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Integral p-adic Hodge Theory

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Abstract.

We give an overview of some questions and results in integral p-adic Hodge theory. A few proofs are supplied.

CONTENTS

§0. Introduction

Fix a prime number p, an algebraic closure $\overline{\mathbf{Q}}_p$ of the field $\mathbf{Q}_p$ of p-adic numbers with ring of integers $\overline{\mathbf{Z}}_p$, and a finite extension F of $\mathbf{Q}_p$ inside $\overline{\mathbf{Q}}_p$ with ring of integers $\mathcal{O}_F$.

Roughly speaking, p-adic Hodge theory (over F) is the study of de Rham and p-adic étale cohomologies of (proper smooth) schemes over F. The research for relations between these two cohomology groups gave birth to Fontaine's theory of semi-stable (and potentially semi-stable) p-adic representations of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/F)$ and, in the course of time, p-adic

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Hodge theory also included the study of these Galois representations.

Integral p-adic Hodge theory could be today defined as the study of Galois stable $\mathbf{Z}_p$-*lattices* in semi-stable p-adic representations together with their links with the various *integral* p-adic cohomologies of proper smooth schemes over F. Integral p-adic Hodge theory gives back classical p-adic Hodge theory (by inverting p), but it also gives rise to completely new characteristic p phenomena (by reducing modulo p). Thus, it is richer than p-adic Hodge theory. It is also much more complicated. Although p-adic Hodge theory is now mostly complete (by the work of many people including Tate, Raynaud, Grothendieck, Bloch, Messing, Fontaine, Colmez, Faltings, Kato, Hyodo, Tsuji...), integral p-adic Hodge theory is far from being as well understood and there remains a great deal to be found before one has a complete theory. Of course, if such a theory exists, it should also contain all the results of p-adic Hodge theory.

This text tries to "take stock" of the situation of integral p-adic Hodge theory so far, although it certainly couldn't pretend to be fully exhaustive. It has two aims: the first is to give the best possible conjectures and results up to now, the second is to prove the minor things which were not already proved in order to state these results. It is organized as follows. In section 1, we recall the basic definitions and results on semi-stable p-adic representations. The key role here is played by weakly admissible filtered (φ, N)-modules. In section 2, we give a conjectural description of Galois stable lattices in semi-stable p-adic representations with small Hodge-Tate weights and we explain the known cases of this conjecture. The idea is to define integral structures also on the filtered modules side called strongly divisible lattices (or strongly divisible modules). In section 3, we prove the case where the Hodge-Tate weights are between 0 and 1. The crystalline case is a consequence of a link between strongly divisible modules (in that case) and p-divisible groups over $\mathcal{O}_F$ (Cor. 3.2.4). To deduce the semi-stable case (§3.5), we need a technical result on Galois representations arising from finite flat group schemes that we prove via the theory of norm fields (Theorem 3.4.3). In section 4, we consider the "higher weight" cases and give a link between strongly divisible modules and some cohomology groups H^m's with $m < p-1$. Finally, in section 5, using strongly divisible lattices we compute the reduction modulo p of Galois stable $\mathbf{Z}_p$-lattices in some two dimensional semi-stable p-adic representations and show how variable this reduction can be.

We have restricted ourselves to finite extensions F of $\mathbf{Q}_p$ mainly for simplicity. All the statements of this paper, except those of section 5, should hold *verbatim* for any complete local field of characteristic 0 with perfect residue field of characteristic p.

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§1. Review of semi-stable p-adic representations

Let $\overline{\mathbf{F}}_p$ be the residue field of $\overline{\mathbf{Z}}_p$ (an algebraic closure of the finite field $\mathbf{F}_p$) and $\mathbf{F} \subset \overline{\mathbf{F}}_p$ the residue field of F. Let $f := [\mathbf{F} : \mathbf{F}_p]$ and $e := [F : F_0]$ where $F_0 \subset F$ is the maximal unramified subfield of F. We write G_F for $\mathrm{Gal}(\overline{\mathbf{Q}}_p/F)$ and σ for the arithmetic Frobenius on F_0. If ℓ is any prime number, an ℓ-adic representation of G_F is, by definition, a continuous linear representation of G_F on a finite dimensional $\mathbf{Q}_\ell$-vector space V.

Definition 1.1 ([14]). A p-adic representation V of G_F is called semi-stable if:

$$\dim_{F_0}(B_{st} \otimes_{\mathbf{Q}_p} V)^{G_F} = \dim_{\mathbf{Q}_p} V.$$

Here, B_{st} is Fontaine's ring of p-adic periods defined in [13] (see also [20]). It is endowed with an action of G_F. The exponent G_F on the left hand side means we take the elements of $B_{st} \otimes_{\mathbf{Q}_p} V$ which are fixed by G_F. If V is *any* p-adic representation of G_F, one has only an inequality $\dim_{F_0}(B_{st} \otimes_{\mathbf{Q}_p} V)^{G_F} \leq \dim_{\mathbf{Q}_p} V$ ([14]).

Definition 1.1 is not very explicit. Fortunately, a recent result of Colmez and Fontaine ([10]) gives an alternative description of semi-stable p-adic representations which is very explicit and useful. Define a filtered (φ, N)-module to be a finite dimensional F_0-vector space D endowed with:
- a σ-linear injective map $\varphi : D \to D$ (the "Frobenius")
- a linear map $N : D \to D$ such that $N\varphi = p\varphi N$ (the "monodromy")
- a decreasing filtration $(\mathrm{Fil}^i D_F)_{i \in \mathbf{Z}}$ on $D_F := F \otimes_{F_0} D$ by F-vector subspaces such that $\mathrm{Fil}^i D_F = D_F$ for $i \ll 0$ and $\mathrm{Fil}^i D_F = 0$ for $i \gg 0$.

The conditions on φ and N imply that N is nilpotent. Let D be a filtered (φ, N)-module and define:

• $t_H(D) := \sum_{i\in\mathbf{Z}} i\dim_F \mathrm{gr}^i D_F$ where $\mathrm{gr}^i D_F = \mathrm{Fil}^i D_F/\mathrm{Fil}^{i+1} D_F$

• $t_N(D) := \sum_{\alpha\in\mathbf{Q}} \alpha\dim_{\overline{\mathbf{Q}}_p} \overline{D}_\alpha$ where $\overline{D}_\alpha$ is the sum of the characteristic subspaces of $\overline{\mathbf{Q}}_p \otimes_{F_0} D$ for the eigenvalues of $Id \otimes \varphi^f$ having valuation α (here the valuation is normalized so that p^f has valuation 1).

It is clear that $t_H(D) \in \mathbf{Z}$ and one can prove $t_N(D) \in \mathbf{Z}$ (see e.g. [1]). By definition a filtered (φ, N)-submodule of D is a filtered (φ, N)-module D' equipped with an injection $D' \hookrightarrow D$ that commutes with φ and N and for which $\mathrm{Fil}^i D'_F = D'_F \cap \mathrm{Fil}^i D_F$.

Definition 1.2 ([14]). A filtered (φ, N)-module D is weakly admissible if $t_H(D) = t_N(D)$ and if $t_H(D') \leq t_N(D')$ for any filtered (φ, N)-submodule D' of D.

If V is a semi-stable p-adic representation of G_F, one can prove that the F_0-vector space $D_{st}(V) := (B_{st} \otimes_{\mathbf{Q}_p} V)^{G_F}$ is a weakly admissible filtered (φ, N)-module in a natural, although not quite canonical, way (see [13]). The aforementioned result of Colmez and Fontaine is:

Theorem 1.3 ([10]). *The functor $D_{st} : V \mapsto (B_{st} \otimes_{\mathbf{Q}_p} V)^{G_F}$ establishes an equivalence of categories between the category of semi-stable p-adic representations of G_F and the category of weakly admissible filtered (φ, N)-modules.*

Note that the functor D_{st} is not canonical since it depends on a filtration on $F \otimes_{F_0} B_{st}$ (or equivalently of an embedding $F \otimes_{F_0} B_{st} \hookrightarrow B_{dR}$ since the filtration is induced via such an embedding by the filtration on B_{dR}) which itself depends on the choice of a uniformizer π in F. When $N = 0$ on $D_{st}(V)$, V is said to be crystalline and in that case $D_{st}(V)$ is independant of any choice.

In the sequel, we will instead use the contravariant functor $D^*_{st}(V) := D_{st}(V^*)$, where V^* is the dual representation of V (crystalline/semi-stable if and only if V is). The reason for this is that the Hodge-Tate weights of V are exactly the $i \in \mathbf{Z}$ such that $\mathrm{gr}^i D^*_{st}(V)_F \neq 0$ (with D_{st}, it would be the $-i$ such that $\mathrm{gr}^i D_{st}(V)_F \neq 0$, see [14]). A quasi-inverse to D^*_{st} is then given by:

$$V^*_{st}(D) := \mathrm{Hom}_{\varphi,N}(D, B_{st}) \cap \mathrm{Hom}_{\mathrm{Fil}^\cdot}(D_F, F \otimes_{F_0} B_{st}),$$

that is to say the $\mathbf{Q}_p$-vector space of F_0-linear maps $f : D \to B_{st}$ being compatible with all the structures (G_F acting by $(g \cdot f)(x) := g(f(x))$).

We will use this quasi-inverse in the sequel.

To finish this section, we remind the reader that a description similar to 1.3 also exists for semi-stable ℓ-adic representations of G_F with $\ell \neq p$ (i.e. ℓ-adic representations such that the inertia acts unipotently) and that it is essentially trivial: they are described by finite dimensional $\mathbf{Q}_\ell$-vector spaces endowed with a continuous linear action of $\mathrm{Gal}(F^{nr}/F)$ (which plays the role of the Frobenius) and with a nilpotent endomorphism N (the monodromy) such that $N\varphi = p^f \varphi N$ where φ is the geometric Frobenius of $\mathrm{Gal}(F^{nr}/F)$ and F^{nr} the maximal unramified extension of F inside $\overline{\mathbf{Q}}_p$. Recall that $g = \exp(Nt_\ell(g))$ if $g \in I_F := \mathrm{Gal}(\overline{\mathbf{Q}}_p/F^{nr})$ and $t_\ell : I_F \to \mathbf{Z}_\ell(1) \simeq \mathbf{Z}_\ell$ is the tame ℓ-component of I_F.

§2. Lattices in semi-stable representations with low Hodge-Tate weights

On the side of p-adic representations of G_F, there is an obvious integral structure, namely the $\mathbf{Z}_p$-lattices that are preserved by the action of G_F (which always exist because G_F is compact). Thus, granting Theorem 1.3, one can ask whether there also exists a corresponding integral structure on the filtered module side.

2.1. Basic assumptions

Let us first examine the ℓ-adic situation. Let V be a semi-stable ℓ-adic representation of G_F and D the associated $(\mathrm{Gal}(F^{nr}/F), N)$-vector space defined at the end of the previous section. *If* $N^\ell = 0$, there are nice integral structures on D that correspond to G_F-stable lattices in V, namely the $\mathbf{Z}_\ell$-lattices in D that are preserved by $\mathrm{Gal}(F^{nr}/F)$ and N. But if $N^\ell \neq 0$, this doesn't work anymore because we cannot use the operators $\frac{N^i}{i!}$ when $i \geq \ell$ to rebuild the unipotent action of inertia on the Galois side (and in that case, one usually works directly with Galois lattices). As the p-adic side is much more involved than the ℓ-adic one, one can expect to need, *at least*, the assumption $N^p = 0$ on D .

Definition 2.1.1. (1) A weakly admissible filtered (φ, N)-module D such that $\mathrm{Fil}^0 D_F = D_F$, $\mathrm{Fil}^m D_F \neq 0$, and $\mathrm{Fil}^{m+1} D_F = 0$ (for some $m \in \mathbf{N}$) has no m-component if, inside the abelian category of weakly admissible modules, D has no non zero weakly admissible quotient $\overline{D}$ such that $\mathrm{Fil}^m \overline{D}_F = \overline{D}_F$.
(2) A semi-stable p-adic representation V of G_F with Hodge-Tate weights between 0 and m has no m-component if $D^*_{st}(V)$ has no m-component.

Note that filtered (φ, N)-modules arising from unipotent p-divisible groups over $\mathcal{O}_F$ (i.e. p-divisible groups with connected Cartier dual) have no 1-component.

