Lie}\,P$, $f_K\big((x_l)_l, (\theta(x_l))_l\big) \in \mathrm{Lie}\,K$, *and*

(ii) $\exp(x) = \exp\big(f_P\big((x_l)_l, (\theta(x_l))_l\big)\big)\exp\big(f_K\big((x_l)_l, (\theta(x_l))_l\big)\big)$.

Proof. Note that $\mathrm{Lie}(P) = \bigoplus_{-l \in V} \mathfrak{g}(l)$. We write

$$\begin{aligned} x &= \textstyle\sum_l x_l = \sum_{-l\in V} x_l + \sum_{-l\notin V} x_l \\ &= \textstyle\sum_{-l\in V} x_l - \sum_{-l\notin V}\theta(x_l) + \sum_{-l\notin V}(x_l + \theta(x_l)), \\ \exp(x) &= \exp\big(\textstyle\sum_{-l\in V} x_l - \sum_{-l\notin V}\theta(x_l) + A\big) \\ &\qquad\qquad \cdot \exp\big(\textstyle\sum_{-l\notin V}(x_l + \theta(x_l))\big), \end{aligned}$$

where A is the convergent Lie power series of order ≥ 2 determined by the above. Then

$$\textstyle\sum_{-l\in V} x_l - \sum_{-l\notin V} \theta(x_l) \in \mathrm{Lie}(P), \quad \sum_{-l\notin V}(x_l + \theta(x_l)) \in \mathrm{Lie}(K).$$

Do the same for A replacing x, and iterate. In the end, we obtain the desired convergent Lie power series f_P and f_K. Q.E.D.

4.5. *The filter on D associated to a point of $D_{\mathrm{SL}(2)}$.* We use the terminology *filter* in the sense of [NB, §6]. We introduce, in Definition 4.6 below, the filter $\mathcal{F}_x$ on D associated to a point $x \in D_{\mathrm{SL}(2)}$. We shall see later that this $\mathcal{F}_x$ coincides with the filter

$$\{U \cap D \mid U \text{ is a neighborhood of } x \text{ in } D_{\mathrm{SL}(2)}\} \quad (4.15 \text{ below}).$$

Definition 4.6. Let $x \in D_{\mathrm{SL}(2),n}$ and (ρ, φ) be a representative of x. Put $\mathbf{r} = \varphi(\mathbf{i})$. For

$$(1) \qquad \begin{cases} U & : \text{a neighborhood of 1 in } K_{\mathbf{r}}, \\ U' & : \text{a neighborhood of 0 in } \mathbf{R}^n_{\geq 0}, \\ U'' & : \text{a neighborhood of 1 in } G_{\mathbf{R}}, \end{cases}$$

we denote

$$(2) \qquad A(U, U', U'') := \left\{ g\tilde{\rho}(t)k\mathbf{r} \;\middle|\; \begin{array}{l} k \in U,\ t \in \mathbf{R}^n_{>0} \cap U', \\ g \in U'',\ \theta_{\tilde{\rho}(t)k\mathbf{r}}(g) = g^{-1} \end{array} \right\}$$

where $\tilde{\rho}$ is as in 3.1 and $\theta_{\tilde{\rho}(t)k\mathbf{r}}$ is the Cartan involution of $G_{\mathbf{R}}$ associated to the maximal compact subgroup $K_{\tilde{\rho}(t)k\mathbf{r}}$. We define $\mathcal{F}_x$ associated to x as the filter on D whose basis is given by the $A(U, U', U'')$ where U, U' and U'' run over all such neighborhoods as in (1).

As is easily seen, $\mathcal{F}_x$ is independent of the choice of a representative (ρ, φ) of x. Note that, since $\theta_{\tilde{\rho}(t)k\mathbf{r}} = \mathrm{Int}(\tilde{\rho}(t))\theta_{\mathbf{r}}\, \mathrm{Int}(\tilde{\rho}(t))^{-1}$, we have

$$(3) \quad A(U, U', U'') = \left\{ \tilde{\rho}(t)gk\mathbf{r} \;\middle|\; \begin{array}{l} k \in U,\ t \in \mathbf{R}^n_{>0} \cap U', \\ g \in \mathrm{Int}(\tilde{\rho}(t))^{-1}(U''),\ \theta_{\mathbf{r}}(g) = g^{-1} \end{array} \right\}.$$

Lemma 4.7. *A basis of the filter $\mathcal{F}_x$ is also given by the following family of sets:*

$$(1) \quad B(U, U', U'') := \left\{ g\tilde{\rho}(t)k\mathbf{r} \;\middle|\; \begin{array}{l} k \in U,\ t \in \mathbf{R}^n_{>0} \cap U', \\ \mathrm{Int}(\tilde{\rho}(t))^j(g) \in U'' \ (j = 0, \pm 1) \end{array} \right\}$$

where U, U' and U'' run over all such neighborhoods as 4.6 (1).

Proof. We prove that, for given U, U' and U'', and for sufficiently small V, V' and V'', such as in 4.6 (1), we have

$$A(V,V',V'') \subset B(U,U',U''). \tag{2}$$

$$B(V,V',V'') \subset A(U,U',U''). \tag{3}$$

We prove (2). By the remark just after Definition 4.6, any element of $A(V,V',V'')$ can be written as

$\tilde{\rho}(t)gk\mathbf{r}$ such that $k \in V$, $t \in \mathbf{R}^n_{>0} \cap V'$, $\mathrm{Int}(\tilde{\rho}(t))(g) \in V''$, $\theta_{\mathbf{r}}(g) = g^{-1}$.

When V'' is sufficiently small, there exists $a \in \mathfrak{g}_{\mathbf{R}}$ near 0 such that

$$\mathrm{Int}(\tilde{\rho}(t))(g) = \exp(a), \quad \big(\mathrm{Ad}(\tilde{\rho}(t))\theta_{\mathbf{r}}\,\mathrm{Ad}(\tilde{\rho}(t))^{-1}\big)(a) = -a.$$

Decompose

$$a = \textstyle\sum_{l\in\mathbf{Z}^n} a_l \in \mathfrak{g}_{\mathbf{R}} = \bigoplus_{l\in\mathbf{Z}^n} \mathfrak{g}(l) \tag{4}$$

under the action of $\mathrm{Ad}(\tilde{\rho}(t))$ ($t \in (\mathbf{G}_{\mathbf{m},\mathbf{R}})^n$). Then we have

$$\mathrm{Int}(\tilde{\rho}(t))(g) = \exp(a), \tag{5}$$

$$g = \mathrm{Int}(\tilde{\rho}(t))^{-1}(\exp(a)) = \exp\big(\textstyle\sum_l t_1^{-l_1}\dots t_n^{-l_n} a_l\big), \tag{6}$$

$$\mathrm{Int}(\tilde{\rho}(t))^{-1}(g) = \exp\big(\textstyle\sum_l t_1^{-2l_1}\dots t_n^{-2l_n} a_l\big). \tag{7}$$

We want to see $\mathrm{Int}(\tilde{\rho}(t))^j(g) \in U''$ for $j = 0, \pm 1$. $\mathrm{Int}(\tilde{\rho}(t))(g) \in U''$ is obvious by definition. In order to see $\mathrm{Int}(\tilde{\rho}(t))^{-1}(g) \in U''$, we compute as follows.

$$\begin{aligned} a &= -\big(\mathrm{Ad}(\tilde{\rho}(t))\theta_{\mathbf{r}}\,\mathrm{Ad}(\tilde{\rho}(t))^{-1}\big)(a) = -\textstyle\sum_l \big(\mathrm{Ad}(\tilde{\rho}(t))\theta_{\mathbf{r}}\big)(t_1^{-l_1}\dots t_n^{-l_n} a_l) \\ &= -\textstyle\sum_l \big(\theta_{\mathbf{r}}\,\mathrm{Ad}(\tilde{\rho}(t))^{-1}\big)(t_1^{-l_1}\dots t_n^{-l_n} a_l) = -\textstyle\sum_l t_1^{-2l_1}\dots t_n^{-2l_n}\theta_{\mathbf{r}}(a_l). \end{aligned}$$

Since $\theta_{\mathbf{r}}$ transforms $\mathfrak{g}(l)$ to $\mathfrak{g}(-l)$, we have $a_{-l} = -t_1^{-2l_1}\dots t_n^{-2l_n}\theta_{\mathbf{r}}(a_l)$, that is,

$$t_1^{-2l_1}\dots t_n^{-2l_n} a_l = -\theta_{\mathbf{r}}(a_{-l}) \quad (\forall l \in \mathbf{Z}^n). \tag{8}$$

Since a is sufficiently near 0, so is each component a_{-l} and hence so is each $-\theta_{\mathbf{r}}(a_{-l})$. Therefore, by (8) and (7), we have $\mathrm{Int}(\tilde{\rho}(t))^{-1}(g) \in U''$. Finally, $g \in U''$ is proved as follows. Take a basis $\{e_{l,j}\}$ of $\mathfrak{g}_{\mathbf{R}}$ subordinate to the decomposition (4) and write $a_l = \sum_j a_{l,j}e_{l,j}$. Since

$$t_1^{-l_1}\dots t_n^{-l_n} a_{l,j} = \sqrt{a_{l,j}(t_1^{-2l_1}\dots t_n^{-2l_n} a_{l,j})} \tag{9}$$

and since $a_{l,j}$ and $t_1^{-2l_1} \dots t_n^{-2l_n} a_{l,j}$ are sufficiently near 0, so is the left-hand-side of (9). Thus we have $g \in U''$ by (6).

We prove (3). By definition, any element of $B(V, V', V'')$ can be written as

$$\tilde{\rho}(t)gk\mathbf{r} \text{ such that } k \in V,\ t \in \mathbf{R}^n_{>0} \cap V',\ \mathrm{Int}(\tilde{\rho}(t))^j(g) \in V'' \text{ for } j = 0, \pm 1.$$

Since V'' is sufficiently small, there exists $b \in \mathfrak{g}_{\mathbf{R}}$ with

(10) $\quad g = \exp(b)$

such that $\mathrm{Ad}(\tilde{\rho}(t))^j(b)$ are sufficiently near 0 for $j = 0, \pm 1$.

Let

$$b = b^- + b^+ \in \mathfrak{g}_{\mathbf{R}} = \mathfrak{g}_{\mathbf{R}}^- \oplus \mathfrak{g}_{\mathbf{R}}^+, \quad \mathfrak{g}_{\mathbf{R}}^{\pm} := \{x \in \mathfrak{g}_{\mathbf{R}} \mid \theta_{\mathbf{r}}(x) = \pm x\},$$

be the Cartan decomposition. Then, by 4.3, we have

$$g = \exp(b) = \exp(f_-(b^-, b^+)) \exp(f_+(b^-, b^+)), \quad f_{\pm}(b^-, b^+) \in \mathfrak{g}_{\mathbf{R}}^{\pm}.$$

Since

$$\tilde{\rho}(t)gk\mathbf{r} = \tilde{\rho}(t) \exp(f_-(b^-, b^+)) \exp(f_+(b^-, b^+))k\mathbf{r},$$

it is enough to show

$$\exp(f_+(b^-, b^+))k \in U \text{ and } \mathrm{Int}(\tilde{\rho}(t))(\exp(f_-(b^-, b^+))) \in U''.$$

Since b is sufficiently near 0, $\exp(f_+(b^-, b^+)) \in K_{\mathbf{r}}$ is sufficiently near 1. Hence

$$\exp(f_+(b^-, b^+))k \in U.$$

Since $b^{\pm} = (b \pm \theta_{\mathbf{r}}(b))/2$, we have

$$\mathrm{Ad}(\tilde{\rho}(t))(b^{\pm}) = \frac{\mathrm{Ad}(\tilde{\rho}(t))(b) \pm \theta_{\mathbf{r}}(\mathrm{Ad}(\tilde{\rho}(t))^{-1}(b))}{2}.$$

These are sufficiently near 0 by (10), and hence so is

$$f_-\big(\mathrm{Ad}(\tilde{\rho}(t))(b^-), \mathrm{Ad}(\tilde{\rho}(t))(b^+)\big).$$

Thus

$$\mathrm{Int}(\tilde{\rho}(t))\big(\exp(f_-(b^-, b^+))\big) \in U''. \quad \text{Q.E.D.}$$

4.8. *Topology $\mathcal{T}$ on $D_{\mathrm{SL}(2)}(W)$.* As in 3.12, let $W = (W^{(j)})_{1 \le j \le n}$ be a compatible family of $\mathbf{Q}$-rational increasing filtrations $W^{(j)}$ of $H_{0,\mathbf{R}}$.

For $x \in D_{\mathrm{SL}(2)}(W)$, let $\mathcal{F}_x$ be the filter on D associated to x in 4.6. For an open set U of D, denote

$$\tilde{U} := \{x \in D_{\mathrm{SL}(2)}(W) \mid U \in \mathcal{F}_x\}.$$

We define the *topology* $\mathcal{T}$ *on* $D_{\mathrm{SL}(2)}(W)$ so that its basis of open sets is given by

$$\{\tilde{U} \mid U \text{ is an open set on } D\}.$$

We denote by $D_{\mathrm{SL}(2)}(W)_{\mathcal{T}}$ the topological space whose underlying set coincides with the one of $D_{\mathrm{SL}(2)}(W)$ but whose topology is $\mathcal{T}$. By construction, for $x \in D_{\mathrm{SL}(2)}(W)$, the filter $\mathcal{F}_x$ on D associated to x coincides with the filter

$$\{V \cap D \mid V \text{ is a neighborhood of } x \text{ in } D_{\mathrm{SL}(2)}(W)_{\mathcal{T}}\}.$$

We recall here the definition of 'regular spaces' and a property of a map into a regular space, which will be used in the proofs of 4.12, 4.14 below.

Definition 4.9 ([NB, Ch. 1, §8, no. 4, Definition 2]). A topological space is called *regular* if it is Hausdorff and satisfies the following axiom: Given any closed subset F of X and any point $x \notin F$, there is a neighborhood of x and a neighborhood of F which are disjoint.

We will see that $D_{\mathrm{SL}(2)}(W)$ and $D_{\mathrm{SL}(2),\mathrm{val}}(W)$ are regular spaces. (On the other hand, $D_{\mathrm{SL}(2)}$ and $D_{\mathrm{SL}(2),\mathrm{val}}$ are not necessarily regular.)

Lemma 4.10 ([NB, Ch. 1, §8, no. 5, Theorem 1]). *Let X be a topological space, A a dense subset of X, $f : A \to Y$ a map from A into a regular space Y. A necessary and sufficient condition for f to extend to a continuous map $\overline{f} : X \to Y$ is that, for each $x \in X$, $f(y)$ tends to a limit in Y when y tends to x while remaining in A. The continuous extension $\overline{f}$ of f to X is then unique.*

4.11. Let $W = (W^{(j)})_{1 \le j \le n}$ be a compatible family of **Q**-rational increasing filtrations $W^{(j)}$ of $H_{0,\mathbf{R}}$ as in 3.12. For valuative submonoid V of $X((\mathbf{G}_\mathbf{m})^n) = \mathbf{Z}^n$ containing $X((\mathbf{G}_\mathbf{m})^n)_+ = \mathbf{N}^n$, we define a **Q**-parabolic subgroup P_V of $G_\mathbf{R}$ as follows.

Note that the quotient group $G_{W,\mathbf{R}}/G_{W,\mathbf{R},u}$ has the induced effective action on

$$(1) \qquad \bigoplus_{l \in \mathbf{Z}^n} \left(\bigcap_{1 \le j \le n} W^{(j)}_{l_j} / \left(\sum_{1 \le k \le n} \bigcap_{1 \le j \le n} W^{(j)}_{l_j - \delta_{jk}}\right)\right),$$

where δ_{jk} is the Kronecker's symbol. Define an action on (1) of $(t_1,\dots,t_n) \in (\mathbf{G_{m,R}})^n$ by $\bigoplus_{l\in\mathbf{Z}^n} t_1^{l_1}\dots t_n^{l_n}$. This induces a $\mathbf{Q}$-rational group homomorphism

$$\mu : (\mathbf{G_{m,R}})^n \to G_{W,\mathbf{R}}/G_{W,\mathbf{R},u}. \tag{2}$$

Let $\nu : (\mathbf{G_{m,R}})^n \to G_\mathbf{R}$ be a splitting of W, that is, ν is a homomorphism of algebraic groups over $\mathbf{R}$ such that $W_k^{(j)} = \bigoplus_{l\in\mathbf{Z}^n, l_j\le k} H(l)$ for all j and all k, where $H(l) := \{v \in H_{0,\mathbf{R}} \mid \nu(t)v = \prod_j t_j^{l_j} v \ (\forall t \in (\mathbf{G_{m,R}})^n)\}$. Then such ν corresponds to a lifting $(\mathbf{G_{m,R}})^n \to G_{W,\mathbf{R}}$ of μ in (2). Let $T \subset (\mathbf{G_{m,R}})^n$ be the annihilator of $V^\times$, and let V' be the image of V under $X((\mathbf{G_m})^n) \to X(T) \simeq X(\nu(T))$. (So $V' \simeq V/V^\times$.) Define the parabolic subgroup P_V of $G_\mathbf{R}$ by $P_V := P_{\nu(T),V'}$ (see 2.8). Then P_V is independent of the choice of ν as is easily seen. By taking ν defined over $\mathbf{Q}$, we see that P_V is $\mathbf{Q}$-rational.

