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Constant Mean Curvature 1 Surfaces with Low Total Curvature in Hyperbolic 3-Space

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Abstract.

Surfaces of constant mean curvature one in hyperbolic 3-space have quite similar properties to minimal surfaces in Euclidean 3-space. We shall list the possibilities of constant mean curvature one surfaces in hyperbolic 3-space with low total absolute curvature, or low dual total absolute curvature, and compare them with the known classification of minimal surfaces with low total curvature. Complete proofs of the new results will be published in two forthcoming papers (listed in the bibliography).

§ Introduction

Recent developments in the study of constant mean curvature 1 (CMC-1) surfaces in hyperbolic 3-space H^3 (of constant sectional curvature -1) have led to many recently-discovered examples of such surfaces, and it is now well-known that CMC-1 surfaces in H^3 share quite similar properties with minimal surfaces in Euclidean 3-space $\boldsymbol{R}^3$. (See [1], [3], [7], [8], [11], [12], [13], [14], [15] and [16].)

The total absolute curvature of a complete minimal surface in $\boldsymbol{R}^3$ is a 4π-multiple of a nonnegative integer and is equal to the area of the Gauss image of the surface. All such surfaces with finite total absolute curvature less than or equal to 8π have been classified by Lopez [4]. Here we consider the corresponding problem for CMC-1 surfaces in H^3.

Classifying CMC-1 surfaces in H^3 with low total absolute curvature turns out to be more difficult and subtle than Lopez's classification, for the following reasons: Unlike the case of minimal surfaces in $\boldsymbol{R}^3$, the Bryant representation formula, which is an analogy of the Weierstrass representation formula, is not formulated by using line integration, but rather uses parallel transport along a path in the non-commutative group

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$\mathrm{SL}(2, \boldsymbol{C})$. Moreover, also unlike the case of minimal surfaces in $\boldsymbol{R}^3$, CMC-1 surfaces in H^3 have two Gauss maps, the hyperbolic Gauss map G and the secondary Gauss map g. The total absolute curvature of CMC-1 surfaces in H^3 is equal to the area of the image of the secondary Gauss map g, but since g might not be single-valued on the surface, the total absolute curvature might not be a 4π-multiple of an integer. The hyperbolic Gauss map G, on the other hand, does not relate to the total absolute curvature of the surface directly, but it has much clearer geometric meaning, namely the image $G(p)$ lies in the ideal boundary S^2 of the hyperbolic space at the point corresponding to the end of the normal geodesic emanating from the point p on the surface. Therefore, unlike the secondary Gauss map g, the hyperbolic Gauss map G is single-valued on the surface, but it may have essential singularities, even when the total absolute curvature is finite.

There is a natural notion of *dual* total absolute curvature for CMC-1 surfaces in H^3. A duality for CMC-1 surfaces is introduced in [13, Remark 1.8], [15], and Yu [17] (called *inverse surfaces* in [17]), which interchanges the role of the hyperbolic Gauss map and the secondary Gauss map (see Section 1.2). The total absolute curvature of the dual CMC-1 surfaces, i.e., the dual total absolute curvature, is equal to the area of the image of the hyperbolic Gauss map G. In particular, the dual total absolute curvature is always a 4π-multiple of an integer. Though the total absolute curvature of CMC-1 surfaces satisfies only the Cohn-Vossen inequality, the dual total absolute curvature has a much stronger lower bound, which is an analogue of the Osserman inequality for minimal surfaces (cf. [15], [18]).

The purpose of this note is to list the possibilities of CMC-1 surfaces in H^3 with low total absolute curvature, or low dual total absolute curvature, and compare them with Lopez's classification. Complete proofs of the new results will be published in forthcoming papers [9], [10]. Though the results at present do not achieve a full classification of CMC-1 surfaces with total absolute curvature or dual total absolute curvature less than or equal to 8π, the authors hope the results might be of help to readers interested in this subject.

§1. Preliminaries

1.1. Total absolute curvature

Let $f\colon M \to H^3$ be a conformal immersion with CMC-1 of a Riemann surface M into H^3. Denote the Gaussian curvature, the induced metric, and the induced area element by K, ds^2, and dA, respectively.

Then K is non-positive and $d\sigma^2 := (-K)ds^2$ is a conformal pseudometric of constant curvature 1 on M. We call the developing map $g\colon \widetilde{M} \to \boldsymbol{CP}^1$ the *secondary Gauss map* of f, where $\widetilde{M}$ is the universal cover of M. Namely, g is a conformal map such that the pull-back of the Fubini-Study metric of $\boldsymbol{CP}^1$ coincides with $d\sigma^2$.