It turns out that in the p-adic setting, one is naturally led to either the hypothesis:

Basic Assumption 2.1.2. *Either the Hodge-Tate weights of the semi-stable p-adic representation V are between 0 and m with $m < p-1$ or they are between 0 and $p-1$ and V has no $p-1$-component.*

or its equivalent filtered variant:

Basic Assumption 2.1.3. *Either the filtration on the weakly admissible filtered module D is such that* $\mathrm{Fil}^0 D_F = D_F$ *and* $\mathrm{Fil}^{m+1} D_F = 0$ *with $m < p-1$ or it is such that* $\mathrm{Fil}^0 D_F = D_F$ *and* $\mathrm{Fil}^p D_F = 0$ *and D has no $p-1$-component.*

Equivalently, one could just say $\mathrm{Fil}^{p-1} D_F = 0$ in the first case of 2.1.3, but it's convenient to have an integer m as in 2.1.2 and 2.1.3. Twisting by the cyclotomic character, one could also weaken Assumption 2.1.2 (resp. 2.1.3) to just require that the *difference* between the extreme Hodge-Tate weights (resp. the *length* of the filtration) is smaller than m. Without assumption on m, it is not yet known how Galois lattices can be described in general in terms of integral structures on the filtered (φ, N)-modules. The link with our ℓ-adic prelude is provided by:

Lemma 2.1.4. *Let D be a weakly admissible filtered (φ, N)-module such that* $\mathrm{Fil}^0 D_F = D_F$ *and* $\mathrm{Fil}^p D_F = 0$. *Then $N^p = 0$ on D.*

Proof. Let $P_H(D)$ (resp. $P_N(D)$) be the Hodge (resp. Newton) polygon associated to D, i.e. the convex polygon such that the part of slope $i \in \mathbf{N}$ (resp. $\alpha \in \mathbf{Q}^+$) is of length $\dim_F \mathrm{gr}^i D_F$ (resp. $\dim_{\overline{\mathbf{Q}}_p} \overline{D}_\alpha$, see §1). The weak admissibility condition implies that $P_H(D)$ lies under $P_N(D)$ and that they have the same endpoints (see [15]). From the corresponding drawing and the assumptions on D, one must have $\alpha \leq p-1$ if $\overline{D}_\alpha \neq 0$ (since $p-1$ is the highest possible slope on $P_H(D)$). But from $N\varphi = p\varphi N$, we get $N(\overline{D}_\alpha) \subset \overline{D}_{\alpha-1}$ if $\alpha \geq 1$ and $N(\overline{D}_\alpha) = 0$ otherwise. Thus, $N^p(\overline{D}_\alpha) = 0$ for all α, i.e. $N^p = 0$. Q.E.D.

Note that Assumptions 2.1.2 or 2.1.3 here are really *stronger* than just $N^p = 0$ (for instance, in the crystalline case, $N = 0$ but m can be arbitrary).

2.2. Strongly divisible modules

In this section, we define integral structures for filtered (φ, N)-modules satisfying Assumption 2.1.3 and we state the main conjecture.

From now on, we fix a uniformizer π in F and denote by $E(u)$ its minimal polynomial (an Eisenstein polynomial of degree e). Let S be the p-adic completion of $W(\mathbf{F})[u, \frac{u^{ie}}{i!}]_{i\in\mathbf{N}}$ where u is an indeterminate and endow S with the following structures:
• a continuous σ-linear Frobenius still denoted $\sigma : S \to S$ such that $\sigma(u) = u^p$
• a continuous linear derivation $N : S \to S$ such that $N(u) = -u$
• a decreasing filtration $(\mathrm{Fil}^i S)_{i\in\mathbf{N}}$ where $\mathrm{Fil}^i S$ is the p-adic completion of $\sum_{j\geq i} S\frac{E(u)^j}{j!}$ (one checks $\frac{E(u)^j}{j!} \in S$).
Note that $N\sigma = p\sigma N$, $N(\mathrm{Fil}^{i+1}S) \subset \mathrm{Fil}^i S$ for $i \in \mathbf{N}$ and $\sigma(\mathrm{Fil}^i S) \subset p^i S$ for $i \in \{0, ..., p-1\}$.

Let D be a weakly admissible filtered (φ, N)-module and assume that $\mathrm{Fil}^0 D_F = D_F$. Let:

$$\mathcal{D} := S \otimes_{W(\mathbf{F})} D$$

and define :
• $\varphi := \sigma \otimes \varphi : \mathcal{D} \to \mathcal{D}$
• $N := N \otimes Id + Id \otimes N : \mathcal{D} \to \mathcal{D}$
• $\mathrm{Fil}^0\mathcal{D} := \mathcal{D}$ and, by induction:

$$\mathrm{Fil}^{i+1}\mathcal{D} := \{x \in \mathcal{D} \mid N(x) \in \mathrm{Fil}^i\mathcal{D} \text{ and } f_\pi(x) \in \mathrm{Fil}^{i+1}D_F\}$$

where $f_\pi : \mathcal{D} \twoheadrightarrow D_F$ is defined by $s(u) \otimes x \mapsto s(\pi)x$.

One can show the map f_π induces surjections $\mathrm{Fil}^i\mathcal{D} \twoheadrightarrow \mathrm{Fil}^i D_F$ ([8]). The filtered module $\mathcal{D}$ has the advantage over the filtered module D that all of its data are defined at the same level (no need to extend scalars to F). Moreover, one can prove that the knowledge of $\mathcal{D}$ is equivalent to that of D ([8]). It turns out the integral structures will naturally live inside the $\mathcal{D}$'s. But first, we note that there is a "B_{st}-counterpart" to this construction, i.e. there is a "period S-algebra", first introduced by Kato in [22] and that the author named $\widehat{B_{st}}$, such that the couple $(B_{st}, \widehat{B_{st}})$ is somewhat analogous to the couple $(D, \mathcal{D})$. More precisely, if t denotes Fontaine's analogue of $2\pi i$ (see [13]), then $\widehat{B_{st}} = \widehat{A_{st}}[1/t]$ where $\widehat{A_{st}}$ is (non canonically) isomorphic to the p-adic completion of

$A_{cris}[X, \frac{X^i}{i!}]_{i\in \mathbf{N}}$. Here, A_{cris} is the integral version of B_{cris} ([13]), X is an indeterminate, and $A_{cris}[X, \frac{X^i}{i!}]_{i\in \mathbf{N}}$ is an S-module via the map $u \mapsto [\underline{\pi}](1+X)^{-1}$ where $[\underline{\pi}]$ is a specific element of A_{cris} made out of a compatible system of p^n-th roots of π in $\overline{\mathbf{Q}}_p$. See [3] for details, where it is also explained how to endow $A_{cris}[X, \frac{X^i}{i!}]_{i\in \mathbf{N}}$ with a continuous action of G_F (which is non trivial on X), Frobenius and monodromy maps, and a decreasing filtration, with all of these structures inducing the previous structures on S, the usual structures on A_{cris}, and ultimately only depending, up to isomorphism, on the choice of π and not on any other choice.

Now, let us go back to the initial problem of defining integral structures:

Definition 2.2.1. Let D be a weakly admissible filtered (φ, N)-module such that $\mathrm{Fil}^0 D_F = D_F$ and $\mathrm{Fil}^{m+1} D_F = 0$ with $m < p$. A strongly divisible lattice (or module) in $\mathcal{D}$ is an S-submodule $\mathcal{M}$ of $\mathcal{D}$ such that:
(1) $\mathcal{M}$ is free of finite rank over S and $\mathcal{M}[\frac{1}{p}] \xrightarrow{\sim} \mathcal{D}$
(2) $\mathcal{M}$ is stable under φ and N
(3) $\varphi(\mathrm{Fil}^m \mathcal{M}) \subset p^m \mathcal{M}$ where $\mathrm{Fil}^m \mathcal{M} := \mathcal{M} \cap \mathrm{Fil}^m \mathcal{D}$.

One can show this definition doesn't depend on m (provided of course $\mathrm{Fil}^{m+1} D_F = 0$ and $m < p$). Using the weak admissibility of D, one can also show that condition (3) in Definition 2.2.1 is actually equivalent to the apparently stronger condition that $\varphi(\mathrm{Fil}^m \mathcal{M})$ spans $p^m \mathcal{M}$ over S (see §2.1 of [3]).

Examples 2.2.2. (1) Let D be the trivial filtered module (i.e. $D = F_0$ with $\mathrm{Fil}^1 D_F = 0$, $N = 0$ and $\varphi = \sigma$). Then S is a strongly divisible lattice in $\mathcal{D} = S[\frac{1}{p}]$.

(2) Let D be as in 2.2.1 and assume $F = F_0 = W(\mathbf{F})[1/p]$ and $N = 0$. Recall ([17]) that a strongly divisible module in the sense of Fontaine and Laffaille is a $W(\mathbf{F})$-lattice M in D such that $\varphi(\mathrm{Fil}^i M) \subset p^i M$ for all $i \in \mathbf{N}$ where $\mathrm{Fil}^i M := M \cap \mathrm{Fil}^i D$. As previously, because D is weakly admissible, this is equivalent to $M = \sum_i \frac{\varphi}{p^i}(\mathrm{Fil}^i M)$. Let $\mathcal{M} := S \otimes_{W(\mathbf{F})} M \subset \mathcal{D}$, then $\mathcal{M}$ is a strongly divisible lattice in $\mathcal{D}$ in the sense of 2.2.1.

(3) Assume $F = F_0$ and let $D = F_0 e_1 \oplus F_0 e_2$ with $\varphi(e_1) = p^r e_1$, $\varphi(e_2) = p^{r-1} e_1$ ($r \in \mathbf{N}$, $2r \leq (p-1)$), $N(e_1) = e_2$, $N(e_2) = 0$, $\mathrm{Fil}^i D =$

$F_0(e_1+\mathcal{L}e_2)$ if $1\leq i\leq 2r-1$ ($\mathcal{L}\in W(\mathbf{F})$) and $\mathrm{Fil}^iD=0$ if $i\geq 2r$. Then one can check that $Se_1\oplus S(e_2/p)$ is a strongly divisible lattice in $\mathcal{D}$.

(4) Assume $F=F_0(\pi)$ with $\pi^{p-1}=-p$ and let $D=F_0e_1\oplus F_0e_2$ with $\varphi(e_1)=pe_1$, $\varphi(e_2)=pe_2$, $N=0$, $\mathrm{Fil}^1D_F=\mathrm{Fil}^2D_F=F(e_1+\pi e_2)$, $\mathrm{Fil}^iD_F=0$ if $i\geq 3$ and assume $p\geq 5$. Then one can check that $Se_1\oplus S\big(e_2+\frac{u^{p(p-2)}}{pU}e_1\big)$ is a strongly divisible lattice in $\mathcal{D}$ where $U=\frac{p-2}{p-1}\big(\frac{u^{p(p-1)}}{p}+1\big)-1\in S^\times$.

For $m\in\mathbf{N}$ consider the category $\mathcal{C}_m$ of S-modules $\mathcal{M}$ endowed with a σ-linear endomorphism φ, a $W(\mathbf{F})$-linear endomorphism N satisfying $N(sx)=N(s)x+sN(x)$ ($s\in S$, $x\in\mathcal{M}$), and an S-submodule $\mathrm{Fil}^m\mathcal{M}$, with morphisms being S-linear maps that preserve Fil^m and commute with φ and N. For $m<p$, we define the category of *strongly divisible modules of weight* $\leq m$ as the full subcategory of $\mathcal{C}_m$ consisting of objects that are isomorphic to a strongly divisible module in some $S\otimes_{W(\mathbf{F})}D$ for D weakly admissible as in 2.2.1. It turns out one can directly describe this category:

Theorem 2.2.3. *The category of strongly divisible modules of weight* $\leq m$ ($m<p$) *is the full subcategory of* $\mathcal{C}_m$ *of objects* $\mathcal{M}$ *satisfying the following conditions*:
(1) $\mathcal{M}$ *is free of finite rank over* S
(2) $(\mathrm{Fil}^mS)\mathcal{M}\subset\mathrm{Fil}^m\mathcal{M}$
(3) $\mathrm{Fil}^m\mathcal{M}\cap p\mathcal{M}=p\mathrm{Fil}^m\mathcal{M}$
(4) $\varphi(\mathrm{Fil}^m\mathcal{M})$ *spans* $p^m\mathcal{M}$
(5) $N\varphi=p\varphi N$
(6) $(\mathrm{Fil}^1S)N(\mathrm{Fil}^m\mathcal{M})\subset\mathrm{Fil}^m\mathcal{M}$.