Let $y = ([\rho,\varphi], Z, V) \in D_{\mathrm{SL}(2),\mathrm{val}}(W)$, let $W' = (W^{(s_j)})_{1\le j\le m}$ ($0 \le m \le n, 1 \le s_1 < \dots < s_m \le n$) be the families of weight filtrations associated to $[\rho,\varphi]$, and let $V' \subset X((\mathbf{G_m})^n)$ be the inverse image of V under $X((\mathbf{G_m})^n) \to X((\mathbf{G_m})^m)$, $(a_j)_{1\le j\le n} \mapsto (a_{s_j})_{1\le j\le n}$. Then, the $\mathbf{Q}$-parabolic subgroup of $G_\mathbf{R}$ associated to the image of y under $D_{\mathrm{SL}(2),\mathrm{val}} \to D_{\mathrm{BS},\mathrm{val}} \to D_{\mathrm{BS}}$ coincides with $P_{V'}$.

Proposition 4.12. *Let $W = (W^{(j)})_{1\le j\le n}$ be a compatible family of $\mathbf{Q}$-rational increasing filtrations $W^{(j)}$ of $H_{0,\mathbf{R}}$.*

(i) *There exists a continuous map $\beta : D \to \mathbf{R}^n_{>0}$ with the following property. For any splitting $\nu : (\mathbf{G_{m,R}})^n \to G_\mathbf{R}$ of W, we have*

$$\beta(\nu(t)x) = t\beta(x) \quad (\forall x \in D, \forall t \in \mathbf{R}^n_{>0}).$$

(ii) *Let β be as in* (i). *Then the map β extends uniquely to a continuous map $\overline{\beta} : D_{\mathrm{SL}(2)}(W)_\mathcal{T} \to \mathbf{R}^n_{\ge 0}$.*

Proof. We prove (i). Take a valuative submonoid V of $X((\mathbf{G_m})^n)$ such that $V \supset X((\mathbf{G_m})^n)_+$ and $V^\times = \{1\}$. (Such V exists. For example, identifying $X((\mathbf{G_m})^n) = \mathbf{Z}^n$ and $X((\mathbf{G_m})^n)_+ = \mathbf{N}^n$, let V be the subset of $X((\mathbf{G_m})^n)$ corresponding to the set of all elements of $\mathbf{Z}^n$ which are ≥ 0 in the lexicographical order of $\mathbf{Z}^n$.) Let $P := P_V$ (see 4.11), let $h : (\mathbf{G_{m,R}})^n \to S_P \subset P/P_u$ be the canonical homomorphism, and let $\theta : A_P \to \mathbf{R}^n_{>0}$ be a continuous homomorphism such that the composite map $\mathbf{R}^n_{>0} \xrightarrow{h} \to A_P \xrightarrow{\theta} \to \mathbf{R}^n_{>0}$ is the identity map. Fix a maximal compact subgroup K of $G_\mathbf{R}$. We define $\beta : D \to \mathbf{R}^n_{>0}$ as the composite map

$$D \to \mathcal{X} \simeq G_\mathbf{R}/K \xrightarrow{||} A_P \xrightarrow{\theta} \mathbf{R}^n_{>0},$$

where $|\ |$ is as in 2.16. We show that β has the property stated in (i). Let $x_1 \in D$ be a point lying over $K \in \mathcal{X}$. Let $x \in D$ and write $x = pkx_1$ with $p \in P$ and $k \in K$. Then, for $t \in \mathbf{R}^n_{>0}$, we have

$$\beta(\nu(t)x) = \beta(\nu(t)pkx_1) = \theta(|\nu(t)p|) = \theta(|\nu(t)|)\theta(|p|) = t\theta(|p|) = t\beta(x).$$

We prove (ii). Let $x \in D_{\mathrm{SL}(2)}(W)$, let (ρ, φ) be a representative of x, and let $\mathbf{r} := \varphi(\mathbf{i})$ ($\mathbf{i} = (i, \ldots, i) \in \mathfrak{h}^m$). Let $W' = (W^{(s_j)})_{1 \le j \le m}$ $(1 \le s_1 < \cdots < s_m \le n)$ be the family of weight filtrations associated to x, let

$$G_{W',\mathbf{R}} := \{g \in (G^\circ)_{\mathbf{R}} \mid gW^{(s_j)} = W^{(s_j)}\ (1 \le j \le m)\}$$

Put $s := (s_j)_{1 \le j \le m}$. Denote by $\iota_s : (\mathbf{G}_{\mathbf{m},\mathbf{R}})^m \to (\mathbf{G}_{\mathbf{m},\mathbf{R}})^n$ the injection to the s-components, that is, $\iota_s : (a_j)_{1 \le j \le m} \mapsto (b_l)_{1 \le l \le n}$; $b_l = a_j$ if $l = s_j$ $(1 \le j \le m)$ and $b_l = 1$ otherwise. Put $\nu_s := \nu \circ \iota_s$. Since both $\tilde{\rho}$ and ν_s split W', there exists a unique

$$(1) \qquad u \in G_{W',\mathbf{R},u} \quad \text{such that} \quad \tilde{\rho} = \mathrm{Int}(u)\nu_s.$$

Since the target $\mathbf{R}^n_{\ge 0}$ is a regular space, it is enough to prove, by 4.10, that, for x, (ρ, φ), $\mathbf{r}$ as above, and for directed families $(t_\lambda)_\lambda$, $(g_\lambda)_\lambda$, $(k_\lambda)_\lambda$ such that $t_\lambda \in \mathbf{R}^m_{>0}$, $g_\lambda \in G_{\mathbf{R}}$, $k_\lambda \in K_{\mathbf{r}}$, $\lim_\lambda t_\lambda = 0$, $\lim_\lambda g_\lambda = 1$, $\lim_\lambda k_\lambda = 1$, there exists a limit

$$(2) \qquad \lim_\lambda \beta(\tilde{\rho}(t_\lambda)g_\lambda k_\lambda \mathbf{r}) \in \mathbf{R}^n_{\ge 0}$$

Let W' and $u \in G_{W',\mathbf{R},u}$ be as above, and let $u_\lambda := \nu_s(t_\lambda)^{-1}u\nu_s(t_\lambda)$. Then

$$\begin{aligned}\beta(\tilde{\rho}(t_\lambda)g_\lambda k_\lambda \mathbf{r}) &= \beta(u\nu_s(t_\lambda)u^{-1}g_\lambda k_\lambda \mathbf{r}) \\ &= \beta(\nu_s(t_\lambda)u_\lambda u^{-1}g_\lambda k_\lambda \mathbf{r}) = \iota_s(t_\lambda)\beta(u_\lambda u^{-1}g_\lambda k_\lambda \mathbf{r}).\end{aligned}$$

Since ν_s splits W' and $u \in G_{W',\mathbf{R},u}$, we have $\lim_\lambda u_\lambda = 1$. Hence $\lim_\lambda \beta(u_\lambda u^{-1}g_\lambda k_\lambda \mathbf{r}) = \beta(u^{-1}\mathbf{r})$. This proves the existence of the limit (2). Q.E.D.

By the proof of Proposition 4.12 (ii), $\overline{\beta}(x) = \beta(u^{-1}\mathbf{r})$ for $x = [\rho, \varphi] \in D_{\mathrm{SL}(2)}(W)$, where $\mathbf{r} = \rho(\mathbf{i})$ and u is as in (1) above.

Lemma 4.13. *Let $W = (W^{(j)})_{1 \le j \le n}$ be a compatible family of $\mathbf{Q}$-rational increasing filtrations $W^{(j)}$ of $H_{0,\mathbf{R}}$. Then the topology of $D_{\mathrm{SL}(2),\mathrm{val}}(W)$ (as a subspace of $D_{\mathrm{SL}(2),\mathrm{val}}$) coincides with the topology as a subspace of $D_{\mathrm{BS},\mathrm{val}}$.*

Proof. It is sufficient to prove that, for any compatible family W' of $\mathbf{Q}$-rational increasing filtrations of $H_{0,\mathbf{R}}$, there exits an open set U of $D_{\mathrm{BS,val}}$ such that $D_{\mathrm{SL}(2),\mathrm{val}}(W) \cap D_{\mathrm{SL}(2),\mathrm{val}}(W') = D_{\mathrm{SL}(2),\mathrm{val}}(W) \cap U$. As is easily seen, $D_{\mathrm{SL}(2),\mathrm{val}}(W) \cap D_{\mathrm{SL}(2),\mathrm{val}}(W')$ is a finite union of sets of the form $D_{\mathrm{SL}(2),\mathrm{val}}(W'')$ where $W'' = (W^{(s_j)})_{1\le j\le m}$ for some m with $0 \le m \le n$ and for some $s_1, \ldots, s_m$ with $1 \le s_1 < \cdots < s_m \le n$. Hence we may assume that W' itself has the form $W' = (W^{(s_j)})_{1\le j\le m}$ $(0 \le m \le n,\ 1 \le s_1 < \cdots < s_m \le n)$. Assume this. Let $\mathcal{V}$ be the set of all valuative submonoids V of $X((\mathbf{G_m}))^n)$ such that $V \supset X((\mathbf{G_m})^n)_+$ and $V^\times = \{1\}$. Then

$$D_{\mathrm{SL}(2),\mathrm{val}}(W) \subset \textstyle\bigcup_{V\in\mathcal{V}} D_{\mathrm{BS,val}}(P_V),$$

where P_V is as in 4.11. In fact, for $([\rho,\varphi], Z, V) \in D_{\mathrm{SL}(2),\mathrm{val}}(W)$, let $W' = (W^{(s_j)})_{1\le j\le m}$ $(0 \le m \le n,\ 1 \le s_1 < \cdots < s_m \le n)$ be the family of weight filtrations associated to $[\rho,\varphi]$, and let $V' \in X((\mathbf{G_m})^n)$ be the inverse image of V under $X((\mathbf{G_m})^n) \to X((\mathbf{G_m})^m)$, $(a_j)_{1\le j\le n} \mapsto (a_{s_j})_{1\le j\le m}$. Take $V'' \in \mathcal{V}$ with $V'' \subset V'$. Then $P_{V'} \supset P_{V''}$ and hence $([\rho,\varphi], Z, V) \in D_{\mathrm{BS,val}}(P_{V'}) \subset D_{\mathrm{BS,val}}(P_{V''})$. We show the existence of V''. Let $l = \mathrm{rank}_{\mathbf{Z}}(V')^\times$ and fix any isomorphism $h : (V')^\times \xrightarrow{\sim} \mathbf{Z}^l$. Define

$$\begin{aligned} V'' := \{x \in V' \mid x \notin (V')^\times\} \\ \cup\{x \in (V')^\times \mid h(x) \ge 0 \text{ for lexicographical order of } \mathbf{Z}^l\}. \end{aligned}$$

Then, $V'' \in \mathcal{V}$ and $V'' \subset V'$.

For $V \in \mathcal{V}$, we define an open subset U_V of $D_{\mathrm{BS,val}}(P_V)$ such that

$$(1) \qquad D_{\mathrm{SL}(2),\mathrm{val}}(W') \cap D_{\mathrm{BS,val}}(P_V) = D_{\mathrm{SL}(2),\mathrm{val}}(W) \cap U_V.$$

This will show

$$D_{\mathrm{SL}(2),\mathrm{val}}(W') = D_{\mathrm{SL}(2),\mathrm{val}}(W) \cap \big(\textstyle\bigcup_{V\in\mathcal{V}} U_V\big).$$

The definition of U_V is as follows. Let $(\mathbf{G}_{\mathbf{m},\mathbf{R}})^n \to S_{P_V}$ be the canonical injective homomorphism. Let $(e_j)_{1\le j\le n}$ be the standard base of $X((\mathbf{G_m})^n)$. Take $a_j \ge 1$ such that $e_j^{a_j}$ is the restriction of an element $\psi_j \in X(S_{P_V})$ to $(\mathbf{G}_{\mathbf{m},\mathbf{R}})^n$ and fix such a_j and ψ_j $(1 \le j \le n)$. Let U'_V be the A_{P_V}-stable open set of $(\bar{A}_{P_V})_{\mathrm{val}}$ consisting of all elements (V', h) such that $\psi_{s_j} \in (V')^\times$ for $1 \le j \le m$, and let $U_V := D \times^{A_{P_V}} U'_V \subset D \times^{A_{P_V}} (\bar{A}_{P_V})_{\mathrm{val}} = D_{\mathrm{BS,val}}(P_V)$. Then we have (1), as is easily seen. Q.E.D.

Proposition 4.14. *Let $W = (W^{(j)})_{1\le j\le n}$ be a compatible family of $\mathbf{Q}$-rational increasing filtrations $W^{(j)}$ of $H_{0,\mathbf{R}}$. Let $\beta : D \to \mathbf{R}^n_{>0}$ and $\overline{\beta} : D_{\mathrm{SL}(2)}(W)_{\mathcal{T}} \to \mathbf{R}^n_{\ge 0}$ be as in 4.12. Then there exists a unique homeomorphism*

$$D_{\mathrm{SL}(2),\mathrm{val}}(W) \xrightarrow{\sim} D_{\mathrm{SL}(2)}(W)_{\mathcal{T}} \times_{\mathbf{R}^n_{\ge 0}} (\mathbf{R}^n_{\ge 0})_{\mathrm{val}}$$

which extends the identity map of D.

Proof. Take a splitting $\nu : (\mathbf{G}_{\mathbf{m},\mathbf{R}})^n \to G_{\mathbf{R}}$ of W. The homeomorphism in 4.14 is defined by

(1) $\quad \psi : (x, Z, \overline{V}) \mapsto (x, (V, h)) \quad (x \in D_{\mathrm{SL}(2)}(W), (V, h) \in (\mathbf{R}^n_{\ge 0})_{\mathrm{val}})$

as follows. Let $W_x = (W^{(s_j)})_{1\le j\le m}$ be the family of weight filtrations associated to x. Let (ρ, φ) be a representative of x, and let $\mathbf{r} = \varphi(\mathbf{i})$ $(\mathbf{i} = (i, \dots, i) \in \mathfrak{h}^m)$. Put $s := (s_j)_{1\le j\le m}$. Let

$$\iota_s : (\mathbf{G}_{\mathbf{m},\mathbf{R}})^m \hookrightarrow (\mathbf{G}_{\mathbf{m},\mathbf{R}})^n, \quad (a_1, \dots, a_m) \mapsto (b_1, \dots, b_n)$$

be the map defined by $b_l := a_{s_j}$ if $l = s_j$ $(1 \le j \le m)$ and $b_l := 1$ otherwise. Put $\nu_s := \nu \circ \iota_s$. Since both ν_s and $\tilde{\rho}$ split W_x, there exists a unique element u of $G_{W_x,\mathbf{R},u}$ such that $\tilde{\rho} = \mathrm{Int}(u)(\nu_s)$. For $(x, Z, \overline{V}) \in D_{\mathrm{SL}(2),\mathrm{val}}(W)$, the image $(x, (V, h))$ under ψ is defined as follows. First, V is the inverse image of $\overline{V}$ under

$$\mathbf{Z}^n = X((\mathbf{G}_{\mathbf{m}})^n) \to X((\mathbf{G}_{\mathbf{m}})^m), \quad \chi \mapsto \chi \circ \iota_s.$$

Take an element $t \in \mathbf{R}^m_{>0}$ such that $\tilde{\rho}(t\beta(u^{-1}\mathbf{r}))^{-1}\mathbf{r} \in Z$. Define $h : V^\times \to \mathbf{R}_{>0}$ by $h(\chi) := \chi(\iota_s(t))$.

The inverse map ψ^{-1} of ψ is given as follows. Let $(x, (V, h)) \in D_{\mathrm{SL}(2)}(W)_{\mathcal{T}} \times_{\mathbf{R}^n_{\ge 0}} (\mathbf{R}^n_{\ge 0})_{\mathrm{val}}$. We define Z and $\overline{V}$ as follows. Let $(e_j)_{1\le j\le n}$ be the standard basis of $X((\mathbf{G}_{\mathbf{m}})^n)$. The fact that the image of $(V, h) \in (\mathbf{R}^n_{\ge 0})_{\mathrm{val}}$ in $\mathbf{R}^n_{\ge 0}$ coincides with $\overline{\beta}(x)$ implies the following two assertions.

(2) $e_j \in V - V^\times$ if $j \in \{s_1, \dots, s_m\}$, and $e_j \in V^\times$ otherwise.

(3) $\overline{\beta}(x)_j = 0$ if $j \in \{s_1, \dots, s_m\}$, and
$\overline{\beta}(x)_j = \beta(u^{-1}\mathbf{r})_j = h(e_j) \ne 0$ otherwise.

Here $(\ )_j$ denotes the j-th component of an element of $\mathbf{R}^n_{\ge 0}$. By (2), we see that there exists a unique valuative submonoid $\overline{V}$ of $X((\mathbf{G}_{\mathbf{m}})^m)$ such that $\overline{V} \supset X((\mathbf{G}_{\mathbf{m}})^m)_+$, $\overline{V} \cap X((\mathbf{G}_{\mathbf{m}})^m)_+ = \{1\}$, and such that V is the inverse image of $\overline{V}$ under $X((\mathbf{G}_{\mathbf{m}})^n) \to X((\mathbf{G}_{\mathbf{m}})^m)$, $\chi \mapsto \chi \circ \iota_s$. By (3), we have

(4) $(\beta(u^{-1}\mathbf{r})^{-1}h)(e_j) = 1$ if $1 \leq j \leq n$ and $j \notin \{s_1, \dots, s_m\}$, and $\beta(u^{-1}\mathbf{r})^{-1}h : V^\times \to \mathbf{R}_{>0}$ factors through $V^\times \to \overline{V}^\times$.