In addition to the secondary Gauss map, the following two holomorphic invariants G and Q are closely related to the geometric properties of CMC-1 surfaces. The *hyperbolic Gauss map* $G\colon M \to \boldsymbol{CP}^1$ is defined as a holomorphic map on M as follows: Identifying the ideal boundary of H^3 with $\boldsymbol{CP}^1$, $G(p)$ is the asymptotic class of the normal geodesic of $f(M)$ starting at $f(p)$ and oriented in the mean curvature vector's direction. The *Hopf differential* Q is the $(2,0)$-part of the complexified second fundamental form, and is a symmetric holomorphic 2-differential on the Riemann surface M.

As K is a non-positive number, we can define the *total absolute curvature*

$$\mathrm{TA} := \int_M (-K)\, dA \in [0, +\infty].$$

Then TA is the area of the image in $\boldsymbol{CP}^1$ of the secondary Gauss map. The value of TA might not be an integral multiple of 4π — for example, the total curvature of the catenoid cousins [1, Example 2] admits *any* positive real number except 4π.

If the induced metric ds^2 is complete and of finite total absolute curvature (i.e., $\mathrm{TA} < +\infty$), then there exists a compact Riemann surface $\overline{M}$ and a finite set of points $\{p_1, \dots, p_n\} \subset \overline{M}$ such that M is biholomorphic to $\overline{M} \setminus \{p_1, \dots, p_n\}$. We call the p_j's the *ends* of f.

For CMC-1 surfaces, equality never holds in the Cohn-Vossen inequality [11]:

$$\frac{\mathrm{TA}}{2\pi} > -\chi(M) = n - 2 + 2\gamma\,, \tag{1.1}$$

where $\chi(M)$ denotes the Euler characteristic of M, and γ is the genus of $\overline{M}$.

1.2. Dual total absolute curvature

The *dual* CMC-1 immersion of a conformal CMC-1 immersion is defined as follows ([15], [17]): For a conformal CMC-1 immersion $f\colon M \to H^3$, there exists a holomorphic null immersion $F\colon \widetilde{M} \to \mathrm{SL}(2, \boldsymbol{C})$ such that $f = FF^*$, where $\widetilde{M}$ is the universal cover of M and $F^* = {}^t\overline{F}$. Here, we consider $H^3 = \mathrm{SL}(2,\boldsymbol{C})/\mathrm{SU}(2) = \{aa^* \,|\, a \in \mathrm{SL}(2,\boldsymbol{C})\}$ in the Hermitian model. We call F the *lift* of f. Then, the inverse matrix F^{-1} is also a holomorphic null immersion, and hence we have a new CMC-1 immersion $f^{\#} = F^{-1}(F^{-1})^* : \widetilde{M} \to H^3$, which is called the *dual* of f. The hyperbolic Gauss map (resp. secondary Gauss map, Hopf differential) of the dual immersion $f^{\#}$ is the secondary Gauss map g (resp. hyperbolic Gauss map G, sign-changed Hopf differential $-Q$) of f.

Although the dual immersion might only be defined on the universal cover $\widetilde{M}$ of M, the induced metric $ds^{\#2}$ and the Gaussian curvature $K^{\#}$ are well-defined on M itself. Hence we can define the *dual total absolute curvature* as

$$\mathrm{TA}^{\#} := \int_M (-K^{\#})\, dA^{\#} ,$$

where $dA^{\#}$ is the area element induced by $ds^{\#2}$. Since the secondary Gauss map of $f^{\#}$ is the hyperbolic Gauss map G of f, $d\sigma^{\#2} := (-K^{\#})ds^{\#2}$ is a pseudo-metric of constant curvature 1 with developing map G. Hence $\mathrm{TA}^{\#}$ is the area of the image of G on $\boldsymbol{CP}^1$.

As shown in [15], [17], the induced metric ds^2 of f is complete if and only if the dual metric $ds^{\#2}$ is complete. If we assume the immersion f is complete and of finite dual total absolute curvature (i.e., $\mathrm{TA}^{\#} < +\infty$), then, as in the finite total curvature case, M is biholomorphic to a finitely punctured compact Riemann surface: $M = \overline{M} \setminus \{p_1, \ldots, p_n\}$. Unlike the minimal surface case, the hyperbolic Gauss map might not extend to a meromorphic function on $\overline{M}$. The dual total absolute curvature $\mathrm{TA}^{\#}$ is finite if and only if the hyperbolic Gauss map can be extended to a meromorphic function on $\overline{M}$, and in this case, $\mathrm{TA}^{\#} = 4\pi \deg G$. In particular, $\mathrm{TA}^{\#}$ is an integral multiple of 4π.