The point is to prove that $\mathcal{M}[1/p]\simeq S\otimes_{W(\mathbf{F})}D$ for a (unique) filtered (φ,N)-module D and that this D is weakly admissible. This is done in [8] and [3] for $m<p-1$ but the proof readily extends to the case $m<p$. Of course, when m grows, these categories are full subcategories one of the other.

Definition 2.2.4. A strongly divisible module of weight $\leq m$ has no m-component if the corresponding weakly admissible D has no m-component (cf. 2.1.1).

To a strongly divisible module $\mathcal{M}$ of weight $\leq m$ one can associate the $\mathbf{Z}_p[G_F]$-module:

$$T^*_{st}(\mathcal{M}):=\mathrm{Hom}_{S,\varphi,N,\mathrm{Fil}^m}(\mathcal{M},\widehat{A_{st}})$$

where one considers S-linear maps from $\mathcal{M}$ to $\widehat{A_{st}}$ that commute with φ, N and preserve Fil^m (this doesnt depend on $m < p$ such that $\mathcal{M}$ is of weight $\leq m$). The group G_F acts by $(g \cdot f)(x) := g(f(x))$.

Proposition 2.2.5. *Let $\mathcal{M}$ be a strongly divisible module of weight $\leq m$ $(m < p)$ and D the corresponding weakly admissible filtered (φ, N)-module. Then $T^*_{st}(\mathcal{M})$ is a Galois stable $\mathbf{Z}_p$-lattice in $V^*_{st}(D)$ (see §1 for V^*_{st}).*

Proof. We only give a sketch here and refer the reader to [3] or [8] for details. Let $\mathcal{D} := \mathcal{M}[1/p] = S \otimes_{W(\mathbf{F})} D$. As $T^*_{st}(\mathcal{M})$ is clearly a Galois stable $\mathbf{Z}_p$-lattice in $V^*_{st}(\mathcal{D}) := \mathrm{Hom}_{S,\varphi,N,\mathrm{Fil}^m}(\mathcal{D}, \widehat{A_{st}}[1/p])$, the real issue is to prove that $V^*_{st}(\mathcal{D})$ is isomorphic as a Galois representation to $V^*_{st}(D)$. Note first that an S-linear map $f : \mathcal{D} \to \widehat{A_{st}}[1/p]$ preserves Fil^m if and only if it preserves Fil^i for $0 \leq i \leq m$. There is a ring morphism commuting with G_F and compatible with the filtration $\widehat{A_{st}}[1/p] \to B_{dR}$, $X \mapsto \frac{[\underline{\pi}]}{\pi} - 1$ where $[\underline{\pi}]$ is the "specific" element of A_{cris} previously mentionned. Using that $D = \{x \in \mathcal{D} \mid N^n(x) = 0 \text{ for some } n \in \mathbf{N}\}$, one gets any $f \in V^*_{st}(\mathcal{D})$ sends D to $B^+_{cris}[\log(1+X)] \subset \widehat{A_{st}}[1/p]$. Composing with the above ring morphism and using the surjectivity of $\mathrm{Fil}^i\mathcal{D} \to \mathrm{Fil}^i D_F$, one ends up with an F_0-linear map $\overline{f} : D \to B^+_{st} \subset B_{dR}$ that commutes with φ and N, preserves the filtration after extending scalars to F and is such that the diagram:

$$\begin{array}{ccccc}
\mathcal{D} & & \xrightarrow{f} & & \widehat{A_{st}}[1/p] \\
f_\pi \downarrow & & & & \downarrow \\
D_F & \overset{1\otimes\overline{f}}{\to} & F \otimes_{F_0} B^+_{st} & \hookrightarrow & B_{dR}
\end{array}$$

commutes. This gives an injective $\mathbf{Q}_p$-linear map $V^*_{st}(\mathcal{D}) \to V^*_{st}(D)$ which is easily checked to be surjective. Q.E.D.

Our main conjecture is:

Conjecture 2.2.6. (1) If $m < p - 1$, the functor $\mathcal{M} \mapsto T^*_{st}(\mathcal{M})$ establishes an anti-equivalence of categories between the category of strongly divisible modules of weight $\leq m$ and the category of G_F-stable lattices in semi-stable representations of G_F with Hodge-Tate weights in $\{0, ..., m\}$.
(2) If $m = p-1$, the functor $\mathcal{M} \mapsto T^*_{st}(\mathcal{M})$ establishes an anti-equivalence of categories between the category of strongly divisible modules of weight $\leq p - 1$ that have no $p - 1$-component and the category of G_F-stable lattices in semi-stable representations of G_F with Hodge-Tate weights in $\{0, ..., p - 1\}$ that have no $p - 1$-component.

In particular, if V is a semi-stable p-adic representation of G_F with Hodge-Tate weights in $\{0, ..., m\}$ (without m-component if $m = p-1$), then Galois lattices in V should *exactly* correspond to strongly divisible modules in the associated $S \otimes_{W(\mathbf{F})} D$. The following theorem summarizes the known cases of conjecture 2.2.6:

Theorem 2.2.7. *Conjecture* 2.2.6 *is true in the following two cases*:
(1) $m < p-1$ *and* $e = 1$
(2) $m = 0$ *or* $m = 1$ *and* $p \neq 2$.

Case (1) is proven in [4] using results of [3]. The method is a generalization of that of Fontaine and Laffaille who did the subcase $m < p-1$, $e = 1$, $N = 0$ ([17]). At the time of [17], the ring S and S-modules like $\mathcal{D}$ and $\mathcal{M}$ were not yet defined, but in that case one can manage with $W(\mathbf{F})$-lattices only, namely those lattices defined in Example (2) of 2.2.2. In the other cases, one can not dispense with S, which makes the theory much more complicated, even when $e = 1$. Case (2) is proven in the next section using the theory of p-divisible groups.

There are two other partial results in the direction of 2.2.6. The first is that if $em < p-1$, then $\mathcal{D}$ at least *always* contains a strongly divisible lattice ([3]). The second is that for $m < p-1$ the restriction of T^*_{st} to the subcategory of "filtered-free" strongly divisible modules of weight $\leq m$ is *fully faithful* ([12]). Here, by filtered free, we mean there is a *basis* $(e_i)_{1 \leq i \leq d}$ of the underlying S-module $\mathcal{M}$ and integers $0 \leq r_1 \leq ... \leq r_d \leq m$ such that:

$$\mathrm{Fil}^m \mathcal{M} = \Big(\bigoplus_i E(u)^{r_i} S e_i\Big) + (\mathrm{Fil}^m S)\mathcal{M}.$$

Unfortunately, most of the strongly divisible modules are not filtered free, but they are if $m \leq 1$ ([5]) or if $e = 1$ and $N = 0$ ([17]). In particular, the full faithfulness of T^*_{st} in case (2) of 2.2.7 was thus proven in [12] (we will derive it below from Tate's full faithfulness theorem).

§3. Finite flat group schemes, p-divisible groups, and norm fields

In this section, we prove statement (2) of Theorem 2.2.7. We first deal with the case of lattices in crystalline representations using results on p-divisible groups (§3.2). Then we derive the general case using the theory of norm fields (§3.3, §3.4 and §3.5).

3.1. The case $m = 0$

This is the case of unramified p-adic representations of G_F. Define the category of *étale* $(\varphi, W(\mathbf{F}))$*-modules* as the category of free $W(\mathbf{F})$-modules of finite rank equipped with a bijective σ-linear endomorphism φ. Then it has long been known (see [16] for instance) that the functor $M \mapsto \mathrm{Hom}_{W(\mathbf{F}),\varphi}(M, W(\overline{\mathbf{F}}_p))$ establishes an anti-equivalence of categories between étale $(\varphi, W(\mathbf{F}))$-modules and G_F-stable lattices in unramified p-adic representations of G_F.

View $W(\mathbf{F})$ as an S-module by sending u and its divided powers to 0. To a strongly divisible module $\mathcal{M}$ of weight 0, one can associate $M := \mathcal{M} \otimes_S W(\mathbf{F})$ and endow it with the image of φ (the image of N being 0). It is clear M is then an étale $(\varphi, W(\mathbf{F}))$-module. Statement (2) of 2.2.7 in the case $m = 0$ comes down to:

Proposition 3.1.1. *The functor $\mathcal{M} \mapsto \mathcal{M} \otimes_S W(\mathbf{F})$ establishes an equivalence of categories between strongly divisible modules of weight* 0 *and étale* $(\varphi, W(\mathbf{F}))$*-modules.*

This is the well-known "Dwork's trick" ([21]) in a divided power context.

3.2. Classification of group schemes and consequences

From now on, we assume $m = 1$ and $p \neq 2$. We connect some of the strongly divisible modules of weight ≤ 1 to p-divisible groups over $\mathcal{O}_F$.

As with Galois lattices, it is tempting, using the alternative definition of strongly divisible modules given by Theorem 2.2.3, to reduce strongly divisible modules of weight $\leq m$ modulo arbitrary powers of p. For $m = 1$, we are led to the following category $\underline{\mathcal{M}}_0^1$.

An object of $\underline{\mathcal{M}}_0^1$ is a triple $(\mathcal{M}, \mathrm{Fil}^1\mathcal{M}, \varphi_1)$ where:
(1) $\mathcal{M}$ is an S-module of finite type isomorphic to $\oplus_{n \in \mathbf{Z}_{>0}}(S/p^nS)^{r_n}$ for integers r_n which are almost all equal to 0
(2) $\mathrm{Fil}^1\mathcal{M}$ is an S-submodule of $\mathcal{M}$ containing $(\mathrm{Fil}^1S)\mathcal{M}$
(3) $\varphi_1 : \mathrm{Fil}^1\mathcal{M} \to \mathcal{M}$ is an additive map such that:

$$\varphi_1(sx) = \frac{\frac{\sigma}{p}(s)}{\frac{\sigma}{p}(E(u))}\varphi_1(E(u)x)$$

where $s \in \mathrm{Fil}^1S$ and $x \in \mathcal{M}$ (note that $\frac{\sigma}{p}(E(u)) \in S^\times$) and such that $\mathcal{M}$ is generated by $\varphi_1(\mathrm{Fil}^1\mathcal{M})$ as an S-module.

A morphism between two objects of $\underline{\mathcal{M}}_0^1$ is an S-linear map sending Fil^1 to Fil^1 and commuting with φ_1. The map φ_1 has to be thought as the p-torsion version of the map $\frac{\varphi}{p}|_{\mathrm{Fil}^1}$. The condition $\mathrm{Fil}^1\mathcal{M} \cap p\mathcal{M} = p\mathrm{Fil}^1\mathcal{M}$ turns out to be automatically satisfied on an object of $\underline{\mathcal{M}}_0^1$. We could define a similar category by requiring the existence of a "monodromy map" N on the S-modules $\mathcal{M}$ (as for strongly divisible modules), but Lemma 3.2.1 below shows that the objects of $\underline{\mathcal{M}}_0^1$ are *already endowed* with a canonical N, and there will be no need here to consider more general torsion objects.

Lemma 3.2.1. *Let $\mathcal{M}$ be an object of $\underline{\mathcal{M}}_0^1$. There is a unique additive map $N : \mathcal{M} \to \mathcal{M}$ such that:*
(1) $N(sx) = N(s)x + sN(x)$ *for* $s \in S$ *and* $x \in \mathcal{M}$
(2) $\varphi_1\big(E(u)N(x)\big) = \frac{\sigma}{p}(E(u))N\big(\varphi_1(x)\big)$ *for* $x \in \mathrm{Fil}^1\mathcal{M}$
(3) $N(\mathcal{M}) \subset u\mathcal{M}$.

Proof. Assume two such N exist and let Δ be their difference. Let $x \in \mathrm{Fil}^1\mathcal{M}$, from conditions (2) and (3) we get:

$$\Delta(\varphi_1(x)) = \Big(\frac{\sigma}{p}\big(E(u)\big)\Big)^{-1}\varphi_1\big(E(u)\Delta(x)\big) \in u^p\mathcal{M}.$$

Since $\mathcal{M}$ is spanned by the image of φ_1 and Δ is S-linear, one has $\Delta(\mathcal{M}) \subset u^p\mathcal{M}$. An obvious induction then yields $\Delta = 0$. For the existence of N, there are 3 possible proofs: (1) one can (tediously) build it by pure linear algebra; (2) one can use 3.2.2 below which implies by [2] that there *must exist* a connection $\nabla : \mathcal{M} \to \mathcal{M} \otimes_S Sdu$ and N is defined by $u\nabla(x) = -N(x) \otimes du$; (3) one easily builds explicitly such an N when $\mathcal{M}$ is free over S ([5]), then, using 3.2.2 below and the fact any commutative finite flat group scheme is the kernel of an isogeny between p-divisible groups, one gets that any object of $\underline{\mathcal{M}}_0^1$ is the quotient of two strongly divisible modules and one takes the quotient N. Q.E.D.