Hence there exists $t \in \mathbf{R}^m_{>0}$ such that

$$\chi(\iota_s(t)) = (\beta(u^{-1}\mathbf{r})^{-1}h)(\chi) \quad (\forall \chi \in V^\times).$$

The Z is defined by

$$Z := \{\tilde{\rho}(t)\mathbf{r} \mid t \in \mathbf{R}^m_{>0}, \chi(\iota_s(t)) = (\beta(u^{-1}\mathbf{r})^{-1}h)(\chi)\ (\forall \chi \in V^\times)\}.$$

It is easy to see that the maps ψ and ψ^{-1} are the inverse to each other and hence ψ is bijective. For the proof of 4.14, it is enough to show that both ψ and ψ^{-1} are continuous.

Assume $\psi(x, Z, \overline{V}) = (x, (V, h))$, let V_0 be a valuative submonoid of $X((\mathbf{G_m})^n)$ such that $V \supset V_0 \supset X((\mathbf{G_m})^n)_+$ and $V_0^\times = \{1\}$, and let $P = P_{V_0}$. Then, since the $\mathbf{Q}$-parabolic subgroup of $G_\mathbf{R}$ associated to $(x, Z, \overline{V})$ is P_V (4.11) and P_V contains P, we have $(x, Z, \overline{V}) \in D_{\mathrm{BS,val}}(P)$. The following (5) is checked easily.

(5) The image of $(x, Z, \overline{V})$ under $D_{\mathrm{BS,val}}(P) \xrightarrow{\sim} D \times^{A_P} (\overline{A}_P)_{\mathrm{val}}$ is $(\mathbf{r}, (\hat{V}, \beta(u^{-1}\mathbf{r})^{-1}\hat{h}))$,

where $\hat{V}$ is the inverse image of V under the restriction map $X(S_P) \to X((\mathbf{G_m})^n)$ for the canonical embedding $(\mathbf{G_m})^n \hookrightarrow S_P$, and $\hat{h}$ is the composite $\hat{V}^\times \to V^\times \xrightarrow{h} \mathbf{R}_{>0}$.

Now fix $(x, Z, \overline{V}) \in D_{\mathrm{SL(2),val}}(W)$ and let

$$(x, (V, h)) \in D_{\mathrm{SL(2)}}(W) \times_{\mathbf{R}^n_{\geq 0}} (\mathbf{R}^n_{\geq 0})_{\mathrm{val}}$$

be its image under ψ. Fix V_0 as above, and let $P = P_{V_0}$. Fix a representative (ρ, φ) of x, and let $\mathbf{r} = \varphi(\mathbf{i})$.

We prove first that the map ψ^{-1} is continuous. By the fact that $D_{\mathrm{BS,val}}$ is a regular space, and by 4.10 and 4.13, it is enough to prove the following. Let $(y_\lambda)_\lambda$ be a directed family of elements of D which converges to $(x, (V, h))$ in $D_{\mathrm{SL(2)}}(W)_\mathcal{T} \times_{\mathbf{R}^n_{\geq 0}} (\mathbf{R}^n_{\geq 0})_{\mathrm{val}}$. Then $(y_\lambda)_\lambda$ converges to $(\mathbf{r}, (\hat{V}, \beta(u^{-1}\mathbf{r})^{-1}\hat{h}))$ in $D \times^{A_P} (\overline{A}_P)_{\mathrm{val}}$. Since $(y_\lambda)_\lambda$ converges to x in $D_{\mathrm{SL(2)}}(W)_\mathcal{T}$,

$$y_\lambda = \tilde{\rho}(t_\lambda)g_\lambda k_\lambda \mathbf{r}, \quad t_\lambda \in \mathbf{R}^m_{>0},\ g_\lambda \in G_\mathbf{R},\ k_\lambda \in K_\mathbf{r},$$
$$t_\lambda \to 0,\ \mathrm{Int}(\tilde{\rho}(t_\lambda))^j(g_\lambda) \to 1\ (j = 0, \pm 1),\ k_\lambda \to 1.$$

By 4.4, $g_\lambda = p_\lambda k'_\lambda$, $p_\lambda \in P$, $k'_\lambda \in K_{\mathbf{r}}$, $\mathrm{Int}(\tilde\rho(t_\lambda))^j(g_\lambda) \to 1$ $(j = 0, \pm 1)$, $k'_\lambda \to 1$. Put $k''_\lambda := k'_\lambda k_\lambda \in K_{\mathbf{r}}$. We have

$$\begin{aligned} y_\lambda &= \tilde\rho(t_\lambda)p_\lambda k''_\lambda \mathbf{r} = \tilde\rho(t_\lambda)p_\lambda\tilde\rho(t_\lambda)^{-1}\tilde\rho(t_\lambda)k''_\lambda\mathbf{r} \\ &= (\tilde\rho(t_\lambda) \bmod P_u) \circ \tilde\rho(t_\lambda)p_\lambda\tilde\rho(t_\lambda)^{-1}k''_\lambda\mathbf{r}, \end{aligned}$$

where $\circ$ is the Borel-Serre action for the $\mathbf{Q}$-parabolic subgroup P. Note that

$$\tilde\rho(t_\lambda)p_\lambda\tilde\rho(t_\lambda)^{-1} \to 1, \quad k''_\lambda \to 1.$$

It suffices to prove

$$(\tilde\rho(t_\lambda) \bmod P_u) \to (\hat V, \beta(u^{-1}\mathbf{r})^{-1}\hat h) \quad \text{in } (\overline{A}_P)_{\mathrm{val}}.$$

Since $\beta(y_\lambda) \to (V, h)$ in $(\mathbf{R}^n_{\geq 0})_{\mathrm{val}}$ and

$$\begin{aligned} \beta(y_\lambda) &= \beta(\tilde\rho(t_\lambda)p_\lambda k''_\lambda\mathbf{r}) = \beta(\nu_s(t_\lambda)\nu_s(t_\lambda)^{-1}u\nu_s(t_\lambda)u^{-1}p_\lambda k''_\lambda\mathbf{r}) \\ &= \iota_s(t_\lambda)\beta(\nu_s(t_\lambda)^{-1}u\nu_s(t_\lambda)) = \iota_s(t_\lambda)\beta(\nu_s(t_\lambda)^{-1}u\nu_s(t_\lambda)u^{-1}p_\lambda k''_\lambda\mathbf{r}), \\ &\nu_s(t_\lambda)^{-1}u\nu_s(t_\lambda) \to 1 \text{ (by } u \in G_{W_x,\mathbf{R},u}), \ \ p_\lambda \to 1, \ \ k''_\lambda \to 1, \end{aligned}$$

we have

$$\iota_s(t_\lambda) \to (V, \beta(u^{-1}\mathbf{r})^{-1}h) \quad \text{in } (\mathbf{R}^n_{\geq 0})_{\mathrm{val}}.$$

This implies

$$\begin{cases} \chi(\iota_s(t_\lambda)) \to 0 & \text{if } \chi \in V - V^\times, \\ \chi(\iota_s(t_\lambda)) \to (\beta(u^{-1}\mathbf{r})^{-1}h)(\chi) & \text{if } \chi \in V^\times. \end{cases}$$

This shows

$$\begin{cases} \chi(\tilde\rho(t_\lambda) \bmod P_u) \to 0 & \text{if } \chi \in \hat V - \hat V^\times, \\ \chi(\tilde\rho(t_\lambda) \bmod P_u) \to (\beta(u^{-1}\mathbf{r})^{-1}\hat h)(x) & \text{if } \chi \in \hat V^\times. \end{cases}$$

Hence $(\tilde\rho(t_\lambda) \bmod P_u) \to (\hat V, \beta(u^{-1}\mathbf{r})^{-1}\hat h)$ in $(\overline{A}_P)_{\mathrm{val}}$.

Next we prove that ψ is continuous. Let $(y_\lambda)_\lambda$ be a directed family of elements of $D_{\mathrm{SL}(2),\mathrm{val}}(W)$ converging to $(x, Z, \overline{V})$. Write $y_\lambda = (x_\lambda, Z_\lambda, \overline{V}_\lambda)$, and let $(x_\lambda, (V_\lambda, h_\lambda))$ be the image of y_λ under ψ. We show that x_λ converges to x and (V_λ, h_λ) converges to (V, h). We assume $y_\lambda \in D_{\mathrm{SL}(2),\mathrm{val}}(W_x) \cap D_{\mathrm{BS},\mathrm{val}}(P)$ without loss of generality. Define $(\hat V_\lambda, \hat h_\lambda) \in (\overline{A}_P)_{\mathrm{val}}$ just as in the definition of $(\hat V, \hat h)$. Since there exist only finitely many possible families of weight filtrations for points in $D_{\mathrm{SL}(2),\mathrm{val}}(W_x)$, we may assume that all x_λ have a common family of weight filtrations $(W^{(s'_j)})_{1\le j\le m'}$. Put $s' := (s'_j)_{1\le j\le m'}$. Let $\nu_{s'} := \nu\circ\iota_{s'}$ and let $\iota' : (\mathbf{G_m})^{m'} \to (\mathbf{G_m})^m$ be the unique homomorphism with $\iota_{s'} = \iota_s\circ\iota'$. Let $(\rho_\lambda,\varphi_\lambda)$ be a representative of x_λ, and let $\mathbf{r}'_\lambda = \varphi_\lambda(\mathbf{i}')$ ($\mathbf{i}' = (i,\dots,i)\in\mathfrak{h}^{m'}$). Let u_λ be the unique element of $G_{W',\mathbf{R},u}$ with $\tilde\rho_\lambda = \mathrm{Int}(u_\lambda)(\nu_{s'})$. The image of y_λ in $D\times^{A_P}(\overline{A}_P)_{\mathrm{val}}$ coincides with $(\mathbf{r}'_\lambda,(\hat V_\lambda,\beta(u_\lambda^{-1}\mathbf{r}_\lambda)^{-1}\hat h_\lambda))$.

Since $(\mathbf{r}'_\lambda,(\hat V_\lambda,\beta(u_\lambda^{-1}\mathbf{r}_\lambda)^{-1}\hat h_\lambda))$ converges to $(\mathbf{r},(\hat V,\beta(u^{-1}\mathbf{r})\hat h))$ in $D\times^{A_P}(\overline{A}_P)_{\mathrm{val}}$, there exist $a_\lambda\in A_P$ satisfying the following two conditions.

(6) $\quad (\hat V_\lambda, a_\lambda\beta(u_\lambda^{-1}\mathbf{r}_\lambda)^{-1}\hat h_\lambda)\to(\hat V,\beta(u^{-1}\mathbf{r})^{-1}\hat h)\quad$ in $(\overline{A}_P)_{\mathrm{val}}$.

(7) $\quad a_\lambda^{-1}\circ\mathbf{r}_\lambda\to\mathbf{r}\quad$ in D.

By (6) and (4) (applied also by replacing u, $\mathbf{r}$, h, s by u_λ, $\mathbf{r}_\lambda$, h_λ, s', respectively), the j-th component of the image of a_λ in $\mathbf{R}^n_{>0}$ converges to 1 if $j\notin\{s_1,\dots,s_m\}$. Hence

$$a_\lambda = \iota_s(t_\lambda)b_\lambda,\ t_\lambda\in\mathbf{R}^m_{>0},\ b_\lambda\in A_P,\ b_\lambda\to 1.$$

(We identify here an element of $\mathbf{R}^n_{>0}$ with its canonical image in A_P.) Hence (6) and (7) are rewritten as

(8) $\quad (\hat V_\lambda,\iota_s(t_\lambda)\beta(u_\lambda^{-1}\mathbf{r}_\lambda)^{-1}\hat h_\lambda)\to(\hat V,\beta(u^{-1}\mathbf{r})^{-1}\hat h)\quad$ in $(\overline{A}_P)_{\mathrm{val}}$.

(9) $\quad \iota_s(t_\lambda)^{-1}\circ\mathbf{r}_\lambda\to\mathbf{r}\quad$ in D.

By (9), we can write

$$\iota_s(t_\lambda)^{-1}\circ\mathbf{r}_\lambda = p_\lambda k_\lambda\mathbf{r},\ \ p_\lambda\in P,\ \ p_\lambda\to 1,\ \ k_\lambda\in K_{\mathbf{r}},\ \ k_\lambda\to 1.$$

Hence

(10) $\quad \mathbf{r}_\lambda = p_\lambda\tilde\rho(t_\lambda)k_\lambda\mathbf{r},\ \ p_\lambda\in P,\ \ p_\lambda\to 1,\ \ k_\lambda\in K_{\mathbf{r}},\ \ k_\lambda\to 1.$

In (8), we have

$$\begin{aligned}\iota_s(t_\lambda)\beta(u_\lambda^{-1}\mathbf{r}_\lambda)^{-1} &= \beta(\nu_s(t_\lambda)^{-1}u_\lambda^{-1}\mathbf{r}_\lambda)^{-1}\\ &= \beta(\nu_s(t_\lambda)^{-1}u_\lambda^{-1}p_\lambda\tilde\rho(t_\lambda)k_\lambda\mathbf{r})^{-1}\\ &= \beta(\nu_s(t_\lambda)^{-1}u_\lambda^{-1}p_\lambda u\nu_s(t_\lambda)u^{-1}k_\lambda\mathbf{r})^{-1}.\end{aligned}$$

We will show

(11) $$\nu_s(t_\lambda)^{-1}u_\lambda^{-1}p_\lambda u\nu_s(t_\lambda)\to 1.$$

This will show

$$\iota_s(t_\lambda)\beta(u_\lambda^{-1}\mathbf{r}_\lambda)^{-1} \to \beta(u^{-1}\mathbf{r})^{-1},$$

and hence, by (8),

$$(\hat{V}_\lambda, \hat{h}_\lambda) \to (\hat{V}, \hat{h}).$$

It follows

$$(V_\lambda, h_\lambda) \to (V, h).$$

We prove (11). Let

$$G_{W',\mathbf{R}} = G_{W',\mathbf{R},u} \rtimes L \tag{12}$$

be the semi-direct decomposition, where

$$L := \{g \in (G^\circ)_\mathbf{R} \mid \nu_{s'}(t)g = g\nu_{s'}(t) \ \ (\forall t \in (\mathbf{G}_{\mathbf{m},\mathbf{R}})^{m'})\}.$$

We claim

$$\begin{aligned}(13) \qquad & p_\lambda, p_\lambda u \in G_{W',\mathbf{R}}, \quad u_\lambda^{-1} p_\lambda u \in L, \\ & p_\lambda u = u_\lambda \cdot (u_\lambda^{-1} p_\lambda u) \ \text{ under the decomposition (12).}\end{aligned}$$

In fact, since the induced homomorphisms $(\mathbf{G}_{\mathbf{m},\mathbf{R}})^{m'} \to P/P_u$ from $\tilde{\rho}_\lambda$ and $\tilde{\rho} \circ \iota'$ coincide and since $\tilde{\rho}_\lambda$ (resp. $\tilde{\rho} \circ \iota'$) is the Borel-Serre lifting at $K_{\mathbf{r}_\lambda}$ (resp. $K_\mathbf{r}$) of this induced homomorphism, we have

$$\tilde{\rho}_\lambda = \mathrm{Int}(p_\lambda)(\tilde{\rho} \circ \iota').$$

Since $\tilde{\rho}_\lambda = \mathrm{Int}(u_\lambda)\nu_{s'}$ and $\tilde{\rho} \circ \iota' = \mathrm{Int}(u)\nu_{s'}$, this proves $u_\lambda^{-1} p_\lambda u \in L$. Since $u_\lambda, u \in G_{W',\mathbf{R}}$, it follows $p_\lambda \in G_{W',\mathbf{R}}$. Hence we have (13).

We prove (11) by using (13). For $\chi \in X((\mathbf{G_m})^n)$, let

$$H(\chi) := \{v \in H_{0,\mathbf{R}} \mid \nu(t)v = \chi(t)v \ \ (\forall t \in (\mathbf{G}_{\mathbf{m},\mathbf{R}})^n)\}.$$

Let $\chi_1, \chi_2 \in X((\mathbf{G_m})^n)$, $v \in H_{0,\mathbf{R}}(\chi_1)$. For the proof of (11), it is enough to show

$$(\nu_s(t_\lambda)^{-1} u_\lambda^{-1} p_\lambda u \nu_s(t_\lambda) v)(\chi_2) \to \begin{cases} v & \text{if } \chi_2 = \chi_1, \\ 0 & \text{otherwise,} \end{cases}$$

where (χ_2) means the χ_2-component in the eigenspace decomposition $H_{0,\mathbf{R}} = \bigoplus_{\chi \in X((\mathbf{G_m})^n)} H(\chi)$. We have

$$(\nu_s(t_\lambda)^{-1} u_\lambda^{-1} p_\lambda u \nu_s(t_\lambda) v)(\chi_2) = \chi_1(\iota_s(t_\lambda))\chi_2(\iota_s(t_\lambda))^{-1}(u_\lambda^{-1} p_\lambda u v)(\chi_2),$$

and the decomposition of $p_\lambda u$ in (13) implies

$$(u_\lambda^{-1}p_\lambda uv)(\chi_2) = \begin{cases} (p_\lambda uv)(\chi_2) & \text{if } (\chi_1\chi_2^{-1})\circ\iota_{s'} = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Since $p_\lambda u \in P$, we have

$$(p_\lambda uv)(\chi_2) = 0 \quad \text{unless} \quad \chi_1\chi_2^{-1} \in V.$$

Furthermore, since $p_\lambda \to 1$, we have

$$(p_\lambda uv)(\chi_2) \to (uv)(\chi_2).$$

Since $u \in G_{W_x,\mathbf{R},u} \subset P_{V,u}$, we have

$$(uv)(\chi_2) = \begin{cases} v & \text{if } \chi_2 = \chi_1, \\ 0 & \text{if } \chi_2 \neq \chi_1 \text{ and } \chi_1\chi_2^{-1} \in V - V^\times. \end{cases}$$

By these facts, (11) is reduced to the following (14) which we apply to $\chi = \chi_1\chi_2^{-1}$.