For $\mathrm{TA}^{\#}$, a hyperbolic analogue of the Osserman inequality holds [15], namely

$$\frac{\mathrm{TA}^{\#}}{2\pi} \geq 2n - 2 + 2\gamma . \tag{1.2}$$

1.3. Notation

Assume f is complete with $\mathrm{TA} < \infty$ or $\mathrm{TA}^{\#} < \infty$, and let $M = \overline{M} \setminus \{p_1, \ldots, p_n\}$, where $\overline{M}$ is a compact Riemann surface. Then Q extends to a meromorphic differential on $\overline{M}$ [1]. We say an end p_j $(j = 1, \ldots, n)$ of a CMC-1 immersion is *regular* if the hyperbolic Gauss map is holomorphic at p_j. When $\mathrm{TA} < \infty$, an end is regular if and only if the order of the Hopf differential Q at p_j is at least -2. Otherwise, the hyperbolic Gauss map has an essential singularity at the end [1].

In this way, the orders of the Hopf differential at the ends are closely related to properties of the surface, so we now introduce a notation for these orders. In the following discussion, we say a surface

Type	TA	The surface	cf.
$\mathbf{O}(0)$	0	Plane	
$\mathbf{O}(-4)$	4π	Enneper's surface	
$\mathbf{O}(-5)$	8π		[4, Theorem 6]
$\mathbf{O}(-6)$	8π		[4, Theorem 6]
$\mathbf{O}(-2,-2)$	4π	Catenoid	
	8π	Double cover of the catenoid	
$\mathbf{O}(-1,-3)$	8π		[4, Theorem 5]
$\mathbf{O}(-2,-3)$	8π		[4, Theorem 4, 5]
$\mathbf{O}(-2,-4)$	8π		[4, Theorem 5]
$\mathbf{O}(-3,-3)$	8π		[4, Theorem 4]
$\mathbf{O}(-1,-2,-2)$	8π		[4, Theorem 5]
$\mathbf{O}(-2,-2,-2)$	8π		[4, Theorem 5]
$\mathbf{I}(-4)$	8π	Chen-Gackstatter surface	[4, Theorem 5], [2]

Table 1. Classification of minimal surfaces in $\boldsymbol{R}^3$ with TA $\leq 8\pi$ [4].

is of *type* $\mathbf{\Gamma}(d_1,\dots,d_n)$ if the surface is given as an immersion $f\colon \overline{M}\setminus\{p_1,\dots,p_n\}\to H^3$, where the order of the Hopf differential at p_j is d_j for each $j=1,\dots,n$. We use $\mathbf{\Gamma}$ because it is the capitalized letter corresponding to γ, the genus of $\overline{M}$. For instance, the class $\mathbf{I}(-4)$ means the class of surfaces of genus 1 with 1 end so that Q has a pole of order 4 at the end, and the class $\mathbf{O}(-2,-3)$ is the class of surfaces of genus 0 with two ends so that Q has a pole of order 2 at one end and a pole of order 3 at the other.

1.4. Minimal surfaces with TA $\leq 8\pi$

Using the above notation, the classification of complete minimal surfaces in $\boldsymbol{R}^3$ with TA $\leq 8\pi$ (Lopez [4]) is listed in Table 1.

§2. Complete CMC-1 surfaces with TA $\leq 4\pi$

It is well-known that the only complete minimal surfaces in $\boldsymbol{R}^3$ of total curvature less than or equal to 4π are the plane, the Enneper surface, and the catenoid. In this section, we shall introduce a complete classification of CMC-1 surfaces in H^3 with TA $\leq 4\pi$.

Assume $f\colon M\to H^3$ is a complete conformal immersion of TA $\leq 4\pi$. Then, by the Cohn-Vossen inequality (1.1), the genus γ and the number

Type	TA	The surface	cf.
$\mathbf{O}(0)$	0	Horosphere	
$\mathbf{O}(-4)$	4π	Enneper cousins	[1, Example 1]
$\mathbf{O}(-2,-2)$	$4\pi\mu$ $(0<\mu<1)$	Catenoid cousins and their m-fold covers	[1, Example 2] [11, Theorem 6.2]
$\mathbf{O}(-2,-2)$	4π	$\star$	[11, Theorem 6.2]

Table 2. Classification of CMC-1 surfaces in H^3 with TA $\leq 4\pi$.

of ends n are restricted to the following cases:

$$(\gamma, n) = (0,1), \quad (0,2), \quad (0,3), \quad (1,1).$$

However, the cases $(\gamma, n) = (0,3)$ and $(1,1)$ do not occur. More precisely, the following theorem holds:

Theorem 1 ([10]). *Any complete* CMC-1 *surface in* H^3 *with* TA $\leq 4\pi$ *is one of those in Table* 2.