Note that any morphism in $\underline{\mathcal{M}}_0^1$ automatically commutes with the respective N given by 3.2.1. The main purpose in defining the category $\underline{\mathcal{M}}_0^1$ lies in:

Theorem 3.2.2 ([5]). *There is an anti-equivalence of categories between $\underline{\mathcal{M}}_0^1$ and the category of commutative finite flat group schemes G over $\mathcal{O}_F$ such that $\mathrm{Ker}(p_G^n) = G$ for some $n \in \mathbf{N}$ and $\mathrm{Ker}(p_G^n)$ is flat over $\mathcal{O}_F$ for all $n \in \mathbf{N}$ (where p_G^n is multiplication by p^n on G).*

One can dispense with the last flatness assumption on the kernels $\mathrm{Ker}(p_G^n)$, but the price is that one has to consider more complicated

S-modules than just $\oplus(S/p^nS)^{r_n}$ for which I do not know the explicit structure (see [5]). This assumption is automatically satisfied if $e < p-1$.

Remark 3.2.3. More general objects than those of $\underline{\mathcal{M}}_0^1$, e.g. objects of $\underline{\mathcal{M}}_0^1$ endowed with an additive map N satisfying (1) and (2) of 3.2.1 but not (3), may correspond to "log-group schemes" (i.e. group objects in the category of log-schemes).

Taking the projective limit in 3.2.2 and using 3.2.1 yields:

Corollary 3.2.4 ([5]). *There is an anti-equivalence of categories between the category of strongly divisible modules $\mathcal{M}$ of weight ≤ 1 such that $N(\mathcal{M}) \subset u\mathcal{M}$ and the category of p-divisible groups over $\mathcal{O}_F$.*

Using this corollary, one can prove the following special case of (2), 2.2.7:

Theorem 3.2.5. *The functor $\mathcal{M} \mapsto T_{st}^*(\mathcal{M})$ establishes an anti-equivalence of categories between the category of strongly divisible modules $\mathcal{M}$ of weight ≤ 1 such that $N(\mathcal{M}) \subset u\mathcal{M}$ and the category of G_F-stable lattices in crystalline representations of G_F with Hodge-Tate weights in $\{0, 1\}$.*

Proof. The full faithfulness is a well-known theorem of Tate ([27]). By [5], one knows that any crystalline V with Hodge-Tate weights in $\{0, 1\}$ contains at least one lattice which is isomorphic to the Tate module of some p-divisible group over $\mathcal{O}_F$. But Raynaud's argument ([24]) then shows this must hold for *any* lattice in such a V. Using 3.2.4, this ensures the essential surjectivity. Q.E.D.

The rest of §3 will be devoted to the rest of the proof of (2), 2.2.7, i.e. the case of semi-stable *non*-crystalline representations.

3.3. Group schemes of type $(p, ..., p)$ and norm fields

In this section, we state a variant of 3.2.2 for group schemes killed by p in terms of modules over the ring of integers of the norm field of an infinite wildly ramified extension of F. This variant will be used in the next section to prove a result on representations of G_F coming from group schemes. Recall that a group scheme of type $(p, ..., p)$ is by definition a commutative finite flat group scheme killed by p.

Choose $(\pi_n)_{n\in\mathbf{N}} \in \overline{\mathbf{Q}}_p^{\mathbf{N}}$ such that $\pi_0 = \pi$, $\pi_{n+1}^p = \pi_n$ and let $F_n := F(\pi_n)$, $\mathcal{O}_{F_n}$ its ring of integers, $F_\infty := \cup F_n$ and $G_{F_\infty} := \mathrm{Gal}(\overline{\mathbf{Q}}_p/F_\infty)$ (in particular $F_0 = F$). It is proven in [30] that the projective limit

$\varprojlim F_n$ (resp. $\varprojlim \mathcal{O}_{F_n}$) with the norms as transition maps is in a natural way a field (resp. a ring) of characteristic p which can be identified with $\mathbf{F}((\underline{\pi}))$ (resp. $\mathbf{F}[[\underline{\pi}]]$). Here, $\underline{\pi}$ is the element $(..., \pi_n, \pi_{n-1}, ..., \pi_0) \in \varprojlim \mathcal{O}_{F_n}$. Such fields as $\varprojlim F_n$ are called *norm fields* in [30]. Let $\mathbf{F}((\underline{\pi}))^{\mathrm{sep}}$ be a separable closure of $\mathbf{F}((\underline{\pi}))$. The main result of [30] is a canonical identification $G_{F_\infty} \simeq \mathrm{Gal}\big(\mathbf{F}((\underline{\pi}))^{\mathrm{sep}}/\mathbf{F}((\underline{\pi}))\big)$ which gives a surprising alternative description of the Galois group G_{F_∞}.

Let σ be the Frobenius on $\mathbf{F}((\underline{\pi}))$ and $\mathbf{F}[[\underline{\pi}]]$. We introduce two kinds of modules:

• The category of *étale* $(\varphi, \mathbf{F}((\underline{\pi})))$*-modules* is the category of finite dimensional $\mathbf{F}((\underline{\pi}))$-vector spaces $\mathfrak{D}$ endowed with a σ-linear map $\varphi : \mathfrak{D} \to \mathfrak{D}$ inducing an isomorphism (or equivalently a surjection):

$$\mathbf{F}((\underline{\pi})) \otimes_{\sigma, \mathbf{F}((\underline{\pi}))} \mathfrak{D} \xrightarrow{1 \otimes \varphi} \mathfrak{D}$$

(with obvious morphisms between objects).

• The category of $(\varphi, \mathbf{F}[[\underline{\pi}]])$*-modules of height* ≤ 1 is the category of free $\mathbf{F}[[\underline{\pi}]]$-modules of finite rank $\mathfrak{M}$ endowed with a σ-linear map $\varphi : \mathfrak{M} \to \mathfrak{M}$ such that $\underline{\pi}^e \mathfrak{M}$ is contained in the $\mathbf{F}[[\underline{\pi}]]$-submodule of $\mathfrak{M}$ generated by $\varphi(\mathfrak{M})$ (ibid.).

If $\mathfrak{M}$ is an object of the second category, then $\mathfrak{M}[1/\underline{\pi}]$ is obviously an object of the first. The following two theorems give alternative descriptions of these two categories:

Theorem 3.3.1 ([16]). *The functor*:

$$\mathfrak{D} \mapsto T^*(\mathfrak{D}) := \mathrm{Hom}_{\mathbf{F}((\underline{\pi})), \varphi}(\mathfrak{D}, \mathbf{F}((\underline{\pi}))^{\mathrm{sep}})$$

establishes an anti-equivalence of categories between the category of étale $(\varphi, \mathbf{F}((\underline{\pi})))$*-modules and the category of continuous representations of* $G_{F_\infty} \simeq \mathrm{Gal}\big(\mathbf{F}((\underline{\pi}))^{\mathrm{sep}}/\mathbf{F}((\underline{\pi}))\big)$ *on finite dimensional* $\mathbf{F}_p$*-vector spaces.*

Theorem 3.3.2. *There is an anti-equivalence of categories between the category of* $(\varphi, \mathbf{F}[[\underline{\pi}]])$*-modules of height* ≤ 1 *and the category of group schemes of type* $(p, ..., p)$ *over* $\mathcal{O}_F$. *Moreover, if* G *is such a group scheme and* $\mathfrak{M}(G)$ *is the corresponding* $\mathbf{F}[[\underline{\pi}]]$*-module, one has an* $\mathbf{F}_p[G_{F_\infty}]$*-module isomorphism:* $T^*(\mathfrak{M}[1/\underline{\pi}]) \simeq G(\overline{\mathbf{Q}}_p)|_{G_{F_\infty}}$.

Proof. Granting 3.2.2, it is enough to prove that there is an equivalence of categories between $(\varphi, \mathbf{F}[[\underline{\pi}]])$-modules of height ≤ 1 and objects of $\underline{\mathcal{M}}_0^1$ killed by p, commuting with the functors to Galois representations. Let $\mathfrak{M}$ be a $(\varphi, \mathbf{F}[[\underline{\pi}]])$-module of height ≤ 1 and view S/pS as an

$\mathbf{F}[[\underline{\pi}]]$-algebra via $\Pi : \mathbf{F}[[\underline{\pi}]] \to S/pS$, $\sum x_i \underline{\pi}^i \mapsto \sum x_i u^i$. One associates to $\mathfrak{M}$ an object $\mathcal{M}$ of $\underline{\mathcal{M}}_0^1$ as follows:

- as an S-module, $\mathcal{M} := S/pS \otimes_{\sigma\circ\Pi,\mathbf{F}[[\underline{\pi}]]} \mathfrak{M}$
- $\mathrm{Fil}^1\mathcal{M} := \{y \in \mathcal{M} \mid (Id \otimes \varphi)(y) \in \mathrm{Fil}^1 S/pS \otimes_{\mathbf{F}[[\underline{\pi}]]} \mathfrak{M}\}$
- $\varphi_1 : \mathrm{Fil}^1\mathcal{M} \to \mathcal{M}$ is defined as the composite:

$$\mathrm{Fil}^1\mathcal{M} \xrightarrow{Id\otimes\varphi} \mathrm{Fil}^1 S/pS \otimes_{\mathbf{F}[[\underline{\pi}]]} \mathfrak{M} \xrightarrow{\frac{\sigma}{p}\otimes Id} S/pS \otimes_{\sigma\circ\Pi,\mathbf{F}[[\underline{\pi}]]} \mathfrak{M} \simeq \mathcal{M}.$$

Using the fact $\mathfrak{M}$ is of height ≤ 1, it is easy to see that the image of φ_1 generates everything. This process obviously defines a functor to $\underline{\mathcal{M}}_0^1$. It turns out this functor is an equivalence of categories on objects killed by p. Using [5], Lemma 2.1.2.1 and [5], Proposition 2.1.2.2, the proof is almost *verbatim* the proof of [6], Theorem 4.1.1. The only difference is that here $\mathrm{Ker}(\Pi) = (\pi^{ep})$ and $\mathrm{Ker}(\sigma\circ\Pi) = (\pi^e)$ instead of (π^p) and (π) in *loc.cit.* and this doesn't change the argument. For the Galois actions, let R be the projective limit

$$\ldots \xrightarrow{\mathrm{Frob}} \overline{\mathbf{Z}}_p/p\overline{\mathbf{Z}}_p \xrightarrow{\mathrm{Frob}} \overline{\mathbf{Z}}_p/p\overline{\mathbf{Z}}_p \xrightarrow{\mathrm{Frob}} \overline{\mathbf{Z}}_p/p\overline{\mathbf{Z}}_p$$

and R^{DP} the Divided Power envelope of R with respect to the ideal generated by the image of $\underline{\pi}^e$ i.e. by the element $(\ldots,\overline{\pi}_2^e,\overline{\pi}_1^e,\overline{\pi}_0^e) \in R$ where $\overline{\pi}_i$ is the image of π_i in $\overline{\mathbf{Z}}_p/p\overline{\mathbf{Z}}_p$ (see [29] for instance). One can endow R^{DP} with a Fil^1 and a φ_1 ([29]) and view it as an S-module via $u \mapsto \mathrm{image}(\underline{\pi})$. By [5] and [3] Lemma 2.3.1.1, the restriction to G_{F_∞} of the Galois representation associated to $\mathcal{M}$ is isomorphic to $\mathrm{Hom}_{S,\varphi_1,\mathrm{Fil}^1}(\mathcal{M},R^{DP})$ (with left action of G_{F_∞} on R^{DP} and obvious notations). Thus, one has to compare $\mathrm{Hom}_{\mathbf{F}((\underline{\pi})),\varphi}(\mathfrak{M}[1/\underline{\pi}],\mathbf{F}((\underline{\pi}))^{\mathrm{sep}})$ and $\mathrm{Hom}_{S,\varphi_1,\mathrm{Fil}^1}\big(\mathcal{M},R^{DP}\big)$. Using [6], Lemma 2.3.3, the proof is again (almost) *verbatim* the proof of [6], Proposition 4.2.1. Q.E.D.