(14) Let $\chi \in V$ and assume $\chi \circ \iota_{s'} = 1$. Then $\chi(\iota_s(t_\lambda))$ is bounded. If furthermore $\chi \in V - V^\times$, then $\chi(\iota_s(t_\lambda)) \to 0$.

This (14) follows from (8), and from $(\beta(u_\lambda^{-1}\mathbf{r}_\lambda)^{-1}\hat{h}_\lambda)(\chi) = 1$ by (4) (applied replacing u, $\mathbf{r}$, h, s by u_λ, $\mathbf{r}_\lambda$, h_λ, s', respectively).

It remains to prove that $x_\lambda \to x$ for the $\mathcal{T}$-toplogy. Take $U \in \mathcal{F}_x$. It is enough to show that if λ is sufficiently large then there exist a neighborhood $U_{\lambda,1}$ of 0 in $\mathbf{R}^{m'}_{\geq 0}$, a neighborhood $U_{\lambda,2}$ of 1 in $G_\mathbf{R}$ and a neighborhood $U_{\lambda,3}$ of 1 in $K_{\mathbf{r}_\lambda}$ such that

$$\tilde{\rho}_\lambda(t)gk\mathbf{r}_\lambda \in U$$

for any $t \in \mathbf{R}^{m'}_{>0} \cap U_{\lambda,1}$, any $g \in G_\mathbf{R}$ and any $k \in U_{\lambda,3}$ with $\operatorname{Int}(\rho_\lambda(t))^j(g) \in U_{\lambda,2}$ for $j = 0, \pm 1$. For $t \in \mathbf{R}^{m'}_{>0}$, $g \in G_\mathbf{R}$ and $k \in K_{\mathbf{r}_\lambda}$, we have, by (10) and $\tilde{\rho}_\lambda = \operatorname{Int}(p_\lambda)(\tilde{\rho} \circ \iota')$,

$$\begin{aligned} \tilde{\rho}_\lambda(t)gk\mathbf{r}_\lambda = \tilde{\rho}(t_\lambda\iota'(t)) \operatorname{Int}(\tilde{\rho}(t_\lambda\iota'(t)))^{-1}(p_\lambda) \\ \cdot \operatorname{Int}(p_\lambda\tilde{\rho}(t_\lambda))^{-1}(g) \operatorname{Int}(p_\lambda\tilde{\rho}(t_\lambda))^{-1}(k')k_\lambda\mathbf{r}. \end{aligned}$$

Note that $\operatorname{Int}(p_\lambda\tilde{\rho}(t_\lambda))^{-1}(k) \in K_\mathbf{r}$. It is enough to prove that if λ is sufficiently large and if $t \in \mathbf{R}^{m'}_{>0}$ converges to 0 then we have the following two assertions.

(15) $\quad t_\lambda\iota'(t) \to 0,$

(16) $\quad \operatorname{Int}\left(\tilde{\rho}(t_\lambda\iota'(t))\right)^j(p_\lambda) \to 1 \quad \text{for } j = 0, -1, -2.$

(15) follows from (14), and (16) follows from $p_\lambda \in P \cap G_{W',\mathbf{R}}$ (13). Q.E.D.

Corollary 4.15. *The topology $\mathcal{T}$ on $D_{\mathrm{SL}(2)}(W)$ coincides with the one on $D_{\mathrm{SL}(2)}(W)$ as a quotient space of $D_{\mathrm{SL}(2),\mathrm{val}}(W)$.*

Proof. By 4.14, $D_{\mathrm{SL}(2),\mathrm{val}}(W) \to D_{\mathrm{SL}(2)}(W)_{\mathcal{T}}$ is proper surjective, because so is $(\mathbf{R}^n_{\geq 0})_{\mathrm{val}} \to \mathbf{R}^n_{\geq 0}$. Hence $D_{\mathrm{SL}(2),\mathrm{val}}(W) \to D_{\mathrm{SL}(2)}(W)_{\mathcal{T}}$ is a closed surjective continuous map. This proves 4.15. Q.E.D.

Lemma 4.16. *Let $f : X \to Y$ be a continuous proper surjective map of topological spaces.*

(i) ([NB, Ch. 1, §10, no. 1, Corollaire 2]). *If X is Hausdorff, then so is Y.*

(ii) *If X is regular, then so is Y.*

Proof. We prove (ii). Let F be a closed set of Y and y be a point of Y with $y \notin F$. Then $f^{-1}(y) \cap f^{-1}(F) = \emptyset$. Since X is regular, for each point $x \in f^{-1}(y)$, there exist disjoint open neighborhoods U_x and V_x of x and of the closed set $f^{-1}(F)$, respectively. Since f is proper, $f^{-1}(y)$ is quasi-compact. Hence, there exist finite subsets $\{U_j\}_{1\leq j\leq n}$ of $\{U_x\}_{x\in f^{-1}(y)}$, which cover $f^{-1}(y)$. Let $\{V_j\}_{1\leq j\leq n}$ be the corresponding finite subsets of $\{V_x\}_{x\in f^{-1}(y)}$. Put $U := \bigcup_{1\leq j\leq n} U_j$ and $V := \bigcap_{1\leq j\leq n} V_j$. Then, it is easy to see that $Y - f(X - U)$ and $Y - f(X - V)$ are disjoint open neighborhoods of y and of F, respectively. Hence Y is regular. Q.E.D.

4.17. *Proof of Theorem* 3.14. Since, for each W, $D_{\mathrm{SL}(2),\mathrm{val}}(W) \to D_{\mathrm{SL}(2)}(W)$ is continuous proper surjective, so is $D_{\mathrm{SL}(2),\mathrm{val}} \to D_{\mathrm{SL}(2)}$. It follows that $D_{\mathrm{SL}(2)}$ is Hausdorff by 4.16 (i). Q.E.D.

Proposition 4.18. *$D_{\mathrm{SL}(2),\mathrm{val}}(W)$ and $D_{\mathrm{SL}(2)}(W)$ are regular spaces.*

Proof. Since $D_{\mathrm{BS},\mathrm{val}}$ is regular, so is $D_{\mathrm{SL}(2),\mathrm{val}}(W)$ by 4.13 and hence so is $D_{\mathrm{SL}(2)}(W)$ by 4.16 (ii). Q.E.D.

4.19 *Remark.* We prove that, for an SL(2)-orbit (ρ, φ) of rank n,

$$[\rho, \varphi] = \lim \varphi(iy_1, \ldots, iy_n)$$

as $\frac{y_j}{y_{j+1}} \to \infty$ ($\forall j$, $y_{n+1} = 1$) in $D_{\mathrm{SL}(2)}$, which is stated after 3.13.

Let W be the family of weight filtration associated to (ρ, φ). Since

$$\varphi(iy_1, \ldots, iy_n) = \tilde{\rho}\Big(\sqrt{\tfrac{y_2}{y_1}}, \ldots, \sqrt{\tfrac{y_{n+1}}{y_n}}\Big)\varphi(\mathbf{i})$$

and $\sqrt{\frac{y_{j+1}}{y_j}} \to 0$, the right-hand-side converges to $[\rho, \varphi]$ in $D_{\mathrm{SL}(2)}(W)_{\mathcal{T}}$ and hence in the topology of $D_{\mathrm{SL}(2)}(W)$ by 4.15.

§5. Actions of $G_{\mathbf{Z}}$

5.1. *Summary.* In this section, we will transport the good properties (i), (ii) in 2.1 of the quotient space $\Gamma\backslash\mathcal{X}_{\mathrm{BS}}$ to other spaces along the diagram 3.1 (1). The main result of this section is the following Theorem 5.2.

Theorem 5.2. (i) *For any subgroup* Γ *of* $G_{\mathbf{Z}}$, *all the quotient spaces* $\Gamma\backslash D_{\mathrm{BS}}$, $\Gamma\backslash D_{\mathrm{BS,val}}$, $\Gamma\backslash D_{\mathrm{SL}(2),\mathrm{val}}$, $\Gamma\backslash D_{\mathrm{SL}(2)}$ *are Hausdorff.*

(ii) *If* Γ *is a subgroup of* $G_{\mathbf{Z}}$ *of finite index, then* $\Gamma\backslash D_{\mathrm{BS}}$, $\Gamma\backslash D_{\mathrm{BS,val}}$ *are compact.*

(iii) *If* Γ *is a neat subgroup of* $G_{\mathbf{Z}}$, *then all the projections* $D_{\mathrm{BS}} \to \Gamma\backslash D_{\mathrm{BS}}$, $D_{\mathrm{BS,val}} \to \Gamma\backslash D_{\mathrm{BS,val}}$, $D_{\mathrm{SL}(2),\mathrm{val}} \to \Gamma\backslash D_{\mathrm{SL}(2),\mathrm{val}}$, $D_{\mathrm{SL}(2)} \to \Gamma\backslash D_{\mathrm{SL}(2)}$ *are local homeomorphisms.*

Before proving this theorem, we recall the notion of 'proper action' and some related results in [NB] which are needed for our present purpose.

Definition 5.3 ([NB, Ch. 3, §4, no. 1, Definition 1]). Let G be a topological group acting continuously on a topological space X. G is said to act properly on X if the map

$$G \times X \to X \times X, \quad (g, x) \mapsto (x, gx),$$

is proper.

Lemma 5.4 (cf. [NB, Ch. 3, §4, no. 2, Proposition 3]). *If a topological group* G *acts properly on a topological space* X, *then the quotient space* $G\backslash X$ *is Hausdorff.*

Lemma 5.5 (cf. [NB, Ch. 3, §4, no. 4, Corollary]). *If a discrete group* G *acts properly and freely on a Hausdorff space* X, *then the projection* $X \to G\backslash X$ *is a local homeomorphism.*

Lemma 5.6 (cf. [NB, Ch. 3, §2, no. 2, Proposition 5]). *Let* G *be a topological group acting continuously on topological spaces* X *and* X'. *Let* $\psi : X \to X'$ *be an equivariant continuous map.*

(i) *If* ψ *is surjective and proper, and if* G *acts properly on* X, *then* G *acts properly on* X'.

(ii) *If G acts properly on X' and if X is Hausdorff, then G acts properly on X.*

Now we come back to our situation.

Lemma 5.7. *If Γ is a neat subgroup of $G_{\mathbf{Z}}$, then Γ acts on $D_{\mathrm{SL}(2)}$ freely.*

Proof. Let $x \in D_{\mathrm{SL}(2),n}$, $\gamma \in \Gamma$, and assume $\gamma x = x$. We prove $\gamma = 1$. Let (ρ, φ) be a representative of x. Then $\mathrm{Ad}(\gamma)Y_j = Y_j$ $(1 \le j \le n)$. Here the Y_j are the semi-simple elements of $\mathfrak{g}_{\mathbf{R}}$ associated to ρ in 3.1. Put $Y := \sum_{1 \le j \le n} Y_j$. Then γ preserves the l-eigenspace $H(l) \subset H_{0,\mathbf{R}}$ of Y for all l. Put $\mathrm{gr}_k := \mathrm{gr}_k^{W^{(n)}}(H_{0,\mathbf{C}})$ and $\mathrm{gr} := \bigoplus_k \mathrm{gr}_k$. Let $F := \varphi(\mathbf{i}) \in D$ and $F(\mathrm{gr})$ be the filtration of gr induced by F. Then, by the assumption, the automorphism $\mathrm{gr}(\gamma)$ of gr induced by γ satisfies $\mathrm{gr}(\gamma)F(\mathrm{gr}) = F(\mathrm{gr})$. Thus we have the following four statements.

(i) $(W^{(n)}[-w], F)$ is an $(N_1 + \cdots + N_n)$-polarized mixed Hodge structure ([Sc]).

(ii) $\gamma W^{(n)} = W^{(n)}$.

(iii) $\mathrm{gr}(\gamma)F(\mathrm{gr}) = F(\mathrm{gr})$.

(iv) If a is an eigenvalue of $\mathrm{gr}(\gamma)$ and if a is a root of 1, then $a = 1$.

We prove $\mathrm{gr}(\gamma) = 1$. Since $F(\mathrm{gr})$ is polarized, the isotropy group of $F(\mathrm{gr})$ is compact, and so $\mathrm{gr}(\gamma)$ is contained in the intersection of a discrete subgroup and a compact subgroup and hence is of finite order. Therefore $\mathrm{gr}(\gamma) = 1$ by (iv).

Now $\gamma = 1$ follows from $\mathrm{gr}(\gamma) = 1$ and the commutativity of γ and Y. Q.E.D.

5.8. *Proof of Theorem* 5.2. We prove (i). $G_{\mathbf{Z}}$ acts on $\mathcal{X}_{\mathrm{BS}}$ properly by [BS]. Since D_{BS}, $D_{\mathrm{BS,val}}$, $D_{\mathrm{SL}(2),\mathrm{val}}$ are Hausdorff by 2.17 (ii), it follows that $G_{\mathbf{Z}}$ acts on these spaces properly by 5.6 (ii). Since $D_{\mathrm{SL}(2),\mathrm{val}} \to D_{\mathrm{SL}(2)}$ is proper and surjective by 3.14 (i), it follows that $G_{\mathbf{Z}}$ acts on $D_{\mathrm{SL}(2)}$ properly by 5.6 (i). Hence, for any subgroup Γ of $G_{\mathbf{Z}}$, all the quotient spaces $\Gamma\backslash D_{\mathrm{BS}}$, $\Gamma\backslash D_{\mathrm{BS,val}}$, $\Gamma\backslash D_{\mathrm{SL}(2),\mathrm{val}}$, $\Gamma\backslash D_{\mathrm{SL}(2)}$ are Hausdorff by 5.4.

We prove (ii). Let Γ be a subgroup of $G_{\mathbf{Z}}$ of finite index. Then $\Gamma\backslash\mathcal{X}_{\mathrm{BS}}$ is compact by [BS]. Since $D_{\mathrm{BS}} \to \mathcal{X}_{\mathrm{BS}}$ and $D_{\mathrm{BS,val}} \to D_{\mathrm{BS}}$ are proper by 2.17 (i), $\Gamma\backslash D_{\mathrm{BS}} \to \Gamma\backslash\mathcal{X}_{\mathrm{BS}}$ and $\Gamma\backslash D_{\mathrm{BS,val}} \to \Gamma\backslash D_{\mathrm{BS}}$ are proper. Hence $\Gamma\backslash D_{\mathrm{BS}}$ and $\Gamma\backslash D_{\mathrm{BS,val}}$ are compact.

We prove (iii). Let Γ be a neat subgroup of $G_{\mathbf{Z}}$. Since Γ acts on $\mathcal{X}_{\mathrm{BS}}$ freely by [BS], so does Γ on D_{BS}, on $D_{\mathrm{BS,val}}$, and on $D_{\mathrm{SL}(2),\mathrm{val}}$

by 3.11. Γ acts on $D_{\mathrm{SL}(2)}$ freely by 5.7. Moreover, all the spaces D_{BS}, $D_{\mathrm{BS,val}}$, $D_{\mathrm{SL}(2),\mathrm{val}}$, $D_{\mathrm{SL}(2)}$ are Hausdorff and acted by Γ properly. Hence all the projections $D_{\mathrm{BS}} \to \Gamma\backslash D_{\mathrm{BS}}$, $D_{\mathrm{BS,val}} \to \Gamma\backslash D_{\mathrm{BS,val}}$, $D_{\mathrm{SL}(2),\mathrm{val}} \to \Gamma\backslash D_{\mathrm{SL}(2),\mathrm{val}}$, $D_{\mathrm{SL}(2)} \to \Gamma\backslash D_{\mathrm{SL}(2)}$ are local homeomorphisms by 5.5. Q.E.D.