The case marked $\star$ is the class of immersions $f\colon \boldsymbol{C}\setminus\{0\} \to H^3$ given by the Weierstrass data

$$\left(g, \omega := \frac{Q}{dg}\right) = \left(az^l + b, \frac{m^2 - l^2}{4a}\frac{dz}{z^2}\right), \quad a \in \boldsymbol{C}\setminus\{0\},\ b \in \boldsymbol{C}, \tag{2.1}$$

for $l = 1$ and $m = 2, 3, \dots$ as in the equation (6.5) in [11]. When $b = 0$, the surface is a catenoid cousin. However, if $b \neq 0$, the surface is not rotationally symmetric.

Though we do not give the details of the proof here, we remark that the proof is more difficult than for the corresponding case of minimal surfaces in $\boldsymbol{R}^3$. For example, the nonexistence of CMC-1 surfaces in H^3 with $(\gamma, n) = (1,1)$ is shown by applying a flux formula in [8]. The nonexistence of CMC-1 surfaces with $(\gamma, n) = (0,3)$ is shown by first applying the classification of irreducible CMC-1 surfaces of type $\mathbf{O}(-2,-2,-2)$ in [16], and then one can show that TA $\geq 4\pi$ for such surfaces. In [10], we will show the stronger inequality TA $> 4\pi$ for CMC-1 surfaces with $(\gamma, n) = (0,3)$. However, the proof is not simple. In [10], some other results for surfaces with TA $\leq 8\pi$ will also be discussed.

§3. Complete CMC-1 surfaces with TA$^{\#}$ $\leq 8\pi$

In this section, we introduce a partial result on classification of CMC-1 surfaces in H^3 with TA$^{\#}$ $\leq 8\pi$. Note that TA may take the

Type	$\mathrm{TA}^{\#}$	Status	cf.
$\mathbf{O}(0)$	0	classified^0	Horosphere
$\mathbf{O}(-4)$	4π	classified	Dual Enneper cousins [7, Example 5.4]
$\mathbf{O}(-2,-2)$	4π	classified	Catenoid cousins, and $\star\star$ [1, Example 2], [11, Theorem 6.2]
$\mathbf{O}(-5)$	8π	classified	[9]
$\mathbf{O}(-6)$	8π	classified	[9]
$\mathbf{O}(-2,-2)$	8π	classified	Double covers of catenoid cousins, and $\star\star\star$ [11, Theorem 6.2]
$\mathbf{O}(-1,-4)$	8π	classified^0	[9]
$\mathbf{O}(-2,-3)$	8π	classified	[9]
$\mathbf{O}(-2,-4)$	8π	classified	[9]
$\mathbf{O}(-3,-3)$	8π	existence^d	[9]
$\mathbf{O}(-1,-1,-2)$	8π	classified^0	[9]
$\mathbf{O}(-1,-2,-2)$	8π	classified	[9]
$\mathbf{O}(-2,-2,-2)$	8π	existence	Classified for the irreducible case [16]
$\mathbf{I}(-3)$	8π	unknown	
$\mathbf{I}(-4)$	8π	existence^d	[9]
$\mathbf{I}(-1,-1)$	8π	unknown	
$\mathbf{I}(-2,-2)$	8π	existence^1	[6]

Table 3. Classification of CMC-1 surfaces in H^3 with $\mathrm{TA}^{\#} \leq 8\pi$.

value $+\infty$ even if $\mathrm{TA}^{\#}$ is finite. By (1.2), the genus γ and the number n of ends of such surfaces are restricted to the following cases:

$$(\gamma, n) = (0,1), \quad (0,2), \quad (0,3), \quad (1,1), \quad (1,2), \quad (2,1).$$

However, the case $(\gamma, n) = (2,1)$ does not occur, which is a consequence of the flux formula in [8]. A list of possible surfaces with $\mathrm{TA}^{\#} \leq 8\pi$ is shown in Table 3 (for the proof, see [9]). In this table,

- *classified* means the complete list of the surfaces in such a class is known,
- *classified*^{0} means there exists a unique surface (up to isometries of H^3 and deformations that come from its reducibility [7, Theorem 3.2]),
- *existence* means that examples of such surfaces are known to exist, but they are not yet classified,

- *existence*^{d} means that examples can be obtained by deforming from a minimal surface in $\boldsymbol{R}^3$, using the method in [7],
- *existence*^{1} means there exists a 1 parameter family of examples, which is not deformations coming from reducibility,
- *unknown* means that neither existence nor nonexistence is known yet.