Remark 3.3.3. Theorem 3.3.2 implies that representations of G_{F_∞} coming from $(\varphi,\mathbf{F}[[\underline{\pi}]])$-modules of height ≤ 1 can be extended to G_F. We will see in the next section that this extension is essentially unique.

Remark 3.3.4. Let $\underline{\mathbf{Z}/p\mathbf{Z}}$ and $\underline{\mu_p}$ be the usual group schemes of rank p. Using [5] and the above proof, one can see that $\mathfrak{M}(\underline{\mathbf{Z}/p\mathbf{Z}}) = \mathbf{F}[[\underline{\pi}]]e_1$ with $\varphi(e_1) = e_1$ and that $\mathfrak{M}(\underline{\mu_p}) = \mathbf{F}[[\underline{\pi}]]e_2$ with $\varphi(e_2) = -F(0)^{-1}\underline{\pi}^e e_2$ where $F(0) = -\frac{E(0)}{p}$ (recall $E(u)$ is the minimal polynomial of π). One of the problems with the category $\underline{\mathcal{M}}_0^1$ for $p=2$ is that there is no map corresponding to the non-trivial morphism of group schemes $\underline{\mathbf{Z}/2\mathbf{Z}} \to \underline{\mu_2}$ sending 1 to -1. However, for $p=2$, *there is* a non-trivial map $\mathfrak{M}(\underline{\mu_2}) \to \mathfrak{M}(\underline{\mathbf{Z}/2\mathbf{Z}})$ that commutes with φ, namely:

$e_2 \mapsto F(0)^{-1}\underline{\pi}^e e_1$ (this map would give 0 in $\underline{\mathcal{M}}_0^1$ by the functor in the proof of 3.3.2). So, one can ask whether statement 3.3.2 *still holds* for $p = 2$ although statement 3.2.2 doesn't...

3.4. A full faithfulness result

Lemma 3.4.1. *Let $\mathfrak{D}' \hookrightarrow \mathfrak{D}$ (resp. $\mathfrak{D}' \twoheadrightarrow \mathfrak{D}$) be an injection (resp. a surjection) of étale $(\varphi, \mathbf{F}((\underline{\pi})))$-modules and assume $\mathfrak{D}$ (resp. $\mathfrak{D}'$) is generated by a $(\varphi, \mathbf{F}[[\underline{\pi}]])$-module $\mathfrak{M}$ (resp. $\mathfrak{M}'$) of height ≤ 1. Then $\mathfrak{M} \cap \mathfrak{D}'$ (resp.* image$(\mathfrak{M}')$) *is a $(\varphi, \mathbf{F}[[\underline{\pi}]])$-module of height ≤ 1.*

Proof. The surjection case is obvious. For the injection, it is clear that $\mathfrak{M}' := \mathfrak{M} \cap \mathfrak{D}'$ is stable under φ and is a direct factor of $\mathfrak{M}$. Let $(f_1, ..., f_d)$ be a basis of $\mathfrak{M}$ (over $\mathbf{F}[[\underline{\pi}]]$) such that $(f_1, ..., f_{d'})$ is a basis of $\mathfrak{M}'$ and denote by $(\overline{f}_{d'+1}, ..., \overline{f}_d)$ the image basis of $\mathfrak{M}/\mathfrak{M}'$. By assumption, there are $s_{ij} \in \mathbf{F}[[\underline{\pi}]]$ such that $\underline{\pi}^e f_i = \sum_{j=1}^d s_{ij}\varphi(f_j)$ for $1 \leq i \leq d$. For $d' + 1 \leq i \leq d$, this implies that $(\varphi(\overline{f}_{d'+1}), ..., \varphi(\overline{f}_d))$ is a basis of $(\mathfrak{M}/\mathfrak{M}')[1/\underline{\pi}]$ since it generates this module, and for $1 \leq i \leq d'$ this implies $0 = \sum_{j=d'+1}^d s_{ij}\varphi(\overline{f}_j)$; i.e. $s_{ij} = 0$ for $d' + 1 \leq j \leq d$ (and $1 \leq i \leq d'$). Hence, $\varphi(\mathfrak{M}')$ generates $\underline{\pi}^e\mathfrak{M}'$. Q.E.D.

Lemma 3.4.2. *Let G_1 and G_2 be two group schemes of type $(p, ..., p)$ over $\mathcal{O}_F$. Then any $\mathbf{F}_p[G_{F_\infty}]$-isomorphism $G_1(\overline{\mathbf{Q}}_p)|_{G_{F_\infty}} \simeq G_2(\overline{\mathbf{Q}}_p)|_{G_{F_\infty}}$ is an $\mathbf{F}_p[G_F]$-isomorphism.*

Proof. Fix such an $\mathbf{F}_p[G_{F_\infty}]$-isomorphism. Let $\mathfrak{M}_i$ be the $(\varphi, \mathbf{F}[[\underline{\pi}]])$-module of height ≤ 1 associated to G_i by 3.3.2 and let $\mathfrak{D} := \mathfrak{M}_i[1/\underline{\pi}]$, which doesn't depend on $i \in \{1, 2\}$ by assumption and 3.3.1. Then $\mathfrak{M} := \mathfrak{M}_1 + \mathfrak{M}_2 \subset \mathfrak{D}$ is obviously still a $(\varphi, \mathbf{F}[[\underline{\pi}]])$-module of height ≤ 1 and thus corresponds to a group scheme $G/\mathcal{O}_F$. The two injections $\mathfrak{M}_i \hookrightarrow \mathfrak{M}$ give morphisms of group schemes $G \to G_i$ such that $G(\overline{\mathbf{Q}}_p)|_{G_{F_\infty}} \xrightarrow{\sim} G_i(\overline{\mathbf{Q}}_p)|_{G_{F_\infty}}$ by 3.3.2. This implies $G_1(\overline{\mathbf{Q}}_p) \simeq G(\overline{\mathbf{Q}}_p) \simeq G_2(\overline{\mathbf{Q}}_p)$ and all of these isomorphisms obviously commute with G_F since they come from morphisms of group schemes. Q.E.D.

We say a representation of G_F on a finite length $\mathbf{Z}_p$-module is *finite flat* if it is isomorphic to the representation of G_F on $G(\overline{\mathbf{Q}}_p)$ for some commutative finite flat group scheme G over $\mathcal{O}_F$ killed by some power of p. The process of schematic closure ([24]) then shows this category is abelian and stable under formation of subobjects and quotients.

Theorem 3.4.3. *The fonctor "restriction to G_{F_∞}" from finite flat representations of G_F to representations of G_{F_∞} is fully faithful. Its essential image is stable under formation of subobjects and quotients.*

Proof. We first start with the full faithfulness. By a standard devissage, one is reduced to the case of representations on $\mathbf{F}_p$-vector spaces. Let G_1, G_2 be two group schemes of type $(p,...,p)$, $\mathfrak{M}_1$, $\mathfrak{M}_2$ the corresponding $(\varphi, \mathbf{F}[[\underline{\pi}]])$-modules of height ≤ 1 and $\mathfrak{D}_i := \mathfrak{M}_i[1/\underline{\pi}]$ $(i = 1, 2)$. Assume there is an $\mathbf{F}_p[G_{F_\infty}]$-morphism $G_2(\overline{\mathbf{Q}}_p)|_{G_{F_\infty}} \to G_1(\overline{\mathbf{Q}}_p)|_{G_{F_\infty}}$ i.e. by 3.3.1 a morphism $f : \mathfrak{D}_1 \to \mathfrak{D}_2$. By 3.4.1, $f(\mathfrak{M}_1)$ and $\mathfrak{M}_2 \cap f(\mathfrak{D}_1)$ are two $(\varphi, \mathbf{F}[[\underline{\pi}]])$-modules of height ≤ 1 that generate $f(\mathfrak{D}_1)$. They correspond to two group schemes G_1', G_2' such that $G_2'(\overline{\mathbf{Q}}_p)|_{G_{F_\infty}} \simeq G_1'(\overline{\mathbf{Q}}_p)|_{G_{F_\infty}}$ and we have morphisms of group schemes $G_1' \to G_1$ and $G_2 \to G_2'$ by 3.3.2. Hence the morphism $G_2(\overline{\mathbf{Q}}_p)|_{G_{F_\infty}} \to G_1(\overline{\mathbf{Q}}_p)|_{G_{F_\infty}}$ factorizes through:

$$G_2(\overline{\mathbf{Q}}_p)|_{G_{F_\infty}} \longrightarrow G_2'(\overline{\mathbf{Q}}_p)|_{G_{F_\infty}} \simeq G_1'(\overline{\mathbf{Q}}_p)|_{G_{F_\infty}} \longrightarrow G_1(\overline{\mathbf{Q}}_p)|_{G_{F_\infty}}.$$

By 3.4.2, $G_2'(\overline{\mathbf{Q}}_p) \simeq G_1'(\overline{\mathbf{Q}}_p)$ as $\mathbf{F}_p[G_F]$-modules from which we get that the map $G_2(\overline{\mathbf{Q}}_p) \to G_1(\overline{\mathbf{Q}}_p)$ commutes with G_F. This gives the full faithfulness. For the rest of the statement, it is enough to prove that any G_{F_∞}-subrepresentation of a finite flat G_F-representation T is preserved by G_F (and hence is finite flat). We proceed by induction on $n \in \mathbf{N}$ such that $p^n T = 0$. For $n = 1$, this is a consequence of 3.4.1 (together with 3.3.1 and 3.3.2). Assume this holds for $n - 1$ and let $T' \subset T$ be a G_{F_∞}-subrepresentation with $p^n T = 0$. Then $T/(T' + pT)$ is a quotient of T/pT, hence is preserved by G_F by the case $n = 1$. By the full faithfulness, the morphism $T \to T/(T' + pT)$ commutes with G_F hence $T' + pT$ is preserved by G_F. Now $(T' + pT)/T'$ is a quotient of pT, hence is preserved by G_F by the case $n - 1$. By the full faithfulness applied to $T' + pT \to (T' + pT)/T'$, T' is preserved by G_F. Q.E.D.

Corollary 3.4.4. *Let V be a crystalline representation of G_F with Hodge-Tate weights in $\{0, 1\}$ and $T \subset V$ a $\mathbf{Z}_p$-lattice which is stable under G_{F_∞}. Then T is stable under G_F.*

Proof. Let T' be a $\mathbf{Z}_p[G_F]$-lattice containing T and recall that by 3.2.4 and 3.2.5, T' is the Tate module of a p-divisible group over $\mathcal{O}_F$. By 3.4.3 any $\mathbf{Z}_p[G_{F_\infty}]$-submodule of $T'/p^n T'$ is stable under G_F for any $n \in \mathbf{N}$. Thus, $T/T \cap p^n T'$ is stable under G_F, i.e. $g(T) \subset T + p^n T'$ for any $g \in G_F$ and $n \in \mathbf{N}$, which implies $g(T) \subset \cap_n (T + p^n T') = T$. Q.E.D.

3.5. Lattices in semi-stable representations with Hodge-Tate weights in $\{0, 1\}$

We finish the proof of (2), 2.2.7 using Corollary 3.4.4 above. We choose $(\pi_n)_{n \in \mathbf{N}}$ as in §3.3 and define F_∞ and G_{F_∞} in the same way.

Let D be a weakly admissible filtered (φ, N)-module such that $\mathrm{Fil}^0 D_F = D_F$ and $\mathrm{Fil}^2 D_F = 0$. Let $V := V^*_{st}(D)$ as in §1 and $\mathcal{D} := S \otimes_{W(\mathbf{F})} D$ as in §2.2. Recall we have defined $V^*_{st}(\mathcal{D})$ in the proof of Proposition 2.2.5 and shown that $V^*_{st}(\mathcal{D}) \xrightarrow{\sim} V^*_{st}(D)$. Define:

$$V^*_{cris}(D) := \mathrm{Hom}_{\varphi}(D, B_{cris}) \cap \mathrm{Hom}_{\mathrm{Fil}^{\cdot}}(D_F, F \otimes_{F_0} B_{cris})$$

and $V^*_{cris}(\mathcal{D}) := \mathrm{Hom}_{S,\varphi,\mathrm{Fil}^1}(\mathcal{D}, B_{cris})$ where we view B_{cris} as an S-algebra by sending u to the element $[\underline{\pi}]$ corresponding to the p^n-th roots π_n (§2.2). We have ring morphisms $B_{st} \to B_{cris}$ and $\widehat{A_{st}}[1/p] \to B_{cris}$ obtained by sending $\log \frac{[\underline{\pi}]}{\pi}$ and X to 0.