§6. Examples and comments

6.1. *Summary.* In this section, we will first give a criterion in Proposition 6.3 for the existence of the canonical map $D_{\mathrm{SL}(2)} \to D_{\mathrm{BS}}$ by using the family of weight filtrations associated to a point of $D_{\mathrm{SL}(2)}$. This criterion explains the reason why we need to introduce the projective limits of blowing-ups $D_{\mathrm{BS,val}}$, $D_{\mathrm{SL}(2),\mathrm{val}}$ of D_{BS}, $D_{\mathrm{SL}(2)}$, respectively, to relate D_{BS} and $D_{\mathrm{SL}(2)}$. Then we will give the list of 'classical situation' in 6.6, and in this situation we will show that $D_{\mathrm{SL}(2)} = D_{\mathrm{BS}}$ and $D_{\mathrm{SL}(2),\mathrm{val}} = D_{\mathrm{BS,val}}$ except one case in Theorem 6.7. As a corollary, we have in 6.9 the canonical surjection from the Borel-Serre space D_{BS} to the Satake space D_S in the 'classical situation'. This map was defined by Zucker [Z2] by another method. Proposition 6.10 gives examples which do not have the canonical map $D_{\mathrm{SL}(2)} \to D_{\mathrm{BS}}$. We will give an example in 6.11 for which $D_{\mathrm{SL}(2)} \subsetneq D_{\mathrm{BS}}$ (hence $\Gamma\backslash D_{\mathrm{SL}(2)}$ is not compact for Γ of finite index in $G_{\mathbf{Z}}$) because the horizontal tangent bundle T^h_D is trivial. Proposition 6.12 gives examples for which $D_{\mathrm{SL}(2)}$ is not locally compact, that is, $D_{\mathrm{SL}(2)}$ has 'slits' influenced by the fact that the isotropy subgroups $K'_{\mathbf{r}}$ are not maximal compact.

6.2. *The case of the upper half-plane $\mathfrak{h}$.* Let $H_{\mathfrak{h}} := \mathbf{Z}^2 = \mathbf{Z}e_1 + \mathbf{Z}e_2$, let $\langle\ ,\ \rangle_{\mathfrak{h}}$ be the anti-symmetric bilinear form on $H_{\mathfrak{h},\mathbf{C}} \times H_{\mathfrak{h},\mathbf{C}}$ characterized by $\langle e_2, e_1\rangle_{\mathfrak{h}} = 1$, and take $(H_{\mathfrak{h}}, \langle\ ,\ \rangle_{\mathfrak{h}})$ as $(H_0, \langle\ ,\ \rangle_0)$. Then

$$\check{D} = \mathbf{P}^1(\mathbf{C});\ F_z \leftrightarrow z = (z_1 : z_2),$$
$$\text{where } F_z^0 = H_{0,\mathbf{C}},\ F_z^1 = \mathbf{C}(z_1e_1 + z_2e_2),\ F_z^2 = 0.$$

Identify $z \in \mathbf{C}$ with $(z:1) \in \mathbf{P}^1(\mathbf{C})$. Then $D \subset \check{D}$ is identified with the upper-half plane $\mathfrak{h} \subset \mathbf{P}^1(\mathbf{C})$. We have

$$G_{\mathbf{R}} = \mathrm{SL}(2,\mathbf{R}) \supset \mathrm{SO}(2,\mathbf{R}) = K_i = K'_i.$$

The map

$$\mathbf{P}^1(\mathbf{Q}) \to \{P \mid \text{a } \mathbf{Q}\text{-parabolic subgroup of } G_{\mathbf{R}} \text{ with } P \neq G_{\mathbf{R}}\},$$
$$z \mapsto P_z := \{g \in G_{\mathbf{R}} \mid gz = z\},$$

is bijective. We have

$$P_\infty = \left\{ \begin{pmatrix} a^{-1} & b \\ 0 & a \end{pmatrix} \middle| \begin{matrix} a \in \mathbf{R}^\times, \\ b \in \mathbf{R} \end{matrix} \right\}, \quad P_{\infty,u} = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| b \in \mathbf{R} \right\},$$
$$S_{P_\infty} = P_\infty / P_{\infty,u}.$$

The unique element of Δ_{P_∞} sends

$$S_{P_\infty} \ni \left(\begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \bmod P_{\infty,u} \right) \longmapsto a^2.$$

If we identiy A_{P_∞} with $\mathbf{R}_{>0}$ by this element of Δ_{P_∞}, we have

$$a \circ (x+iy) = x + ia^{-1}y \quad (a \in \mathbf{R}_{>0}, x \in \mathbf{R}, y \in \mathbf{R}_{>0}).$$

Hence $D_{\mathrm{BS}}(P_\infty)$ is identified with the topological space $\{x+iy \,|\, x \in \mathbf{R}, 0 < y \leq \infty\}$. We have

$$\mathcal{X}_{\mathrm{BS}} = D_{\mathrm{BS}} \xleftarrow{\sim} D_{\mathrm{BS,val}} \xleftarrow{\sim} D_{\mathrm{SL(2),val}} \xrightarrow{\sim} D_{\mathrm{SL(2)}},$$

and $i\infty \in D_{\mathrm{BS}}(P_\infty)$ is identified with the class of the SL(2)-orbit $(\rho_{\mathfrak{h}}, \varphi_{\mathfrak{h}})$, which we call the *standard* SL(2)*-orbit*, defined by

$$\begin{cases} \rho_{\mathfrak{h}} = \mathrm{id} : \mathrm{SL}(2,\mathbf{C}) \to G_{\mathbf{C}}, \\ \varphi_{\mathfrak{h}}(z) = F_z \quad (z \in \mathbf{P}^1(\mathbf{C})). \end{cases}$$

In fact, it is obvious that $(\mathbf{R}^n_{\geq 0})_{\mathrm{val}} = \mathbf{R}^n_{\geq 0}$ for $n = 1$. Since P is a $\mathbf{Q}$-parabolic subgroup of $G_{\mathbf{R}} = \mathrm{SL}(2,\mathbf{R})$ with $P \neq G_{\mathbf{R}}$, we have $\operatorname{rank} S_P = 1$. Hence, by Proposition 2.15, we have $D_{\mathrm{BS,val}} \xrightarrow{\sim} D_{\mathrm{BS}}$. $D_{\mathrm{SL(2)}} = D_{\mathrm{SL(2)},\leq 1}$, because $\mathfrak{sl}(2,\mathbf{C})^n \to \mathfrak{g}_{\mathbf{C}}$ can not be injective if $n > 1$. Hence, by Proposition 4.14, we have $D_{\mathrm{SL(2),val}} \xrightarrow{\sim} D_{\mathrm{SL(2)}}$.

Proposition 6.3. *We use the notation in* 2.5, 2.6, 3.6, 3.7. *Let* $x = [\rho, \varphi] \in D_{\mathrm{SL(2)}}$ *and let* $W = (W^{(j)})_{1 \leq j \leq n}$ *be the associated family of weight filtrations, where* $W^{(j)} = W(\sigma_j)$ (3.5).

(i) *The following conditions* (a), (b), (c) *are equivalent.*

(a) *For any* $y, y' \in D_{\mathrm{SL(2),val}}$ *lying over* x, *the images of* y, y' *in* D_{BS}, *via* $D_{\mathrm{BS,val}}$, *coincide.*

(b) *The subspaces* $W^{(j)}_k$ $(1 \leq j \leq n,\ k \in \mathbf{Z})$ *are linearly ordered by inclusion.*

(c) *For* $\chi, \chi' \in X((\mathbf{G_m})^n)$ *with* $H(\chi) \neq 0$ *and* $H(\chi') \neq 0$, *either* $\chi\chi'^{-1}$ *or* $\chi'\chi^{-1}$ *is contained in* $X((\mathbf{G_m})^n)_+$.

(ii) *Assume the equivalent conditions in* (i) *are satisfied. Then the* $\mathbf{Q}$*-subgroup* $(G^\circ)_{W,\mathbf{R}} := G_{W,\mathbf{R}} \cap (G^\circ)_\mathbf{R}$ *of* $G_\mathbf{R}$ *is parabolic and, for any lifting* $y \in D_{\mathrm{SL}(2),\mathrm{val}}$ *of* x*, the image of* y *in* D_{BS}*, via* $D_{\mathrm{BS},\mathrm{val}}$*, coincides with* $((G^\circ)_{W,\mathbf{R}}, A_{(G^\circ)_{W,\mathbf{R}}} \circ \varphi(\mathbf{i}))$.

For the proof of Proposition 6.3, we use the following

Lemma 6.4. *Let* $\mathcal{P}$ *be the set of all* $\mathbf{Q}$*-parabolic subgroups of* $G_\mathbf{R}$*, and let* $\mathcal{M}$ *be the set of all finite sets* M *such that* $M = \{M_j \mid 0 \le j \le m\}$*, the* M_j *are* $\mathbf{Q}$*-rational* $\mathbf{R}$*-subspaces of* $H_{0,\mathbf{R}}$*,* $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_m = H_{0,\mathbf{R}}$ *and* $M_j^\perp = M_{m-j}$ $(0 \le j \le m)$*. Let* $p : \mathcal{M} \to \mathcal{P}$ *be the map defined by* $p(M) := \{g \in (G^\circ)_\mathbf{R} \mid gM_j = M_j\ (0 \le j \le m)\}$*. Write*

$$\begin{aligned} &\mathcal{M} = \mathcal{M}_0 \sqcup \mathcal{M}_1 \sqcup \mathcal{M}_2, \quad \textit{where} \\ &\mathcal{M}_0 := \{M \in \mathcal{M} \mid M \textit{ is not exceptional}\}, \\ &\mathcal{M}_1 := \{M \in \mathcal{M} \mid M \textit{ is exceptional with } m \textit{ even}\}, \\ &\mathcal{M}_2 := \{M \in \mathcal{M} \mid M \textit{ is exceptional with } m \textit{ odd}\}. \end{aligned}$$

Here the meaning of 'exceptional' is as in 2.11*. Then*

(i) *Let* $M \in \mathcal{M}_0$ *and* $P = p(M)$*. Then,* M *coincides with the set of all* P*-stable* $\mathbf{R}$*-subspaces of* $H_{0,\mathbf{R}}$*. We have* $p^{-1}(p(M)) = \{M\}$*.*

(ii) *Let* $P \in p(\mathcal{M}_1 \sqcup \mathcal{M}_2) \subset \mathcal{P}$*. Let* N *be the set of all* P*-stable* $\mathbf{R}$*-subspaces of* $H_{0,\mathbf{R}}$*, and let*

$$M := \{L \in N \mid \forall L' \in N, \textit{ either } L \subset L' \textit{ or } L' \subset L\}.$$

Then, $M \in \mathcal{M}_2$*. Write* $M = \{M_j \mid 0 \le j \le m\}$*,* $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_m = H_{0,\mathbf{R}}$*, let* e_1*,* e_2 *be as in* 2.11*, and let* $L_k := M_{(m-1)/2} + \mathbf{R}e_k$ *for* $k = 1, 2$*. Then,* $N = M \cup \{L_1, L_2\}$*,* $M \cup \{L_1\}$ *and* $M \cup \{L_2\}$ *belong to* $\mathcal{M}_1$*, and* $p^{-1}(P)$ *consists of the three elements* M*,* $M \cup \{L_1\}$ *and* $M \cup \{L_2\}$*.*

(iii) $p : \mathcal{M} \to \mathcal{P}$ *is surjective.*

(iv) $p(\mathcal{M}_1) = p(\mathcal{M}_2)$*.*

p *induces a* $2:1$ *map* $p : \mathcal{M}_1 \to p(\mathcal{M}_1)$*, and a bijection* $p : \mathcal{M}_0 \sqcup \mathcal{M}_2 \xrightarrow{\sim} \mathcal{P}$*.*

Proof. The proofs of (i) and (ii) are straightforward. It follows that $p(\mathcal{M}_1) = p(\mathcal{M}_2)$, that the map $p : \mathcal{M}_1 \to p(\mathcal{M}_1)$ is $2:1$ and that the map $p : \mathcal{M}_0 \sqcup \mathcal{M}_2 \to \mathcal{P}$ is injective. It remains to prove $\mathcal{P} = p(\mathcal{M}_0 \sqcup \mathcal{M}_2)$. We divide our considerations into three cases:

(a) w is odd.

(b) w is even and $\langle\ ,\ \rangle_0 : H_{0,\mathbf{Q}} \times H_{0,\mathbf{Q}} \to \mathbf{Q}$ is not hyperbolic.

(c) w is even, and $\langle\ ,\ \rangle_0 : H_{0,\mathbf{Q}} \times H_{0,\mathbf{Q}} \to \mathbf{Q}$ is hyperbolic (that is, $d := \dim H_{0,\mathbf{Q}}$ is even and there exists a basis $(e_j)_{1\le j\le d}$ of $H_{0,\mathbf{Q}}$ such that $\langle e_j, e_k\rangle_0 = 1$ if $j + k = d + 1$ and $\langle e_j, e_k\rangle_0 = 0$ otherwise).

Assume first we are in the case (a). Then there exists $M = \{M_j \mid 0 \le j \le m\} \in \mathcal{M}_0$ (m is even, $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_m = H_{0,\mathbf{R}}$) such that $\dim \mathrm{gr}_j^M = 1$ for $1 \le j \le m$. Let $P := p(M)$, $r := m/2$. Then P is a minimal $\mathbf{Q}$-parabolic subgroup of $G_\mathbf{R}$. This is because $P/P_u \simeq (\mathbf{G_{m,R}})^r$ (see 2.11 (2)) and has no $\mathbf{Q}$-parabolic subgroup other than P/P_u itself. We have $r = \operatorname{rank} S_P = \sharp(\Delta_P)$. We find 2^r elements M' of $\mathcal{M}_0$ such that $M' \subset M$ (in fact, for each finite subset I of $\{1, \dots, r\}$, we find M' defined by $M' := \{0, M_j\ (j \in I), M_{m-j}\ (j \in I), H_{0,\mathbf{R}}\}$). Since $p : \mathcal{M}_0 \to \mathcal{P}$ is injective, this shows that there exist at least 2^r elements P' of $p(\mathcal{M}_0)$ such that $P' \supset P$. By 2.10 (3), this shows that any $\mathbf{Q}$-parabolic subgroup P' of $G_\mathbf{R}$ with $P' \supset P$ belongs to $p(\mathcal{M}_0)$. Now let P' be any element of $\mathcal{P}$. Take any minimal $\mathbf{Q}$-parabolic subgroup P'' of $G_\mathbf{R}$ such that $P' \supset P''$. Then $P = gP''g^{-1}$ for some $g \in G_\mathbf{Q}$. Since $gP'g^{-1} \supset P$, we have $gP'g^{-1} \in p(\mathcal{M}_0)$. Hencce $P' \in p(\mathcal{M}_0)$.

Next assume we are in the case (b). Then there exists $M = \{M_j \mid 0 \le j \le m\} \in \mathcal{M}_0$ (m is odd, $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_m = H_{0,\mathbf{R}}$) such that $\dim \mathrm{gr}_j^M = 1$ if $1 \le j \le m$ and $j \ne (m+1)/2$ and such that the $\mathbf{R}$-bilinear form $\varphi : \mathrm{gr}^M_{(m+1)/2} \times \mathrm{gr}^M_{(m+1)/2} \to \mathbf{R}$ induced by $\langle\ ,\ \rangle_0$ is anisotropic (that is, $\varphi(x, x) \ne 0$ for any $x \in \mathrm{gr}^M_{(m+1)/2} - \{0\}$). Let $P := p(M)$, $r := (m-1)/2$. Then P is a minimal $\mathbf{Q}$-parabolic subgroup of $G_\mathbf{R}$. This is because $P/P_u \simeq (\mathbf{G_{m,R}})^r \times \mathrm{SO}(\varphi)_\mathbf{R}$ (see 2.11 (2)) and has no $\mathbf{Q}$-parabolic subgroup other than P/P_u itself. We have $r = \operatorname{rank} S_P = \sharp(\Delta_P)$. Just as in the case (a), we find 2^r elements M' of $\mathcal{M}_0$ such that $M' \subset M$ and then we can deduce $\mathcal{P} = p(\mathcal{M}_0)$.

Assume lastly we are in the case (c). Then there exists $M = \{M_j \mid 0 \le j \le m\} \in \mathcal{M}_2$ (m is odd, $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_m = H_{0,\mathbf{R}}$) such that $\dim \mathrm{gr}_j^M = 1$ if $1 \le j \le m$ and $j \ne (m+1)/2$. Let $P := p(M)$, $r := (m+1)/2$. Then P is a minimal $\mathbf{Q}$-parabolic subgroup of $G_\mathbf{R}$. This is because $P/P_u \simeq (\mathbf{G_{m,R}})^r$ (see 2.11 (2)) and has no $\mathbf{Q}$-parabolic subgroup other than P/P_u itself. We have $r = \operatorname{rank} S_P = \sharp(\Delta_P)$. Just as in the cases (a), (b), we find 2^r elements M' of $\mathcal{M}_0 \sqcup \mathcal{M}_2$ such that $M' \subset M$ and then we can deduce $\mathcal{P} = p(\mathcal{M}_0 \sqcup \mathcal{M}_2)$. Q.E.D.

6.5. *Proof of Proposition* 6.3. It is easy to see the equivalence of (b) and (c), the implication from (b) and (c) to (a), and the implication from (b) and (c) to the conclusion in (ii).

We prove the implication from (a) to (c). We first prove the following assertion:

(1) Assume (a). Let $\chi, \chi' \in X((\mathbf{G_m})^n)$ which satisfy $H(\chi) \neq 0$, $H(\chi') \neq 0$ and $(\chi\chi'^{-1})^{\pm} \notin X((\mathbf{G_m})^n)_+$. Then, w is even, $\chi' = \chi^{-1}$ and $\dim H(\chi) = \dim H(\chi') = 1$.