The case marked $\star\star$ (resp. $\star\star\star$) is the class of surfaces given by the Weierstrass data (2.1) for $m = 1$ and $l = 2, 3, \dots$ (resp. $m = 2$ and $l = 1, 3, 4, \dots$).

It is interesting to compare Table 3 with Table 1. In the case of minimal surfaces in $\boldsymbol{R}^3$ with TA $\leq 8\pi$, the cases

$$\mathbf{O}(-1,-4), \quad \mathbf{O}(-1,-1,-2), \quad \mathbf{I}(-2,-2)$$

do not occur, whereas these cases really do occur for CMC-1 surfaces in H^3. On the other hand, there is no CMC-1 surface in H^3 of type $\mathbf{O}(-1,-3)$, in spite of the fact that such minimal surfaces exist in $\boldsymbol{R}^3$. Although the existence of CMC-1 surfaces in H^3 of type $\mathbf{I}(-3)$ and $\mathbf{I}(-1,-1)$ is still unknown, Table 3 shows the existence of CMC-1 surfaces in H^3 for which the corresponding minimal surfaces in $\boldsymbol{R}^3$ cannot exist.

References

[1] R. Bryant, *Surfaces of mean curvature one in hyperbolic space*, Astérisque, **154–155** (1987), 321–347.

[2] C. C. Chen and F. Gackstatter, *Elliptische und hyperelliptische Funktionen und vollständige Minimalflächen vom Enneperschen Typ*, Math. Ann., **259** (1982), 359–369.

[3] P. Collin, L. Hauswirth and H. Rosenberg, *The geometry of finite topology surfaces properly embedded in hyperbolic space with constant mean curvature one*, Ann. of Math., to appear.

[4] F. J. Lopez, *The classification of complete minimal surfaces with total curvature greater than* -12π, Trans. Amer. Math. Soc., **334** (1992), 49–74.

[5] R. Osserman, A Survey of Minimal Surfaces, 2nd ed., Dover, 1986.

[6] W. Rossman and K. Sato, *Constant mean curvature surfaces with two ends in hyperbolic space*, Experimental Math., **7(2)** (1998), 101–119.

[7] W. Rossman, M. Umehara and K. Yamada, *Irreducible constant mean curvature* 1 *surfaces in hyperbolic space with positive genus*, Tôhoku Math. J., **49** (1997), 449–484.

[8] ———, *A new flux for mean curvature* 1 *surfaces in hyperbolic* 3*-space, and applications*, Proc. Amer. Math. Soc., **127** (1999), 2147–2154.

[9] ———, *Mean curvature* 1 *surfaces in hyperbolic* 3*-space with low total curvature* I, preprint, `math. DG/0008015`.

[10] ———, *Mean curvature* 1 *surfaces in hyperbolic* 3*-space with low total curvature* II, preprint.

[11] M. Umehara and K. Yamada, *Complete surfaces of constant mean curvature-*1 *in the hyperbolic* 3*-space*, Ann. of Math., **137** (1993), 611–638.

[12] ———, *A parameterization of Weierstrass formulae and perturbation of some complete minimal surfaces of* $\boldsymbol{R}^3$ *into the hyperbolic* 3*-space*, J. Reine Angew. Math., **432** (1992), 93–116.

[13] ———, *Surfaces of constant mean curvature-c in* $H^3(-c^2)$ *with prescribed hyperbolic Gauss map*, Math. Ann., **304** (1996), 203–224.

[14] ———, *Another construction of a* CMC-1 *surface in* H^3, Kyungpook Math. J., **35** (1996), 831–849.

[15] ———, *A duality on* CMC-1 *surface in the hyperbolic* 3*-space and a hyperbolic analogue of the Osserman Inequality*, Tsukuba J. Math., **21** (1997), 229–237.

[16] ———, *Metric of constant curvature one with three conical singularities on* 2*-sphere*, Illinois J. Math., **44(1)** (2000), 72–94.

[17] Z. Yu, *Value distribution of hyperbolic Gauss maps*, Proc. Amer. Math. Soc., **125** (1997), 2997–3001.

[18] ———, *The inverse surface and the Osserman Inequality*, Tsukuba J. Math., **22** (1998), 575–588.

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