Lemma 3.5.1. (1) *The map $f \mapsto f|_D$ induces an isomorphism of $\mathbf{Q}_p$-vector spaces $V^*_{cris}(\mathcal{D}) \xrightarrow{\sim} V^*_{cris}(D)$.*
(2) *The ring homomorphisms $B_{st} \to B_{cris}$ and $\widehat{A_{st}}[1/p] \to B_{cris}$ induce isomorphisms of $\mathbf{Q}_p$-vector spaces $V^*_{st}(D) \xrightarrow{\sim} V^*_{cris}(D)$ and $V^*_{st}(\mathcal{D}) \xrightarrow{\sim} V^*_{cris}(\mathcal{D})$.*
(3) *The diagram*
$$\begin{array}{ccc} V^*_{st}(\mathcal{D}) & \xrightarrow{\sim} & V^*_{cris}(\mathcal{D}) \\ \wr\downarrow & & \downarrow\wr \\ V^*_{st}(D) & \xrightarrow{\sim} & V^*_{cris}(D) \end{array}$$
is commutative.

Proof. (3) follows from the definition of the various maps. To prove (1) and (2), we first note that we can replace everywhere B_{cris} by B^+_{cris} and B_{st} by B^+_{st}. For B_{cris}, this is a direct consequence of [13], Theorem5.3.7(i). For B_{st}, one can argue as follows. Let $f \in V^*_{st}(D)$, $x \in D\backslash\{0\}$ and $r \in \mathbf{Z}_{\geq 0}$ such that $N^{r+1}(x) = 0$ but $N^r(x) \neq 0$. Then $f(\varphi^s(N^r(x))) \in \mathrm{Fil}^0 B_{cris}$ for all $s \in \mathbf{Z}$ which implies $f(N^r(x)) \in B^+_{cris}$ by [13], Theorem5.3.7(i). Hence $f(N^{r-1}(x)) \in \mathrm{Fil}^0 B_{cris} + B^+_{cris} \log \frac{[\underline{\pi}]}{\pi}$. Since $\varphi(\log \frac{[\underline{\pi}]}{\pi}) = p \log \frac{[\underline{\pi}]}{\pi}$, the same argument shows $f(N^{r-1}(x)) \in B^+_{cris} + B^+_{cris} \log \frac{[\underline{\pi}]}{\pi}$ and we deduce $f(x) \in B^+_{st}$ by induction. The isomorphism in (1) comes from the facts that $\mathrm{Fil}^1 \mathcal{D} = f_\pi^{-1}(\mathrm{Fil}^1 D_F)$ and $[\underline{\pi}] - \pi \in \mathrm{Fil}^1(F \otimes_{F_0} B^+_{cris})$ where $f_\pi : \mathcal{D} \to D_F$ is the map of §2.2. Note that it exists because there is just Fil^1. One checks the ring homomorphisms $B^+_{st} \to B^+_{cris}$ and $\widehat{A_{st}}[1/p] \to B^+_{cris}$ deduced from the ones without + commute with φ and preserve Fil^1 and hence induce maps of vector spaces as in (2). Since we know that $V^*_{st}(\mathcal{D}) \xrightarrow{\sim} V^*_{st}(D)$, we only have to prove $V^*_{st}(D) \xrightarrow{\sim} V^*_{cris}(D)$ thanks to (1) and the commutativity in (3). The inverse map $V^*_{cris}(D) \to V^*_{st}(D)$ is given by $f \mapsto f + \log \frac{[\underline{\pi}]}{\pi} f \circ N$ (using $B^+_{cris} \subset B^+_{st}$). Q.E.D.

Remark 3.5.2. One can prove that the above isomorphism $V^*_{st}(\mathcal{D}) \xrightarrow{\sim} V^*_{cris}(\mathcal{D})$ does not require $\mathrm{Fil}^2 D_F = 0$. Also, all the isomorphisms in

Lemma 3.5.1 commute with G_{F_∞} although they do not commute with G_F.

Lemma 3.5.3. *Let D' be the same filtered (φ, N)-module as D but with $N = 0$. Then D' is also weakly admissible.*

Proof. With the notations of §1, we have:

$$\overline{\mathbf{Q}}_p \otimes_{F_0} N(D) = \oplus_\alpha N(\overline{D}_\alpha)$$

with $N(\overline{D}_\alpha) \subset \overline{D}_{\alpha-1}$ (since $N\varphi^f = p^f \varphi^f N$). But $\overline{D}_\alpha = 0$ if $\alpha \notin [0,1]$ (weak admissibility condition) so $N(D) \subset \overline{D}_0$ which implies $t_N(N(D)) = 0$ and also $t_H(N(D)) = 0$ since $0 \leq t_H(N(D)) \leq t_N(N(D))$. Note that $N^2 = 0$ (same proof as for 2.1.4). Let $D^0 \subset D$ be a F_0-vector subspace stable under φ but not necessarily under N with the induced filtration $\mathrm{Fil}^i D_F^0 := D_F^0 \cap \mathrm{Fil}^i D_F$ $(i = 0, 1)$. Define $D^1 := D^0 + N(D^0)$. From the exact sequence $0 \to N(D^0) \to D^1 \to D^0/(D^0 \cap N(D^0)) \to 0$, the weak admissibility condition for $D^1 \subset D$, and the additivity property of t_H and t_N, we have:

$$t_H^1(D^0/(D^0 \cap N(D^0))) \leq t_N(D^0/(D^0 \cap N(D^0)))$$

where by t_H^1 we mean the t_H computed with the filtration on $D^0/(D^0 \cap N(D^0))$ coming from the quotient filtration of D^1. From the exact sequence $0 \to D^0 \cap N(D^0) \to D^0 \to D^0/(D^0 \cap N(D^0)) \to 0$, we deduce $t_H(D^0) = t_H^0(D^0/(D^0 \cap N(D^0)))$ and $t_N(D_0) = t_N(D^0/(D^0 \cap N(D^0)))$ where by t_H^0 we mean the t_H computed with the filtration on $D^0/(D^0 \cap N(D^0))$ coming from the quotient filtration of D^0. From the inclusion $D^0 \subset D^1$ and the above inequality, we get:

$$t_H^0(D^0/(D^0 \cap N(D^0))) \leq t_H^1(D^0/(D^0 \cap N(D^0))) \leq t_N(D^0/(D^0 \cap N(D^0)))$$

hence $t_H(D^0) \leq t_N(D^0)$. This gives the desired result. Q.E.D.

Let $\mathcal{D}' := S \otimes_{W(\mathbf{F})} D'$ (with its usual structures) and note that $\mathcal{D}' \simeq \mathcal{D}$ except for the operator N. We have $V_{cris}^*(D') = V_{cris}^*(D)$ and $V_{cris}^*(\mathcal{D}') = V_{cris}^*(\mathcal{D})$ since the definition of these vector spaces do not use N. Using (2), Lemma 3.5.1, we deduce isomorphisms $V_{st}^*(D') \simeq V_{st}^*(D)$ and $V_{st}^*(\mathcal{D}') \simeq V_{st}^*(\mathcal{D})$ such that the diagram:

$$\begin{array}{ccc} V_{st}^*(\mathcal{D}) & \xrightarrow{\sim} & V_{st}^*(\mathcal{D}') \\ \wr\downarrow & & \downarrow\wr \\ V_{st}^*(D) & \xrightarrow{\sim} & V_{st}^*(D') \end{array}$$

commutes. We call V the commun underlying $\mathbf{Q}_p$-vector space and ρ', ρ the two different Galois actions $G_F \to \mathrm{Aut}(V)$ corresponding to D'

and D respectively. Let $D(-1)$ be the filtered (φ, N)-module defined by $\mathrm{Fil}^m D(-1)_F := \mathrm{Fil}^{m+1} D_F$, $\varphi_{D(-1)} := p^{-1}\varphi_D$ and $N_{D(-1)} := N_D$. The operator N induces a morphism of filtered modules $N : D(-1) \to D$ and thus a morphism of Galois representations:

$$N : (V, \rho) \to (V \otimes \mathbf{Q}_p(-1), \rho \otimes \chi^{-1})$$

where $\mathbf{Q}_p(-1)$ is the $\mathbf{Q}_p$-dual of $\mathbf{Q}_p(1) := \left(\varprojlim \mu_{p^n}(\overline{\mathbf{Q}}_p)\right) \otimes \mathbf{Q}_p$ and χ is the p-adic cyclotomic character. Working out the isomorphism $V^*_{st}(D') \simeq V^*_{st}(D)$ from the proof of 3.5.1, we easily obtain:

Lemma 3.5.4. *Let $t_p : G_F \to \varprojlim \mu_{p^n}(\overline{\mathbf{Q}}_p) = \mathbf{Z}_p(1)$ be the 1-cocycle defined by $t_p(g) := \left(g(\pi_n)/\pi_n\right)_{n\in\mathbf{N}}$. Then:*

$$\rho = (Id + t_p \otimes N) \circ \rho'.$$

From this and the results of §3.4, we obtain the following key corollary:

Corollary 3.5.5. *Let $T \subset V$ be a $\mathbf{Z}_p$-lattice which is stable under ρ. Then T is also stable under ρ', or equivalently $N(T) \subset T \otimes \mathbf{Z}_p(-1)$.*

Proof. Since $t_p(g) = 0$ if $g \in G_{F_\infty}$, it follows from 3.5.4 that T is preserved by $\rho'(G_{F_\infty}) = \rho(G_{F_\infty})$. By 3.4.4, T is stable under ρ'. Q.E.D.

Now, let $\mathcal{M}$ be a strongly divisible lattice in $\mathcal{D}$. We denote by $\mathcal{M}'$ the image of $\mathcal{M}$ in $\mathcal{D}'$ under the identification $\mathcal{D} \simeq \mathcal{D}'$. In particular, as S-modules, $\mathcal{M}$ and $\mathrm{Fil}^1\mathcal{M}$ are just the same as $\mathcal{M}'$ and $\mathrm{Fil}^1\mathcal{M}'$.

Lemma 3.5.6. (1) *The S-module $\mathcal{M}'$ is preserved N in $\mathcal{D}'$, i.e. $\mathcal{M}'$ is a strongly divisible lattice in $\mathcal{D}'$.*
(2) *Under the isomorphism $V^*_{st}(D) \simeq V^*_{st}(D')$, the lattice $T^*_{st}(\mathcal{M})$ corresponds to the lattice $T^*_{st}(\mathcal{M}')$.*

Proof. For (1), we have to prove $N(\mathcal{M}) \subset \mathcal{M}$ with N being $N\otimes 1$ on $\mathcal{D}' = S \otimes D' = S \otimes D$. By Lemma 3.2.1 or by [5] Proposition 5.1.3, there is a unique additive map $N' : \mathcal{M} \to \mathcal{M}$ such that $N'(sx) = N(s)x + sN'(x)$, $N'(\mathcal{M}) \subset u\mathcal{M}$ and $N'\varphi = p\varphi N'$. As $D = \cap_{n\in\mathbf{N}}\varphi^n(\mathcal{D})$ (this is easily checked), the last commutativity condition implies $N'(D) \subset D$ and the condition $N'(\mathcal{M}) \subset u\mathcal{M}$ implies $N'|_D = 0$. Hence, on $\mathcal{M}[1/p] = S \otimes D$, N' is exactly $N \otimes 1$. This proves (1). Recall from Lemma 3.5.1 and the foregoing that we have a commutative diagram:

$$\begin{array}{ccc} V^*_{st}(\mathcal{D}) & \xrightarrow{\sim} & V^*_{st}(\mathcal{D}') \\ \wr\downarrow & & \downarrow\wr \\ V^*_{cris}(\mathcal{D}) & = & V^*_{cris}(\mathcal{D}') \end{array}$$

where the top arrow is the identification $V_{st}^*(D) \simeq V_{st}^*(D')$. In order to prove (2), it is enough to prove that the two lattices $T_{st}^*(\mathcal{M}) \subset V_{st}^*(\mathcal{D})$ and $T_{st}^*(\mathcal{M}') \subset V_{st}^*(\mathcal{D}')$ map to the same $\mathbf{Z}_p$-module in $V_{cris}^*(\mathcal{D}) = V_{cris}^*(\mathcal{D}')$. Define $T_{cris}^*(\mathcal{M}) := \mathrm{Hom}_{S,\varphi,\mathrm{Fil}^1}(\mathcal{M}, A_{cris}) \subset V_{cris}^*(\mathcal{D})$ and likewise for $T_{cris}^*(\mathcal{M}')$. Since N is not involved, we have $T_{cris}^*(\mathcal{M}) = T_{cris}^*(\mathcal{M}')$. By [3] Lemma 2.3.1.1, $T_{st}^*(\mathcal{M})$ (resp. $T_{st}^*(\mathcal{M})$) exactly maps to $T_{cris}^*(\mathcal{M})$ (resp. $T_{cris}^*(\mathcal{M}')$) under $V_{st}^*(\mathcal{D}) \xrightarrow{\sim} V_{cris}^*(\mathcal{D})$ (resp. $V_{st}^*(\mathcal{D}') \xrightarrow{\sim} V_{cris}^*(\mathcal{D}')$). This gives (2). Q.E.D.