Write $\chi = (a_1, \dots, a_n)$, $\chi' = (a'_1, \dots, a'_n)$ under the identification $X((\mathbf{G_m})^n) = \mathbf{Z}^n$. Then, there exist j, k such that $a_j > a'_j$, $a_k < a'_k$. Let $s, s' : \{1, \dots, n\} \to \{1, \dots, n\}$ be bijections satisfying $s(1) = j$, $s(2) = k$, $s'(1) = k$, $s'(1) = j$, and let V (resp. V') be the valuative submonoid of $X((\mathbf{G_m})^n) = \mathbf{Z}^n$ consisting of all $(b_1, \dots, b_n)$ $(b_j \in \mathbf{Z})$ such that $(b_{s(1)}, \dots, b_{s(n)}) \geq 0$ (resp. $(b_{s'(1)}, \dots, b_{s'(n)}) \geq 0$) for the lexicographical order of $\mathbf{Z}^n$. Then, $V \supset X((\mathbf{G_m})^n)_+$, $V' \supset X((\mathbf{G_m})^n)_+$, $V^\times = V'^\times = \{1\}$. By (a), we have $P_V = P_{V'}$ (see 4.11). Let $\mathcal{M} = \mathcal{M}_0 \sqcup \mathcal{M}_1 \sqcup \mathcal{M}_2$, $\mathcal{P}$, and $p : \mathcal{M} \to \mathcal{P}$ be as in 6.4. Let $L := \sum_{\psi \in \chi V'^{-1}} H(\psi)$, $L' := \sum_{\psi \in \chi' V^{-1}} H(\psi)$. Then, since $\chi' \notin \chi V'^{-1}$ and $\chi \notin \chi' V^{-1}$, we have $H(\chi') \not\subset L$, $H(\chi) \not\subset L'$, $H(\chi) \subset L$, $H(\chi') \subset L'$. Hence $L \not\subset L'$ and $L' \not\subset L$. Since L and L' are stable under $P_V = P_{V'}$, we have, by 6.4, that $P_V = P_{V'} = p(M)$. Write $M = \{M_j \mid 0 \leq j \leq m\}$, $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_m = H_{0,\mathbf{R}}$. By 6.4, w is even, $L = H(\chi) + M_{(m-1)/2}$, $L' = H(\chi') + M_{(m-1)/2}$, $\dim H(\chi) = \dim H(\chi') = 1$, and $\langle H(\chi), H(\chi')\rangle_0 \neq 0$. Hence we have (1).

To proceed more, we need the following (2), (3).

(2) Let $\theta : \mathbf{Z}^n \to \mathbf{Z}^n$ be the map defined by $\theta(b_1, \dots, b_n) = (b_1, b_1 + b_2, \dots, b_1 + \cdots + b_n)$, so that

$$H(\theta(b_1, \dots, b_n))$$
$$= \{v \in H_{0,\mathbf{R}} \mid \rho(\Delta(t))v = t_1^{b_1} \dots t_n^{b_n} v \ (t \in (\mathbf{G}_{\mathbf{m},\mathbf{R}})^n)\}$$

(Δ is as in 3.1). Let $b_1, \dots, b_n \in \mathbf{Z}$ so that $H(\theta(b_1, \dots, b_n)) \neq 0$. Then, for any $c_1, \dots, c_n \in \mathbf{Z}$ with $|c_j| \leq |b_j|$ and $c_j \equiv b_j \mod 2$ $(1 \leq j \leq n)$, we have $H(\theta(c_1, \dots, c_n)) \neq 0$.

(3) For $b_1, \dots, b_n \in \mathbf{Z}$, $H(b_1, \dots, b_n)$ has canonically a Hodge structure of weight $w + b_n$.

(2) is deduced from the following (4) and (5) whose proofs are easy.

(4) Let $b_1, \dots, b_n \in \mathbf{Z}$. Then,

$$N_j H(\theta(b_1, \dots, b_n)) \subset H(\theta(b_1, \dots, b_{j-1}, b_j - 2, b_{j+1}, \dots b_n)).$$

(5) Let $b_j \in \mathbf{Z}$, $b_j \geq 0$ $(1 \leq j \leq n)$. Then,

$$\prod_{1\leq j\leq n} N_j^{b_j} : H(\theta(b_1,\ldots,b_n)) \xrightarrow{\sim} H(\theta(-b_1,\ldots,-b_n)).$$

We prove (3). By [Sc], $(W^{(n)}[-w],\varphi(0))$ is a mixed Hodge structure. Hence

$$\varphi(0)(\mathrm{gr}_k^{W^{(n)}}) = \bigoplus\nolimits_{b\in\mathbf{Z}^{n-1}} H(b,k)$$

has a Hodge structure of weight $w+k$. Since $\tilde{\rho}(t)\varphi(0) = \varphi(0)$ for all $t \in \mathbf{R}^n_{>0}$, each $H(b,k)$ $(b \in \mathbf{Z}^{n-1})$ carries a Hodge structure of weight $w+k$.

Now we complete the proof of $(a) \Rightarrow (c)$. Assume (a). By (1), if (c) is not satisfied, then w is even and there exists $\chi \in X((\mathbf{G_m})^n)$ such that $\dim H(\chi) = \dim H(\chi^{-1}) = 1$ and $\chi \notin X((\mathbf{G_m})^n)_+$, $\chi^{-1} \notin X((\mathbf{G_m})^n)_+$. Write $\chi = (a_1,\ldots,a_n) \in \mathbf{Z}^n$. Then there exist j, k such that $a_j > 0$ and $a_k < 0$. By replacing χ by χ^{-1} if necessary, we may assume $j < k$. We prove

$$a_k = -1 \tag{6}$$

Take l such that $j < l \leq k$ and $a_l < a_{l-1}$. Let $b_j := a_j - a_{j-1}$ $(1 \leq j \leq n$, a_0 means 0). Then $(a_1,\ldots,a_n) = \theta(b_1,\ldots,b_n)$, $b_l < 0$. Define $a'_1,\ldots,a'_n,b'_1,\ldots,b'_n \in \mathbf{Z}$ by $b'_j := b_j$ for $j \neq l$ and $b'_l := b_l + 2$, $a'_j - a'_{j-1} := b'_j$ $(1 \leq j \leq n$, $a'_0 = 0)$. Then, $H(a'_1,\ldots,a'_n) = H(\theta(b'_1,\ldots,b'_n)) \neq 0$ by (2). If $a_k < -1$, then $a'_k = a_k + 2 < -a_k$, and $a'_j = a_j > -a_j$. Since $H(a'_1,\ldots,a'_n) \neq 0$ and $H(-a_1,\ldots,-a_n) \neq 0$, (1) shows $a'_j = a_j$ $(1 \leq j \leq n)$. Hence $a_k + 2 = a_k$, a contradiction. Hence we have (6).

We next prove

$$a_l = -1 \quad \text{for} \quad k \leq l \leq n. \tag{7}$$

In fact, if $a_l \neq -1$ for some l with $k \leq l \leq n$, then $(a_1,\ldots,a_n) = \theta(b_1,\ldots,b_n)$ with $b_j \in \mathbf{Z}$ $(1 \leq j \leq n)$, $b_l \neq 0$ for some l with $k \leq l \leq n$. By (5),

$$H(\theta(b_1,\ldots,b_k,-|b_{k+1}|,\ldots,-|b_n|)) \neq 0$$

and hence there exist $a'_{k+1},\ldots,a'_n \in \mathbf{Z}$ such that

$$H(a_1,\ldots,a_k,a'_{k+1},\ldots,a'_n) \neq 0$$

and such that $a'_l < a_k$ for some l with $k < l \leq n$. Since $a_j > 0$ and $a'_l \leq -2$, this contradicts (6).

By (7), $a_n = -1$. Hence, by (3), $H(\chi)$ carries a Hodge structure of weight $w-1$, which is odd. This contradicts $\dim H(\chi) = 1$. Hence we have (a) $\Rightarrow$ (c), and Proposition 6.3 is proved. Q.E.D.

6.6. *Classical situation.* Let $F \in D$, and let $T_D(F)$ and $T^h_D(F)$ be the tangent space and horizontal tangent space of D at F, respectively (1.6).

It can be proved that the following (i) and (ii) are equivalent.

(i) For any $F \in D$, $T^h_D(F) = T_D(F)$ and $\dim K'_F = \dim K_F$ (2.2 (1)).

(ii) One of the following (a), (b) is satisfied:

(a) There is $t \in \mathbf{Z}$ such that $w = 2t+1$ and $h^{p,w-p} = 0$ if $p \neq t, t+1$.

(b) There is $t \in \mathbf{Z}$ such that $w = 2t$, $h^{t+1,t-1} \leq \inf\{1, h^{t,t}\}$, and $h^{p,q} = 0$ if $p \geq t+2$.

Note that the condition (i) is independent of the choice of $F \in D$. The equivalence of (i) and (ii) follows by computing dimensions of the subspaces $F^r(\mathfrak{g}_{\mathbf{C}})$ in 1.6 and of the Lie algebras of the following groups.

$$
G_{\mathbf{R}} \simeq \begin{cases} \mathrm{Sp}(2g, \mathbf{R}) & \text{if } w = 2t+1, \\ O(a,b;\mathbf{R}) & \text{if } w = 2t, \end{cases}
$$

$$
(1) \qquad K_{\mathbf{r}} \simeq \begin{cases} U(g) & \text{if } w = 2t+1, \\ O(a,\mathbf{R}) \times O(b,\mathbf{R}) & \text{if } w = 2t, \end{cases}
$$

$$
K'_{\mathbf{r}} \simeq \begin{cases} \prod_{j\geq 0} U(h^{t+1+j,t-j}) & \text{if } w = 2t+1, \\ \left(\prod_{j>0} U(h^{t+j,t-j})\right) \times O(h^{t,t}, \mathbf{R}) & \text{if } w = 2t, \end{cases}
$$

where $g := \operatorname{rank} H_0/2$ if $w = 2t+1$, and a, b are the signature of $(H_{0,\mathbf{R}}, \langle\ ,\ \rangle_0))$ if $w = 2t$. (cf. NOTATION, [U2]).

We say that we are in the *classical situation* if these equivalent conditions (i), (ii) are satisfied. The polarized Hodge structures in (ii) (a) are Tate twists of the first cohomology of polarized abelian varieties, and the primitive part of the second cohomology of a polarized K3 surface belongs to (ii) (b).

Theorem 6.7. *In the classical situation, except in the case* (i) *below, there exist homeomorphisms* $D_{\mathrm{SL}(2)} \xrightarrow{\sim} D_{\mathrm{BS}}$, $D_{\mathrm{SL}(2),\mathrm{val}} \xrightarrow{\sim} D_{\mathrm{BS},\mathrm{val}}$ *extending the identity map of* D.

(i) *w is even,* $\operatorname{rank} H_0 = 4$, *and there exists a* $\mathbf{Q}$*-basis* $(e_j)_{1\leq j\leq 4}$ *of* $H_{0,\mathbf{Q}}$ *such that* $\langle e_j, e_k\rangle_0 = 1$ *if* $j+k=5$, *and* $=0$ *otherwise.*

Proof. In the case $w = 2t$ and $h^{t+s,t-s} = 0$ for any $s \neq t$, D is a one point set and Theorem 6.7 holds trivially. We assume $h^{t+1,t-1} = 1$ in the case $w = 2t$.

We have the following (1).

(1) Let $[\rho, \varphi] \in D_{\mathrm{SL}(2),n}$, and let $W = (W^{(j)})_{1 \leq j \leq n}$ be the associated family of weight filtrations. Let $1 \leq j \leq n$. In the case 6.6 (ii) (a), we have

$$W^{(j)}_{-2} = 0, \ W^{(j)}_{1} = H_{0,\mathbf{R}}.$$

In the case 6.6 (ii) (b), we have one of the following (b1), (b2).

(b1) $\quad \mathrm{gr}^{W^{(j)}}_k = 0$ unless $k = 0, \pm 1$, and $\dim \mathrm{gr}^{W^{(j)}}_k = 2$ for $k = \pm 1$.

(b2) $\quad \mathrm{gr}^{W^{(j)}}_k = 0$ unless $k = 0, \pm 2$, and $\dim \mathrm{gr}^{W^{(j)}}_k = 1$ for $k = \pm 2$.

This follows from the facts that the filtration $\varphi(\mathbf{i})(\mathrm{gr}^{W^{(j)}}_k)$, induced on $\mathrm{gr}^{W^{(j)}}_k$ by $\varphi(\mathbf{i})$, is a Hodge structure of weight $w+k$ for each $k \in \mathbf{Z}$, and that if we denote the Hodge type of this Hodge structure by $(h^{p,q}_k)_{p,q \in \mathbf{Z}}$ then $h^{p,w-p} = \sum_k h^{p,w+k-p}_k$.

We next prove the following (2).

(2) Let the notation be as in (1). In the case 6.6 (ii) (a), we have

$$\begin{aligned} 0 \subsetneq W^{(1)}_{-1} \subsetneq W^{(2)}_{-1} \subsetneq \cdots \subsetneq W^{(n)}_{-1} \\ \subset W^{(n)}_{0} \subsetneq \cdots \subsetneq W^{(2)}_{0} \subsetneq W^{(1)}_{0} \subsetneq H_{0,\mathbf{R}}. \end{aligned}$$

In the case 6.6 (ii) (b) with $n \geq 2$, we have $n = 2$, $W^{(1)}$ is of type (b1), $W^{(2)}$ is of type (b2), and

$$\begin{aligned} &0 \subsetneq W^{(2)}_{-2} = W^{(2)}_{-1} \subsetneq W^{(1)}_{-1} \subset W^{(1)}_{0} \subsetneq W^{(2)}_{0} = W^{(2)}_{1} \subsetneq H_{0,\mathbf{R}}, \\ &\dim W^{(2)}_{-2} = \dim W^{(1)}_{-1}/W^{(2)}_{-1} = \dim W^{(2)}_{0}/W^{(1)}_{0} \\ &\qquad = \dim H_{0,\mathbf{R}}/W^{(2)}_{1} = 1. \end{aligned}$$

In fact, in the case 6.6 (ii) (a), since $\mathrm{Ker}(a_1N_1 + \cdots + a_jN_j) = W^{(j)}_0$ for any $a_1, \ldots, a_j > 0$ (3.5), we have $W^{(j')}_0 \supset W^{(j)}_0$ for $1 \leq j' \leq j \leq n$, and hence, by taking $(\ )^{\perp}$, we obtain $W^{(j')}_{-1} \subset W^{(j)}_{-1}$. Since $W^{(j')} \neq W^{(j)}$ for $j' \neq j$, this proves (2) in the case 6.6 (ii) (a).

We consider the case 6.6 (ii) (b). Assume $n \geq 2$. If $1 \leq j \leq n$ and $W^{(j)}$ is of type (b1), $(a_1N_1 + \cdots + a_jN_j)^2 = 0$ for any $a_1, \ldots, a_j > 0$ and hence $(a_1N_1 + \cdots + a_{j'}N_{j'})^2 = 0$ for any $j' \leq j$ and any $a_1, \ldots, a_{j'} > 0$.

Hence $W^{(j')}$ for $j' \leq j$ is also of type (b1), and we have $W_0^{(j')} \supsetneq W_0^{(j)}$ for $j' < j$ just as in the case 6.6 (ii) (a). This contradicts the statement about dimensions in (b1) if $j \geq 2$. Hence any $W^{(j)}$ with $j \geq 2$ is of type (b2). If $W^{(j)}$ is of type (b2), $\operatorname{Ker}((a_1N_1 + \cdots + a_jN_j)^2) = W_1^{(j)}$ for any $a_1, \dots, a_j > 0$. Hence, if $j' < j$ and $W^{(j')}$ and $W^{(j)}$ are of type (b2), we have $W_1^{(j')} \supset W_1^{(j)}$. Since $W^{(j')} \neq W^{(j)}$, we have $W_1^{(j')} \supsetneq W_1^{(j)}$ which contradicts the statement about dimensions in (b2). Hence we have $n = 2$, $W^{(1)}$ is of type (b1), and $W^{(2)}$ is of type (b2). By (1), it remains to prove $W_{-1}^{(1)} \supset W_{-2}^{(2)}$ ($W_0^{(1)} \supset W_1^{(2)}$ follows from this by taking $(\ )^{\perp}$). The condition on the Hodge numbers (ii) (b) shows that the signature of $(H_{0,\mathbf{R}}, \langle\ ,\ \rangle_0)$ is $(d-2, 2)$, where $d = \operatorname{rank} H_0$, and hence $H_{0,\mathbf{R}}$ has no 3-dimensional isotropic subspace, *i.e.*, $\mathbf{R}$-subspace on which the restriction of $\langle\ ,\ \rangle_0$ is zero. However, if $W_{-1}^{(1)} \not\supset W_{-1}^{(2)}$, $W_{-1}^{(1)} + W_{-2}^{(2)}$ is a 3-dimensional isotropic subspace as is shown in the following way. From $(aN_1 + bN_2)^3 = 0$ for $a, b \in \mathbf{R}_{>0}$, we have $N_1^iN_2^j = 0$ if $i \geq 0$, $j \geq 0$, $i + j \geq 3$. Since $W_{-1}^{(1)} = \operatorname{Im} N_1$ and $W_{-2}^{(2)} = \operatorname{Im}(N_1 + N_2)^2$, by $\langle hx, y\rangle_0 + \langle x, hy\rangle_0 = 0$ for $h \in \mathfrak{g}_{\mathbf{R}}$, we see that $W_{-1}^{(1)} + W_{-2}^{(2)}$ is an isotropic subspace. This proves (2) in the case 6.6 (ii) (b).