Corollary 3.5.7. *Statement* (2) *of* 2.2.7 *holds.*

Proof. We can assume $m = 1$. We first prove the full faithfulness. Let $\mathcal{M}_1$, $\mathcal{M}_2$ be two strongly divisible modules of weight ≤ 1, T_1, T_2 their corresponding lattices and $f : T_2 \to T_1$ a Galois morphism. Let $V_i := T_i \otimes \mathbf{Q}_p$, $D_i := D_{st}^*(V_i)$, D_i' as before and $V_i' := V_{st}^*(D_i')$ ($i \in \{1,2\}$). Recall $V_i \simeq V_i'$ as vector spaces. The map f induces $f : V_2 \to V_1$ and $f : V_2' \to V_1'$ which is G_F-equivariant for both actions of G_F (look at the corresponding map on D_i and D_i'). By 3.5.5, T_i is Galois stable in V_i' and thus $f : T_2 \to T_1$ commutes with this "crystalline" Galois action. By 3.5.6 and 3.2.5, it induces a morphism $\mathcal{M}_1 \to \mathcal{M}_2$. It remains to prove this morphism commutes with the original N, but this is obvious since this is so for $\mathcal{M}_1[1/p] \to \mathcal{M}_2[1/p]$. Let us now prove the essential surjectivity. Let V be a semi-stable p-adic representation with Hodge-Tate weights in $\{0, 1\}$ and $T \subset V$ a Galois stable lattice. Let $D := D_{st}^*(V)$ and D', V' as before. Since T is also Galois stable in V' (Corollary 3.5.5, this is the key point), by 3.2.5 it corresponds to a strongly divisible lattice $\mathcal{M}$ in $\mathcal{D}' := S \otimes D'$. By statement (2) of 3.5.6, it remains to prove that $\mathcal{M}$ is stable under N in $\mathcal{D} := S \otimes D$. Denote by N' the S-derivation on $\mathcal{M}$ induced by $\mathcal{D}'$, by $N(V)$ the unramified quotient of V corresponding to $N(D) \subset D$ (see the proof of 3.5.3) and by $N(T)$ the image of T in $N(V)$. One has an injection of crystalline representations with Hodge-Tate weights in $\{0, 1\}$, $N(V) \otimes \mathbf{Q}_p(1) \hookrightarrow V$, which induces $N(T) \otimes \mathbf{Z}_p(1) \hookrightarrow T$ by 3.5.5. If $\mathcal{M}_0$ denotes the strongly divisible lattice in $S \otimes N(D)$ corresponding to $N(T)$ (case $m = 0$) and $\mathcal{M}_0(1)$ the obvious one corresponding to $N(T) \otimes \mathbf{Z}_p(1)$, then by 3.2.5 we have morphisms $\mathcal{M} \to \mathcal{M}_0(1)$ and $\mathcal{M}_0 \to \mathcal{M}$, the composite of which is $N - N' : \mathcal{M} \to \mathcal{M}(1)$ (with obvious notation, one checks this by looking over $S[1/p]$). Forgetting the twist "(1)", this implies $N(\mathcal{M}) \subset \mathcal{M}$. Q.E.D.

§4. Integral p-adic cohomologies

In this section, we suggest a cohomological interpretation of strongly divisible modules.

We fix a proper smooth scheme X over $\mathrm{Spec}(F)$ and we assume X admits a proper semi-stable model $\mathcal{X}$ over $\mathcal{O}_F$ (i.e. étale-locally $\mathcal{X}$ is smooth over $\mathcal{O}_F[X_1, ..., X_r]/(X_1 \cdots X_r - \pi)$ for some r). Let $\mathcal{Y} := \mathcal{X} \times_{\mathrm{Spec}(\mathcal{O}_F)} \mathrm{Spec}(\mathbf{F})$ and $\mathcal{X}_1 := \mathcal{X} \times_{\mathrm{Spec}(\mathcal{O}_F)} \mathrm{Spec}(\mathcal{O}_F/p\mathcal{O}_F)$. Endow $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{X}_1$ with their natural log-structure ([22]) and for $m \in \mathbf{N}$ denote by:

$$\begin{aligned} H^m_{\text{ét}}(X \times_F \overline{\mathbf{Q}}_p, \mathbf{Z}_p) &:= \varprojlim H^m((X \times_F \overline{\mathbf{Q}}_p)_{\text{ét}}, \mathbf{Z}/p^n\mathbf{Z}) \\ H^m_{\text{ét}}(X \times_F \overline{\mathbf{Q}}_p, \mathbf{Q}_p) &:= H^m_{\text{ét}}(X \times_F \overline{\mathbf{Q}}_p, \mathbf{Z}_p) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p \end{aligned}$$

the usual p-adic étale cohomology groups of X. By [28], $H^m_{\text{ét}}(X \times_F \overline{\mathbf{Q}}_p, \mathbf{Q}_p)$ is a semi-stable p-adic representation of G_F with Hodge-Tate weights in $\{-m, ..., 0\}$. Moreover, if $V^m := H^m_{\text{ét}}(X \times_F \overline{\mathbf{Q}}_p, \mathbf{Q}_p)^*$ ($\mathbf{Q}_p$-dual) and $D^m := D^*_{st}(V^m)$ is the associated filtered (φ, N)-module (see §1), then:

(1) $$D^m \simeq H^m_{\text{log}-\text{cris}}(\mathcal{Y}/W(\mathbf{F})) \otimes F_0$$

where:

$$H^m_{\text{log}-\text{cris}}(\mathcal{Y}/W(\mathbf{F})) := \varprojlim H^m_{\text{log}-\text{cris}}\big(\mathcal{Y}/\mathrm{Spec}(W_n(\mathbf{F}))\big)$$

is the log-crystalline cohomology of $\mathcal{Y}$ with respect to the base scheme $\mathrm{Spec}(W_n(\mathbf{F}))$ endowed with the log-structure $\big(\mathbf{N} \to W_n(\mathbf{F}), 1 \mapsto 0\big)$. More precisely this cohomology is naturally endowed with operators φ and N and one has an isomorphism (depending on the choice of π):

$$F \otimes_{W(\mathbf{F})} H^m_{\text{log}-\text{cris}}(\mathcal{Y}/W(\mathbf{F})) \simeq H^m_{\text{dR}}(X)$$

where $H^m_{\text{dR}}(X)$ is the usual de Rham cohomology of X endowed with its Hodge filtration. Then (1) is an isomorphism of *filtered* (φ, N)*-modules* (see [19], [22] and [28] for details).

Now, let:

$$\mathcal{D}^m := S \otimes_{W(\mathbf{F})} D^m$$

and endow it with the same structures as in section 2.2. It is shown in [19] that there is an isomorphism of $S[1/p]$-modules:

$$\mathcal{D}^m \simeq H^m_{\text{log}-\text{cris}}(\mathcal{X}_1/S) \otimes F_0$$

where:

$$H^m_{\log-\mathrm{cris}}(\mathcal{X}_1/S) := \varprojlim H^m_{\log-\mathrm{cris}}\big(\mathcal{X}_1/\mathrm{Spec}(S/p^nS)\big)$$

is the log-crystalline cohomology of $\mathcal{X}_1$ with respect to the base scheme $\mathrm{Spec}(S/p^nS)$ endowed with the log-structure $\big(\mathbf{N} \to S/p^nS, 1 \mapsto u\big)$. Here the log-scheme $\mathcal{X}_1$ is viewed over $\mathrm{Spec}(S/p^nS)$ via the embedding $\mathrm{Spec}(\mathcal{O}_F/p\mathcal{O}_F) \hookrightarrow \mathrm{Spec}(S/p^nS)$, $u \mapsto \pi$. Assume $m < p-1$ and consider:

$$T^m := \mathbf{Z}_p\text{−dual of } \big(H^m_{\text{ét}}(X \times_F \overline{\mathbf{Q}}_p, \mathbf{Z}_p)/\text{torsion}\big).$$

Then T^m is a Galois stable lattice in V^m. Conjecture 2.2.6 predicts there should exist a corresponding strongly divisible lattice in $\mathcal{D}^m$. Consider:

$$\mathcal{M}^m := H^m_{\log-\mathrm{cris}}(\mathcal{X}_1/S)/\text{torsion}.$$

One can prove that $\mathcal{M}^m \subset \mathcal{D}^m$ and that it is stable under φ and N ([19]).

Question 4.1. *Assume* $m < p-1$.
(1) Is $\mathcal{M}^m$ a strongly divisible lattice in $\mathcal{D}^m$ in the sense of Definition 2.2.1?
(2) If this is so, is $T^*_{st}(\mathcal{M}^m)$ isomorphic to T^m?

The following theorem summarizes the known answers to these questions:

Theorem 4.2. *The answer to questions* (1) *and* (2) *of* 4.1 *is yes in the following two cases*:
(1) $e = 1$
(2) $m \leq 1$.

Case (1) is proven in [7]. The method is a generalization of that of Fontaine and Messing (syntomic cohomology) who did the subcase $e = 1$, $N = 0$ ([18]). However, the proofs are more involved because strongly divisible modules when $N \neq 0$ are much more complicated than when $N = 0$, even if $e = 1$ (see [9] for instance). Case (2) is a special case of results of Faltings and is proven in [12] using his theory of almost étale extensions.

§5. A glimpse at reduction modulo p

Integral p-adic Hodge theory has the virtue that we can form its reduction modulo p. We provide here some samples of such reductions

(Propositions 5.2, 5.3 and Theorem 5.4). More precisely we reduce modulo p lattices in some 2-dimensional (over $\overline{\mathbf{Q}}_p$) semi-stable representations of G_F for $F = \mathbf{Q}_p$. This is the simplest case, although not so simple! In the sequel, we denote by $I_{\mathbf{Q}_p}$ the inertia subgroup of $G_{\mathbf{Q}_p}$, by val the p-adic valuation normalized by val$(p) = 1$ and by ified character of $G_{\mathbf{Q}_p}$ sending the arithmetic Frobenius to λ.

Let us consider semi-stable p-adic representations V of $G_{\mathbf{Q}_p}$ endowed with an embedding $E \hookrightarrow \mathrm{Aut}_{G_{\mathbf{Q}_p}}(V)$ where E is a finite (arbitrarily large) extension of $\mathbf{Q}_p$ inside $\overline{\mathbf{Q}}_p$ such that $\dim_E V = 2$. In that case, $D := D_{st}(V^*)$ is also a 2-dimensional E-vector space with E-linear φ, N and filtration. We assume moreover $\mathrm{Fil}^0 D = D$ and $\mathrm{Fil}^1 D \neq 0$, and we denote by $k \geq 2$ the smallest integer such that $\mathrm{Fil}^k D = 0$. Since $\dim_E D = 2$, we have $\mathrm{Fil}^1 D = \mathrm{Fil}^2 D = \ldots = \mathrm{Fil}^{k-1} D$. We denote by $\mathcal{O}_E$ the ring of integers of E and by $\mathfrak{m}_E$ its maximal ideal.