By (1) and (2), we have the following.

(3) In the classical situation, the set $\{W_k^{(j)} \mid 1 \leq j \leq n, k \in \mathbf{Z}\}$ is totally ordered, and $(G^\circ)_{W,\mathbf{R}}$ is a parabolic subgroup of $G_{\mathbf{R}}$.

Hence, by 6.3, we have a continuous map $D_{\mathrm{SL}(2)} \to D_{\mathrm{BS}}$ which extends the identity map of D.

We prove

(4) $D_{\mathrm{SL}(2)} \to D_{\mathrm{BS}}$ is injective.

By 3.10, an SL(2)-orbit (ρ, φ) of rank n is characterized by the associated $(W, \mathbf{r})$. Assume that the points of $D_{\mathrm{SL}(2)}$ determined by $(W, \mathbf{r})$, $(W', \mathbf{r}')$ are sent to the same point in D_{BS}. Then we have

$$(G^\circ)_{W',\mathbf{R}} = (G^\circ)_{W,\mathbf{R}}, \quad \mathbf{r}' = a \circ \mathbf{r} \ \ (\exists a \in A_{(G^\circ)_{W,\mathbf{R}}}).$$

The totally ordered set $\{W_k^{(j)} \mid 1 \leq j \leq n, k \in \mathbf{Z}\}$ is not exceptional (2.11). In fact, if it is exceptional, then $w = 2t$ ($\exists t \in \mathbf{Z}$), $d := \operatorname{rank} H_0$ is even, and there exists a $\mathbf{Q}$-basis $(e_j)_{1 \leq j \leq d}$ of $H_{0,\mathbf{Q}}$ such that

$$\langle e_j, e_k\rangle_0 = \begin{cases} 1 & \text{if } j + k = d + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, the signature of $(H_{0,\mathbf{R}}, \langle\ ,\ \rangle_0)$ is $(d/2, d/2)$. On the other hand, it was $(d-2, 2)$, and hence we have $d = 4$ and this would imply that we were in the case (i).

Hence, it follows $W' = W$ from $(G^\circ)_{W',\mathbf{R}} = (G^\circ)_{W,\mathbf{R}}$ and (2). We see also, by (2), that the torus $S_{(G^\circ)_{W,\mathbf{R}}}$ in 2.2 coincides with the torus of the SL(2)-orbit of rank n determined by $(W, \mathbf{r})$. Hence $\mathbf{r}' = a \circ \mathbf{r}$ lies on the torus orbit containing $\mathbf{r}$ of this SL(2)-orbit of rank n. Thus, the points in $D_{\mathrm{SL}(2)}$ determined by $(W, \mathbf{r})$, $(W', \mathbf{r}')$ coincide, as desired.

We prove

(5) $D_{\mathrm{SL}(2)} \to D_{\mathrm{BS}}$ is surjective.

Let $(P, Z) \in D_{\mathrm{BS}}$. Let $M = (0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_m = H_{0,\mathbf{R}}) \in \mathcal{M}$ be a $\mathbf{Q}$-rational increasing filtration of $H_{0,\mathbf{R}}$ such that $P = p(M)$ (see 6.4). Let $n = m/2$ if m is even, and $n = (m-1)/2$ if m is odd.

We first prove (5) in the case 6.6 (ii) (a). Let $e(j) := \dim M_j/M_{j-1}$ for $1 \le j \le n$, and let $e := \sum_{1 \le j \le n} e(j)$. Fix a polarized Hodge structure $(H_1, \langle\ ,\ \rangle_1, F_1)$ of weight 1 whose Hodge type $(h_1^{p,q})_{p,q \in \mathbf{Z}}$ is given by

$$h_1^{p,q} = g - e \ \text{ if } (p,q) = (1,0) \text{ or } (0,1), \quad h_1^{p,q} = 0 \ \text{ otherwise.}$$

Fix an isomorphism

$$\big(H_{\mathfrak{h},\mathbf{Q}}^{\oplus e} \oplus H_{1,\mathbf{Q}}, \langle\ ,\ \rangle_{\mathfrak{h}}^{\oplus e} \oplus \langle\ ,\ \rangle_1\big) \simeq (H_{0,\mathbf{Q}}, \langle\ ,\ \rangle_0),$$

where $(H_{\mathfrak{h},\mathbf{Q}}, \langle\ ,\ \rangle_{\mathfrak{h}})$ is as in 6.2, and take this isomorphism as an identification. Let (ρ, φ) be the SL(2)-orbit of rank n defined by

$$\rho(g_1, \ldots, g_n) := \big(\textstyle\bigoplus_{1 \le j \le n} \rho_{\mathfrak{h}}(g_j)^{\oplus e(j)}\big) \oplus \mathrm{id},$$
$$\varphi(z_1, \ldots, z_n) := \big(\textstyle\bigoplus_{1 \le j \le n} \varphi_{\mathfrak{h}}(z_j)^{\oplus e(j)}\big)(-t) \oplus F_1(-t),$$

where $(\rho_{\mathfrak{h}}, \varphi_{\mathfrak{h}})$ is the standard SL(2)-orbit in 6.2, and $(-t)$ means the Tate twist. Then the family W of weight filtrations of (ρ, φ) satisfies $M = \{W_k^{(j)} \mid 1 \le j \le n, k \in \mathbf{Z}\}$ and hence $(G^\circ)_{W,\mathbf{R}} = P$. Let $\mathbf{r} = \varphi(\mathbf{i})$. Since $K_{\mathbf{r}} = K'_{\mathbf{r}}$, we have $D = P \cdot \mathbf{r}$ by $G_{\mathbf{R}} = PK_{\mathbf{r}}$, and hence tere is $p \in P$ such that $p\mathbf{r} \in Z$. The group $P = (G^\circ)_{W,\mathbf{R}}$ acts on $D_{\mathrm{SL}(2)}(W)$, and the image of $p[\rho, \varphi] \in D_{\mathrm{SL}(2)}(W)$ in D_{BS} is (P, Z).

Next we prove (5) in the case 6.6 (ii) (b). Since $G_{\mathbf{R}} \simeq O(h^{t,t}, 2; \mathbf{R})$, if $P \ne G_{\mathbf{R}}$ then we have one of the following (c), (d), (e).

(c) $n = 1$, $\dim M_1 = 2$.

(d) $n = 1$, $\dim M_1 = 1$.

(e) $n = 2$, $\dim M_1 = \dim M_2/M_1 = 1$.

In the case (d), since $G_{\mathbf{R}} \simeq O(h^{t,t}, 2; \mathbf{R})$ and $\dim M_1 < 2$, there is an element $l \in M_{m-1} \cap H_{0,\mathbf{Q}}$ such that $\langle l, l\rangle_0 < 0$. Fix such l. Take a $\mathbf{Q}$-subspace L of $H_{0,\mathbf{Q}}$ such that

$$\begin{aligned} &M_{m-1} = M_1 \oplus L_{\mathbf{R}} && \text{in the case (c)},\\ &M_{m-1} \cap (l^{\perp}) = M_1 \oplus L_{\mathbf{R}} && \text{in the case (d)},\\ &M_{m-2} = M_2 \oplus L_{\mathbf{R}} && \text{in the case (e)}.\end{aligned}$$

Then, $\dim_{\mathbf{Q}} L = d-4$ in the cases (c), (e), $\dim_{\mathbf{Q}} L = d-3$ in the case (d) ($d = \operatorname{rank} H_0$), and the restriction of $\langle\ ,\ \rangle_0$ to L is non-degenerate in any case. Fix a polarized Hodge structure $(H_1, \langle\ ,\ \rangle_1, F_1)$ of weight w satisfying the following conditions.

$$\begin{aligned} &(H_{1,\mathbf{Q}}, \langle\ ,\ \rangle_1) \simeq (L, \text{the restriction of } \langle\ ,\ \rangle_0),\\ &h_1^{p,q} = 0 \ \text{ for } (p,q) \neq (t,t).\end{aligned}$$

In the case (c), fix also a polarized Hodge structure $(H_2, \langle\ ,\ \rangle_2, F_2)$ of weight 1 whose Hodge type $(h_2^{p,q})_{p,q\in\mathbf{Z}}$ is given by

$$h_2^{p,q} = 1 \ \text{ if } (p,q) = (1,0) \text{ or } (0,1), \quad h_2^{p,q} = 0 \ \text{ otherwise.}$$

Then $(H_{0,\mathbf{Q}}, \langle\ ,\ \rangle_0)$ is isomorphic to

$$\begin{aligned} &\big(H_{\mathfrak{h},\mathbf{Q}} \otimes_{\mathbf{Q}} H_{2,\mathbf{Q}} \oplus H_{1,\mathbf{Q}},\ \langle\ ,\ \rangle_{\mathfrak{h}} \otimes \langle\ ,\ \rangle_2 \oplus \langle\ ,\ \rangle_1\big) && \text{in the case (c)},\\ &\big(\operatorname{Sym}^2_{\mathbf{Q}}(H_{\mathfrak{h},\mathbf{Q}}) \oplus H_{1,\mathbf{Q}},\ -\langle l,l\rangle_0 \operatorname{Sym}^2(\langle\ ,\ \rangle_{\mathfrak{h}}) \oplus \langle\ ,\ \rangle_1\big) && \text{in the case (d)},\\ &\big(H_{\mathfrak{h},\mathbf{Q}}^{\otimes 2} \oplus H_{1,\mathbf{Q}},\ \langle\ ,\ \rangle_{\mathfrak{h}}^{\otimes 2} \oplus \langle\ ,\ \rangle_1\big) && \text{in the case (e)}.\end{aligned}$$

Here $\operatorname{Sym}^k(\langle\ ,\ \rangle_{\mathfrak{h}})$ is defined by

$$\big(\textstyle\prod_{1\le j\le k} x_j, \prod_{1\le j\le k} y_j\big) \mapsto \sum_{\sigma\in\mathfrak{S}_k} \prod_{1\le j\le k} \langle x_j, \sigma y_j\rangle_{\mathfrak{h}},$$

and $-\langle l,l\rangle_0 \operatorname{Sym}^2(\langle\ ,\ \rangle_{\mathfrak{h}})$ means $-\langle l,l\rangle_0$ times $\operatorname{Sym}^2(\langle\ ,\ \rangle_{\mathfrak{h}})$. We fix this isomorphism and take it as an identification. Let (ρ, φ) be the SL(2)-orbit of rank 1 in the cases (c), (d), and the SL(2)-orbit of rank 2 in the case (e), defined respectively by

$$\begin{cases} \rho(g) := \rho_{\mathfrak{h}}(g) \otimes 1_{H_2} \oplus 1_{H_1},\\ \varphi(z) := (\varphi_{\mathfrak{h}}(z) \otimes F_2)(1-t) \oplus F_1 \end{cases} \quad \text{in the case (c)},$$

$$\begin{cases} \rho(g) := \operatorname{Sym}^2(\rho_{\mathfrak{h}}(g)) \oplus 1_{H_1},\\ \varphi(z) := (\operatorname{Sym}^2(\varphi_{\mathfrak{h}}(z)))(1-t) \oplus F_1 \end{cases} \quad \text{in the case (d)},$$

$$\begin{cases} \rho(g_1, g_2) := \rho_{\mathfrak{h}}(g_1) \otimes \rho_{\mathfrak{h}}(g_2) \oplus 1_{H_1},\\ \varphi(z_1, z_2) := (\varphi_{\mathfrak{h}}(z_1) \otimes \varphi_{\mathfrak{h}}(z_2))(1-t) \oplus F_1, \end{cases} \quad \text{in the case (e)}.$$

Then the family W of weight filtrations of (ρ, φ) satisfies $M = \{W_k^{(j)} \mid 1 \le j \le n, k \in \mathbf{Z}\}$ and hence $(G^\circ)_{W,\mathbf{R}} = P$. It can be checked that $G_{\mathbf{R}} = G_{W,\mathbf{R}} K'_{\mathbf{r}}$, where $\mathbf{r} = \varphi(\mathbf{i})$. This implies $D = G_{W,\mathbf{R}} \cdot \mathbf{r}$, and hence there is $p \in G_{W,\mathbf{R}}$ such that $p \cdot \mathbf{r} \in Z$. The group $G_{W,\mathbf{R}}$ acts on $D_{\mathrm{SL}(2)}(W)$, and the image of $p[\rho, \varphi] \in D_{\mathrm{SL}(2)}(W)$ in D_{BS} is (P, Z).

Finally we prove

(6) $D_{\mathrm{SL}(2)} \to D_{\mathrm{BS}}$ and $D_{\mathrm{SL}(2),\mathrm{val}} \to D_{\mathrm{BS},\mathrm{val}}$ are homeomorphisms.

From the coincidence of the tori in the proof of (4), we see that, for $x \in D_{\mathrm{BS}}$, the map from the inverse image of x in $D_{\mathrm{SL}(2),\mathrm{val}}$ to the the inverse image of x in $D_{\mathrm{BS},\mathrm{val}}$ is bijective. Hence $D_{\mathrm{SL}(2),\mathrm{val}} \to D_{\mathrm{BS},\mathrm{val}}$ is bijective. By (3), this map is a homeomorphism. This shows that the bijection $D_{\mathrm{SL}(2)} \to D_{\mathrm{BS}}$ is also a homeomorphism. Q.E.D.

6.8. *Remark.* In the case 6.7 (i), we have $D_{\mathrm{SL}(2),\mathrm{val}} = D_{\mathrm{BS},\mathrm{val}}$ as topological spaces, and we have a continuous surjection $D_{\mathrm{SL}(2)} \to D_{\mathrm{BS}}$ extending the identity map of D. This map $D_{\mathrm{SL}(2)} \to D_{\mathrm{BS}}$ is not injective. In fact,

$$(H_{0,\mathbf{Q}}, \langle\ ,\ \rangle_0) \simeq (H_{\mathfrak{h},\mathbf{Q}} \otimes_{\mathbf{Q}} H_{\mathfrak{h},\mathbf{Q}}, \langle\ ,\ \rangle_{\mathfrak{h}} \otimes \langle\ ,\ \rangle_{\mathfrak{h}}),$$

and if we take this isomorphism as an identification, we have two $\mathrm{SL}(2)^2$-orbits (ρ, φ), (ρ', φ') defined by

$$\rho(g_1, g_2) := \rho_{\mathfrak{h}}(g_1) \otimes \rho_{\mathfrak{h}}(g_2), \quad \varphi(z_1, z_2) := (\varphi(z_1) \otimes \varphi(z_2))(1 - t),$$
$$\rho'(g_1, g_2) := \rho_{\mathfrak{h}}(g_2) \otimes \rho_{\mathfrak{h}}(g_1), \quad \varphi'(z_1, z_2) := (\varphi(z_2) \otimes \varphi(z_1))(1 - t),$$

whose images in D_{BS} coincide but $[\rho, \varphi] \neq [\rho', \varphi']$.

6.9. *Relation with Satake compactifications.* In the classical situation, we have a compactification $\Gamma\backslash D_S$ of $\Gamma\backslash D$ defined by Satake for a subgroup Γ of $G_{\mathbf{Z}}$ of finite index ([Sa]). The space D_S is the set of all pairs (W, F), where W is a $\mathbf{Q}$-rational increasing filtration of $H_{0,\mathbf{R}}$ and $F = (F_{(j)})_{j \in \mathbf{Z}}$ is a family of decreasing filtrations $F_{(j)}$ of the $\mathbf{C}$-vector spaces $\mathbf{C} \otimes_{\mathbf{R}} \mathrm{gr}_j^W$ $(j \in \mathbf{Z})$, satisfying the following condition (i).

(i) There exist an integer $n \ge 0$ and an element $[\rho, \varphi]$ of $D_{\mathrm{SL}(2),n}$ such that the n-th weight filtration $W(N_1 + \cdots + N_n)$ of $[\rho, \varphi]$ coincides with W, and such that, for some $\tilde{F} \in \varphi(\mathbf{C}^n) \subset \check{D}$, the filtration of $\mathbf{C} \otimes_{\mathbf{R}} \mathrm{gr}_j^W$ induced by $\tilde{F}$ (which is independent of the choice of $\tilde{F}$) coincides with $F_{(j)}$ for any $j \in \mathbf{Z}$.

Except the case 6.7 (i), by composing the evident surjection $D_{\mathrm{SL}(2)} \to D_S$ with the isomorphism $D_{\mathrm{SL}(2)} \simeq D_{\mathrm{BS}}$ in 6.7, we obtain a canonical surjection $D_{\mathrm{BS}} \to D_S$. (In the case 6.7 (i), by using (1) and (2) in the proof of Theorem 6.7, we can see that the map $D_{\mathrm{BS}} \to D_S$ factors through the surjection $D_{\mathrm{SL}(2)} \to D_{\mathrm{BS}}$.) This map $D_{\mathrm{BS}} \to D_S$ was defined by Zucker [Z2] by another method.

Propositon 6.10. *Assume one of the following* (i), (ii) *is satisfied for some* $t \in \mathbf{Z}$.

(i) $w = 2t+1$, $h^{t+1,t} \geq 2$, $h^{t+2,t-1} \neq 0$.

(ii) $w = 2t$, $h^{t,t} \geq 3$, $h^{t+1,t-1} \geq 2$, *and there is a* $\mathbf{Q}$*-vector subspace of* $H_{0,\mathbf{Q}}$ *of dimension* 3 *on which the restriction of* $\langle\ ,\ \rangle_0$ *is zero.*

Then there is no continuous map $D_{\mathrm{SL}(2)} \to D_{\mathrm{BS}}$ *which extends the identity map of* D.

Proof. First we consider the case (i). Fix a polarized Hodge structure $(H_1, \langle\ ,\ \rangle_1, F_1)$ of weight w whose Hodge type $(h_1^{p,q})_{p,q\in\mathbf{Z}}$ is given by

$$h_1^{p,q} = h^{p,q} - \begin{cases} 2 & \text{if } (p,q) = (t+1,t) \text{ or } (t,t+1), \\ 1 & \text{if } (p,q) = (t+2,t-1) \text{ or } (t-1,t+2), \\ 0 & \text{otherwise.} \end{cases}$$

Fix an isomorphism

$$\begin{aligned}&\big(H_{\mathfrak{h},\mathbf{Q}} \otimes_{\mathbf{Q}} \mathrm{Sym}^2_{\mathbf{Q}}(H_{\mathfrak{h},\mathbf{Q}}) \oplus H_{1,\mathbf{Q}}, \langle\ ,\ \rangle_{\mathfrak{h}} \otimes \mathrm{Sym}^2(\langle\ ,\ \rangle_{\mathfrak{h}}) \oplus \langle\ ,\ \rangle_1\big)\\ &\simeq (H_{0,\mathbf{Q}}, \langle\ ,\ \rangle_0),\end{aligned}$$

where $(H_{\mathfrak{h},\mathbf{Q}}, \langle\ ,\ \rangle_{\mathfrak{h}})$ is as in 6.2, and take this as an identification. Let (ρ, φ) be the SL(2)-orbit of rank 2 defined by

$$\begin{aligned}\rho(g_1, g_2) &:= \rho_{\mathfrak{h}}(g_1) \otimes \mathrm{Sym}^2(\rho_{\mathfrak{h}}(g_2)) \oplus 1_{H_1},\\ \varphi(z_1, z_2) &:= \varphi_{\mathfrak{h}}(z_1) \otimes \mathrm{Sym}^2(\varphi_{\mathfrak{h}}(z_2))(1-t) \oplus F_1.\end{aligned}$$

Then this SL(2)-orbit of rank 2 does not satisfy the condition 6.3 (i) (c). In fact, $\rho(\Delta(t_1,t_2))$ acts on $(e_1 \otimes e_2^2, 0)$ (resp. $(e_2 \otimes e_1^2, 0)$) by $t_1^{-1}t_2^2$ (resp. $t_1 t_2^{-2}$), hence $\tilde{\rho}(t_1,t_2)$ acts on $(e_1 \otimes e_2^2, 0)$ (resp. $(e_2 \otimes e_1^2, 0)$) by $t_1^{-1}t_2$ (resp. $t_1 t_2^{-1}$). It follows $(e_1 \otimes e_2^2, 0) \in H(-1,1)$ and $(e_2 \otimes e_1^2, 0) \in H(1,-1)$.

Next we consider the case (ii). Let L be a $\mathbf{Q}$-vector subspace of $H_{0,\mathbf{Q}}$ of dimension 3 on which the restriction of $\langle\ ,\ \rangle_0$ is zero. Since $h^{t+1,t-1} + h^{t-1,t+1} > 3$, there is an element $l \in L^\perp \subset H_{0,\mathbf{Q}}$ such that $\langle l, l\rangle_0 < 0$. Let L' be a $\mathbf{Q}$-subspace of $(L + \mathbf{Q}l)^\perp \subset H_{0,\mathbf{Q}}$ such that $L \oplus L' = (L+\mathbf{Q}l)^\perp$. Then we have $\dim_{\mathbf{Q}} L' = \dim_{\mathbf{Q}} H_{0,\mathbf{Q}} - 7$, and the restriction of $\langle\ ,\ \rangle_0$ to $L'_{\mathbf{C}}$ is non-degenerate. Fix polarized Hodge structures $(H_1, \langle\ ,\ \rangle_1, F_1)$ of weight w and $(H_2, \langle\ ,\ \rangle_2, F_2)$ of weight 1 having the following properties: The Hodge types $(h_j^{p,q})_{p,q\in\mathbf{Z}}$ of $(H_j, \langle\ ,\ \rangle_j, F_j)$ for $j = 1, 2$ are given by

$$h_1^{p,q} = h^{p,q} - \begin{cases} 2 & \text{if } (p,q) = (t,t), \\ 1 & \text{if } (p,q) = (t+1, t-1) \text{ or } (t-1, t+1), \\ 0 & \text{otherwise.} \end{cases}$$

$$h_2^{p,q} = \begin{cases} 1 & \text{if } (p,q) = (1,0) \text{ or } (0,1), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$(H_{1,\mathbf{Q}}, \langle\ ,\ \rangle_1) \simeq (L', \text{the restriction of } \langle\ ,\ \rangle_0).$$

Then there is an isomorphism

$$\begin{aligned} &\big(H_{\mathfrak{h},\mathbf{Q}} \otimes_{\mathbf{Q}} H_{2,\mathbf{Q}} \oplus \mathrm{Sym}^2_{\mathbf{Q}}(H_{\mathfrak{h},\mathbf{Q}}) \oplus H_{1,\mathbf{Q}}, \\ &\qquad \langle\ ,\ \rangle_{\mathfrak{h}} \otimes \langle\ ,\ \rangle_2 \oplus (-\langle l, l\rangle_0 \,\mathrm{Sym}^2(\langle\ ,\ \rangle_{\mathfrak{h}})) \oplus \langle\ ,\ \rangle_1\big) \\ &\simeq (H_{0,\mathbf{Q}}, \langle\ ,\ \rangle_0). \end{aligned}$$

We take this as an identification. Let (ρ, φ) be the SL(2)-orbit of rank 2 defined by

$$\begin{aligned} \rho(g_1, g_2) &:= \rho_{\mathfrak{h}}(g_1) \otimes 1_{H_2} \oplus \mathrm{Sym}^2(\rho_{\mathfrak{h}}(g_2)) \oplus 1_{H_1}, \\ \varphi(z_1, z_2) &:= (\varphi_{\mathfrak{h}}(z_1) \otimes F_2)(1-t) \oplus \mathrm{Sym}^2(\varphi_{\mathfrak{h}}(z_2))(1-t) \oplus F_1. \end{aligned}$$

Then this SL(2)-orbit of rank 2 does not satisfy the condition in 6.3 (i) (c). In fact, for any element $x \in H_2$, $(e_2 \otimes x, 0, 0) \in H(1,1)$ and $(0, e_2^2, 0) \in H(0,2)$. Q.E.D.

Example 6.11 and Proposition 6.12 below show that, for a subgroup Γ of $G_{\mathbf{Z}}$ of finite index, $\Gamma\backslash D_{\mathrm{SL}(2)}$ is not necessarily compact in general, and furthermore not necessarily locally compact in general.

6.11. *Example.* Consider the case $h^{5,0} = h^{0,5} = 1$ and $h^{p,q} = 0$ otherwise. This is satisfied by the polarized Hodge structure associated to a modular form of weight 6. In this case, D is identified with the upper half plane $\mathfrak{h}$, which is the Griffiths domain of the case $h^{1,0} = h^{0,1} = 1$ and $h^{p,q} = 0$ otherwise. We have $D_{\mathrm{BS}} = \mathfrak{h}_{\mathrm{BS}}$, but $D_{\mathrm{SL}(2)} = \mathfrak{h}$, as follows from the condition 3.1 (ii).

Propositon 6.12. *Assume one of the following* (i), (ii) *is satisfied for some* $t \in \mathbf{Z}$.

(i) $w = 2t+1$, $h^{t+1,t} \neq 0$, *and* $h^{s,w-s} \neq 0$ *for some* $s > t+1$.

(ii) $w = 2t$, $h^{t,t} \geq 2$, $h^{t+1,t-1} \geq 1$, $\sum_{j\geq 1} h^{t+j,t-j} \geq 2$, *and there is a* $\mathbf{Q}$*-vector subspace of* $H_{0,\mathbf{Q}}$ *of dimension* 2 *on which the restriction of* $\langle\ ,\ \rangle_0$ *is zero.*

Then $D_{\mathrm{SL}(2)}$ *is not locally compact. More precisely, there are an open set* U *of* D_{BS} *and* V *of* $D_{\mathrm{SL}(2)}$ *such that the inverse image* U' *of* U *in* $D_{\mathrm{BS,val}}$ *and the inverse image* V' *of* V *in* $D_{\mathrm{SL}(2),\mathrm{val}}$ *satisfy*

$$U' \xrightarrow{\sim} U, \quad V' \xrightarrow{\sim} V, \quad V' = U' \cap D_{\mathrm{SL}(2),\mathrm{val}},$$

and such that there are integers $m > l \geq 0$ *and a commutative diagram*

$$\begin{array}{ccc} U' & \simeq & \mathbf{R}^m \times \mathbf{R}_{\geq 0} \\ \cup & & \cup \\ V' & \simeq & (\mathbf{R}^m \times \mathbf{R}_{>0}) \cup (\mathbf{R}^l \times 0) \\ \cup & & \cup \\ U' \cap D & \simeq & \mathbf{R}^m \times \mathbf{R}_{>0}. \end{array}$$

Note that the subspace $(\mathbf{R}^m \times \mathbf{R}_{>0}) \cup (\mathbf{R}^l \times 0)$ *of* $\mathbf{R}^m \times \mathbf{R}_{\geq 0}$ *is not locally compact.*

Proof. Fix a $\mathbf{Q}$-rational $\mathbf{R}$-subspace L of $H_{0,\mathbf{R}}$ satisfying the following condition. In the case (i), $\dim L = 1$. In the case (ii), $\dim L = 2$ and $\langle\ ,\ \rangle_0$ is zero on L. Let P be the $\mathbf{Q}$-parabolic subgroup $\{g \in (G^\circ)_\mathbf{R} \mid gL = L\}$ of $G_\mathbf{R}$, and let W be the $\mathbf{Q}$-rational filtration of $H_{0,\mathbf{R}}$ defined by

$$W_{-2} := 0 \subset W_{-1} := L \subset W_0 := L^\perp \subset W_1 := H_{0,\mathbf{R}}.$$

Then we have

$$(G^\circ)_{W,\mathbf{R}} = P, \quad D_{\mathrm{BS,val}}(P) \cap D_{\mathrm{SL}(2),\mathrm{val}} = D_{\mathrm{SL}(2),\mathrm{val}}(W),$$
$$D_{\mathrm{BS,val}}(P) \xrightarrow{\sim} D_{\mathrm{BS}}(P), \quad D_{\mathrm{SL}(2),\mathrm{val}}(W) \xrightarrow{\sim} D_{\mathrm{SL}(2)}(W).$$

On the other hand, in the case (i), fix a polarized Hodge structure $(H_1, \langle\ ,\ \rangle_1, F_1)$ of weight w whose Hodge type $(h_1^{p,q})_{p,q\in\mathbf{Z}}$ is given by

$$h_1^{p,q} = h^{p,q} - \begin{cases} 1 & \text{if } (p,q) = (t+1,t) \text{ or } (t,t+1), \\ 0 & \text{otherwise.} \end{cases}$$

In the case (ii), fix polarized Hodge structures $(H_j, \langle\ ,\ \rangle_j, F_j)$ $(j = 1, 2)$ of weight w for $j = 1$ and of weight 1 for $j = 2$, respectively, having the following properties: The Hodge types $(h_j^{p,q})_{p,q\in\mathbf{Z}}$ of $(H_j, \langle\ ,\ \rangle_j, F_j)$ for

$j = 1, 2$ are given by

$$h_1^{p,q} = h^{p,q} - \begin{cases} 2 & \text{if } (p,q) = (t,t), \\ 1 & \text{if } (p,q) = (t+1, t-1) \text{ or } (t-1, t+1), \\ 0 & \text{otherwise.} \end{cases}$$

$$h_2^{p,q} = \begin{cases} 1 & \text{if } (p,q) = (1,0) \text{ or } (0,1), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$(H_{1,\mathbf{Q}}, \langle\ ,\ \rangle_1) \simeq (L', \text{the restriction of } \langle\ ,\ \rangle_0)$$

for some $\mathbf{Q}$-subspace L' of $H_{0,\mathbf{Q}}$ such that $L \oplus L'_{\mathbf{R}} = L^{\perp}$. Then we have an isomorphism

$$\begin{aligned} &(H_{0,\mathbf{Q}}, \langle\ ,\ \rangle_0) \\ &\simeq \begin{cases} (H_{\mathfrak{h},\mathbf{Q}} \oplus H_{1,\mathbf{Q}}, \langle\ ,\ \rangle_{\mathfrak{h}} \oplus \langle\ ,\ \rangle_1) & \text{in the case (i),} \\ (H_{\mathfrak{h},\mathbf{Q}} \otimes_{\mathbf{Q}} H_{2,\mathbf{Q}} \oplus H_{1,\mathbf{Q}}, \langle\ ,\ \rangle_{\mathfrak{h}} \otimes \langle\ ,\ \rangle_2 \oplus \langle\ ,\ \rangle_1) & \text{in the case (ii),} \end{cases} \end{aligned}$$

which sends L onto $\mathbf{Q}e_1 \oplus \{0\}$ in the case (i) (resp. $e_1 \otimes H_{2,\mathbf{Q}} \oplus \{0\}$ in the case (ii)) and L' onto $H_{1,\mathbf{Q}}$. Fix this isomorphism and take it as an identification. Let (ρ, φ) be the SL(2)-orbit defined by

$$\begin{cases} \rho(g) := \rho_{\mathfrak{h}}(g) \oplus 1_{H_1} \\ \varphi(z) := \varphi_{\mathfrak{h}}(z)(1-t) \oplus F_1 \end{cases} \qquad \text{in the case (i),}$$

$$\begin{cases} \rho(g) := \rho_{\mathfrak{h}}(g) \otimes 1_{H_2} \oplus 1_{H_1}, \\ \varphi(z) := (\varphi_{\mathfrak{h}}(z) \otimes F_2)(1-t) \oplus F_1, \end{cases} \qquad \text{in the case (ii).}$$

We claim

$$(1) \qquad D_{\mathrm{SL}(2),\mathrm{val}}(W) = D \cup P[\rho, \varphi] \quad \text{in } D_{\mathrm{BS},\mathrm{val}}(P).$$

In fact, let (ρ', φ') be an SL(2)-orbit of rank 1 whose weight filtration is W. Since $\tilde{\rho}$ and $\tilde{\rho}'$ split W, there is an element $p \in P_u$ such that $\tilde{\rho}' = \mathrm{Int}(p)\tilde{\rho}$. The Hodge types $\varphi(i)(\mathrm{gr}_j^W)$ and of $\varphi'(i)(\mathrm{gr}_j^W)$ coincide for each j. (In the case (i) (resp. (ii)), it is $(t+1, t+1)$ (resp. $(t+1, t) + (t, t+1)$) for $j = 1$, (t, t) (resp. $(t, t-1) + (t-1, t)$) for $j = -1$, and $(h_1^{p,q})$ for $j = 0$.) Hence by [U1, 3.16 (iii)], there is an element $q \in (G^{\circ})_{\mathbf{R}}$ which commutes with $\tilde{\rho}' = \mathrm{Int}(p)\tilde{\rho}$ and satisfies $\varphi'(\mathbf{i}) = qp\varphi(\mathbf{i})$. By 3.10, we have $\rho' = \mathrm{Int}(qp)\rho$, $\varphi' = qp\varphi$. Since $qp \in P$, this proves (1).

Let D' be the subspace of D defined in 2.16 with respect to the maximal compact subgroup $K_{\mathbf{r}}$ of $G_{\mathbf{R}}$. Then, (1) shows that we have a commutative diagram of topological spaces

$$\begin{array}{ccccc} D_{\mathrm{BS,val}}(P) & = & D_{\mathrm{BS}}(P) & \simeq & D' \times \mathbf{R}_{\geq 0} \\ \cup & & & & \cup \\ D_{\mathrm{SL}(2),\mathrm{val}}(W) & = & D_{\mathrm{SL}(2)}(W) & \simeq & (D' \times \mathbf{R}_{>0}) \cup ({}^{\circ}P \cdot \mathbf{r} \times 0), \end{array}$$

where ${}^{\circ}P$ is as in 2.16 and $[\rho, \varphi] \in D_{\mathrm{SL}(2)}(W)$ corresponds to $\mathbf{r} \times 0$. Let

$$m := \dim D' = \dim D - 1,$$
$$l := \dim({}^{\circ}P \cdot \mathbf{r}) = \dim(P \cdot \mathbf{r}) - 1.$$

By [U2, 3.12 (iii)], we have $\dim(P \cdot \mathbf{r}) < \dim D$ under the assumption of Proposition 6.12. This proves that the statement of 6.12 holds for some neighborhood U of the image of $[\rho, \varphi]$ in D_{BS}. Q.E.D.

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