Examples 5.1. The following three examples exhaust all the possibilities of two dimensional semi-stable representations of $G_{\mathbf{Q}_p}$ with Hodge-Tate weights $(0, k-1)$:

$$(1) \qquad \left\{ \begin{array}{ccc} D & = & Ee_1 \oplus Ee_2 \\ \varphi(e_1) & = & p^{k-1}(\lambda e_1 + \mu e_2) \\ \varphi(e_2) & = & \lambda^{-1} e_2 \\ N & = & 0 \\ \mathrm{Fil}^{k-1} D & = & Ee_1 \\ (\lambda, \mu) & \in & \mathcal{O}_E^\times \times E \end{array} \right.$$

$$(2) \qquad \left\{ \begin{array}{ccc} D & = & Ee_1 \oplus Ee_2 \\ \varphi(e_1) & = & p^{k-1} e_2 \\ \varphi(e_2) & = & -e_1 + \mu e_2 \\ N & = & 0 \\ \mathrm{Fil}^{k-1} D & = & Ee_1 \\ \mu & \in & \mathfrak{m}_E \end{array} \right.$$

$$(3) \qquad \left\{ \begin{array}{ccc} \varphi(e_1) & = & p^{k/2} \lambda e_1 \\ \varphi(e_2) & = & p^{k/2-1} \lambda e_2 \\ \mathrm{Fil}^{k-1} D & = & E(e_1 - \mathcal{L} e_2) \\ N(e_1) & = & e_2 \\ N(e_2) & = & 0 \\ k & \in & 2\mathbf{Z}_{>0} \\ (\lambda, \mathcal{L}) & \in & \{\pm 1\} \times E \end{array} \right.$$

(The reader can check that the above filtered (φ, N)-modules are all weakly admissible.)

Following Serre ([26]), define for $n \in \mathbf{Z}_{>0}$ and $g \in I_{\mathbf{Q}_p}$:

$$\theta_{p^n-1}(g) := \frac{g(p^{1/(p^n-1)})}{p^{1/(p^n-1)}} \in \mu_{p^n-1}(\overline{\mathbf{Q}}_p) \simeq \mathbf{F}_{p^n}^{\times} \hookrightarrow \overline{\mathbf{F}}_p^{\times}.$$

This turns out to be independent of the choice of $p^{1/(p^n-1)}$ and defines a tamely ramified character $\theta_{p^n-1} : I_{\mathbf{Q}_p} \to \overline{\mathbf{F}}_p^{\times}$. Let $T \subset V$ be a Galois stable $\mathcal{O}_E$-lattice and $\overline{T} := T \otimes_{\mathcal{O}_E} (\mathcal{O}_E/\mathfrak{m}_E)$ its reduction "modulo p". By [26] the semi-simplification of this reduction can be described in terms of powers of the characters θ_{p^n-1}. For instance, by noticing that the Galois representations associated to the filtered modules of Example 5.1 (1) are reducible, one immediately gets:

Proposition 5.2 ([17]). *Let V be a semi-stable p-adic representation of $G_{\mathbf{Q}_p}$ such that $D_{st}^*(V)$ is as in Example* 5.1(1). *Let $T \subset V$ be a Galois stable $\mathcal{O}_E$-lattice. Then:*

$$\overline{T} \simeq \begin{pmatrix} \theta_{p-1}^{k-1}\mathrm{Frob}(\overline{\lambda}) & * \\ 0 & \mathrm{Frob}(\overline{\lambda}^{-1}) \end{pmatrix}.$$

For cases (2) and (3) of 5.1, one needs integral p-adic Hodge theory. By computing explicit strongly divisible lattices in $S \otimes D$ for D as in 5.1 (2), (3) and reducing them modulo p, one gets, assuming of course $k < p+1$ (Basic Assumption):

Proposition 5.3 ([17]). *Let V be a semi-stable p-adic representation of $G_{\mathbf{Q}_p}$ such that $D_{st}^*(V)$ is as in Example* 5.1(2) *and assume $k < p+1$. Let $T \subset V$ be a Galois stable $\mathcal{O}_E$-lattice. Then:*

$$\overline{T}|_{I_{\mathbf{Q}_p}} \otimes \overline{\mathbf{F}}_p \simeq \begin{pmatrix} \theta_{p^2-1}^{k-1} & 0 \\ 0 & \theta_{p^2-1}^{p(k-1)} \end{pmatrix}.$$

And, finally, the semi-stable non-crystalline case, which is somewhat more involved:

Theorem 5.4 ([9]). *Let V be a semi-stable p-adic representation of $G_{\mathbf{Q}_p}$ such that $D_{st}^*(V)$ is as in Example* 5.1(3) *and assume $k < p+1$. Let $T \subset V$ be a Galois stable $\mathcal{O}_E$-lattice. Define $\ell :=$ val$(\mathcal{L})$, $[\ell]$ the greatest integer $\leq \ell$, and, if $\ell \in \mathbf{Z}$, $\alpha := \mathcal{L}/p^{\ell}$. Let $H_0 := 0$ and, for $n \in \mathbf{Z}_{>0}$, $H_n := \sum_{i=1}^{n} \frac{1}{i}$. Define also:*

$$a := (-1)^{\frac{k}{2}}\Big(-1 + \frac{k}{2}\Big(\frac{k}{2} - 1\Big)\big(\mathcal{L} + 2H_{k/2-1}\big)\Big)$$

and if $\ell \in \{-\frac{k}{2}+2, -\frac{k}{2}+1, ..., -1\}$:

$$b := (-1)^{\frac{k}{2}-\ell}\left(\frac{k}{2}-\ell\right)\binom{\frac{k}{2}-1-\ell}{-2\ell+1}\alpha.$$

(1) *If* $\mathrm{val}(a) = 0$, *then:*

$$\overline{T} \simeq \begin{pmatrix} \theta_{p-1}^{\frac{k}{2}}\mathrm{Frob}(\overline{a}^{-1}\overline{\lambda}) & * \\ 0 & \theta_{p-1}^{\frac{k}{2}-1}\mathrm{Frob}(\overline{a\lambda}) \end{pmatrix}$$

or

$$\overline{T} \simeq \begin{pmatrix} \theta_{p-1}^{\frac{k}{2}-1}\mathrm{Frob}(\overline{a\lambda}) & * \\ 0 & \theta_{p-1}^{\frac{k}{2}}\mathrm{Frob}(\overline{a}^{-1}\overline{\lambda}) \end{pmatrix}.$$

(2) *If* $\mathrm{val}(a) > 0$, *then:*

$$\overline{T}|_{I_{\mathbf{Q}_p}} \otimes \overline{\mathbf{F}}_p \simeq \begin{pmatrix} \theta_{p^2-1}^{\frac{k}{2}-1+p\frac{k}{2}} & 0 \\ 0 & \theta_{p^2-1}^{\frac{k}{2}+p(\frac{k}{2}-1)} \end{pmatrix}.$$

(3) *If* $\mathrm{val}(a) < 0$ (*i.e.* $\ell < 0$), *then:*

- *if* $\ell < -\frac{k}{2}+2$, *then* $\overline{T}|_{I_{\mathbf{Q}_p}} \otimes \overline{\mathbf{F}}_p \simeq \begin{pmatrix} \theta_{p^2-1}^{k-1} & 0 \\ 0 & \theta_{p^2-1}^{p(k-1)} \end{pmatrix}$,
- *if* $-\frac{k}{2}+2 \leq \ell < 0$ *and* $\ell \notin \mathbf{Z}$, *then:*

$$\overline{T}|_{I_{\mathbf{Q}_p}} \otimes \overline{\mathbf{F}}_p \simeq \begin{pmatrix} \theta_{p^2-1}^{\frac{k}{2}-[\ell]+p(\frac{k}{2}+[\ell]-1)} & 0 \\ 0 & \theta_{p^2-1}^{\frac{k}{2}+[\ell]-1+p(\frac{k}{2}-[\ell])} \end{pmatrix},$$

- *if* $-\frac{k}{2}+2 \leq \ell < 0$ *and* $\ell \in \mathbf{Z}$, *then:*

$$\begin{aligned} \overline{T} &\simeq \begin{pmatrix} \theta_{p-1}^{\frac{k}{2}-\ell}\mathrm{Frob}(\overline{b}^{-1}\overline{\lambda}) & * \\ 0 & \theta_{p-1}^{\frac{k}{2}+\ell-1}\mathrm{Frob}(\overline{b\lambda}) \end{pmatrix} \\ &\textit{or} \\ \overline{T} &\simeq \begin{pmatrix} \theta_{p-1}^{\frac{k}{2}+\ell-1}\mathrm{Frob}(\overline{b\lambda}) & * \\ 0 & \theta_{p-1}^{\frac{k}{2}-\ell}\mathrm{Frob}(\overline{b}^{-1}\overline{\lambda}) \end{pmatrix}. \end{aligned}$$

Remark 5.5. Proposition 5.3 and Theorem 5.4 are wrong in general for $k > p+1$.

These results can be applied to modular forms. Fix an embedding $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$. Let f be a cuspidal eigenform on $\Gamma_0(N)$ of weight $k \geq 2$ and

$\rho_f : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to GL_2(E_f)$ the p-adic global representation associated to f where $E_f \subset \overline{\mathbf{Q}}_p$ is a finite extension of $\mathbf{Q}_p$. Denote by $\overline{\rho}_f$ the semi-simplification modulo p of ρ_f and let $\overline{\rho}_{f,p} := \overline{\rho}_f|_{G_{\mathbf{Q}_p}}$. By [28] and [25], one easily deduces from Propositions 5.2, 5.3 and Theorem 5.4:

Corollary 5.6 (Deligne). *Let f be a cuspidal eigenform of weight k for $\Gamma_0(N)$ with $(p, N) = 1$. Let a_p be the eigenvalue of the Hecke operator T_p and assume* $\mathrm{val}(a_p) = 0$. *Then $\overline{\rho}_{f,p}$ is as in 5.2 with $\lambda \in \mathcal{O}_{E_f}^{\times}$ such that $p^{k-1}\lambda + \lambda^{-1} = a_p$.*

Corollary 5.7 (Fontaine, Serre). *Let f be a cuspidal eigenform of weight k for $\Gamma_0(N)$ with $(p, N) = 1$ and $2 \leq k \leq p$. Let a_p be the eigenvalue of the Hecke operator T_p and assume* $\mathrm{val}(a_p) \neq 0$. *Then $\overline{\rho}_{f,p}$ is as in* 5.3.

Corollary 5.8 ([9]). *Let f be a cuspidal eigenform of weight k for $\Gamma_0(N)$ with $p\|N$ and $2 \leq k \leq p$. Assume f is new at p. Let a_p be the eigenvalue of the Hecke operator T_p and $\mathcal{L}_p(f) \in E_f$ the invariant associated to f ([23]). Then $\overline{\rho}_{f,p}$ is as in* 5.4 *with $\mathcal{L} = \mathcal{L}_p(f)$ and $\lambda = a_p/p^{k/2-1} \in \{\pm 1\}$.*

Remark 5.9. Corollaries 5.6 and 5.7 were originally proven in several letters (letter from Deligne to Serre (28/05/74) for the first and letters from Serre to Fontaine (27/05/79) and Fontaine to Serre (25/06/79 and 10/07/79) for the second). One can find published alternative proofs of these corollaries in [11] which don't use neither p-adic Hodge theory nor integral p-adic Hodge theory, i.e. don't use nor prove Propositions 5.2 and 5.3, but show that Corollary 5.7 also holds in weight $k = p + 1$ (integral p-adic Hodge theory cannot yet deal directly with this case because of Assumption 2.1.2).

As a conclusion, let us mention the following fact. In [9], it is proven that there is a surprising link between the various cases of Theorem 5.4 and the Jordan-Hölder decomposition of the representation $\mathrm{Sym}^{k-2}\overline{\mathbf{F}}_p^2 \otimes_{\overline{\mathbf{F}}_p} \mathrm{St}$ of $\mathrm{GL}_2(\mathbf{Z}_p)$. Here St is the Steinberg representation of $\mathrm{GL}_2(\mathbf{Z}_p)$ in characteristic p, i.e. the inflation to $\mathrm{GL}_2(\mathbf{Z}_p)$ of the natural representation of $\mathrm{GL}_2(\mathbf{F}_p)$ on the space of functions $\mathbf{P}^1(\mathbf{F}_p) \to \overline{\mathbf{F}}_p$ with average value 0 (with $g \in \mathrm{GL}_2(\mathbf{F}_p)$ acting on a function through the usual action of g^{-1} on $\mathbf{P}^1(\mathbf{F}_p)$). This gives a mysterious link between integral p-adic Hodge theory and the representation theory of $\mathrm{GL}_2(\mathbf{Z}_p)$. I hope more is true in that direction.

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