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Discrete Spectrum and Weyl's Asymptotic Formula for Incomplete Manifolds

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Abstract.

Motivated by recent interest in the spectrum of the Laplacian of incomplete surfaces with isolated conical singularities, we consider more general incomplete m-dimensional manifolds with singularities on sets of codimension at least 2. With certain restrictions on the metric, we establish that the spectrum is discrete and satisfies Weyl's asymptotic formula.

§1. Discreteness of the Spectrum

When one studies the Morse index of minimal surfaces in Euclidean 3-space $\mathbb{R}^3$ or of mean curvature 1 surfaces in hyperbolic 3-space $\mathbb{H}^3$, the problem reduces to the study of the number of eigenvalues less than 2 of the spectrum of the Laplace-Beltrami operator on Met_1 surfaces [FC], [UY], [LR]. (Met_1 surfaces are incomplete 2-dimensional manifolds with constant curvature 1 and isolated conical singularities.) Met_1 surfaces are known to have pure point spectrum and satisfy Weyl's asymptotic formula.

Here we will show that the spectrum is discrete and that Weyl's asymptotic formula holds for more general incomplete manifolds. We allow the dimension to be arbitrary; we do not make any specific assumptions about the curvature; and we allow more general singularities, of at least codimension 2 (in a sense to be made precise below). This more general setting allows us to consider singularities such as a product of an $m - n$ dimensional metric cone with a portion of $\mathbb{R}^n$ ($m \geq n + 2$), one of our desired examples. In this example, the incomplete metric is singular only in the direction of the metric cone and not on the portion of $\mathbb{R}^n$ itself, so generally the incomplete manifolds and their metrics $\tilde{g}$ that we consider will not be conformally equivalent to open sets of compact Riemann manifolds, unlike the case of Met_1 surfaces. With this in

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mind, we now define the types of incomplete manifolds and metrics $\tilde{g}$ that we will study here.

Let (M, g) be a compact manifold of dimension m with smooth Riemannian metric g. Let N be a compact submanifold of dimension n with codimension $m - n \geq 2$. Suppose further that in a neighborhood of N the metric g can be diagonalized; that is, there exist local coordinates $(x_1, \dots, x_{m-n}, y_1, \dots, y_n)$, where $(0, \dots, 0, y_1, \dots, y_n)$ are coordinates for N, so that $(dx_1, \dots, dx_{m-n}, dy_1, \dots, dy_n)$ is globally defined in some open neighborhood of N and so that the metric g is written

$$g = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix},$$

where g_1 is an $m - n \times m - n$ positive definite matrix, and g_2 is an $n \times n$ positive definite matrix. (For example, such a case can occur if M has a product structure $M = M_1 \times N$ near N, where M_1 is an $m - n$ dimensional compact Riemannian manifold.)

Theorem 1.1. *Let N be an n-dimensional compact submanifold of an m-dimensional compact manifold (M, g) with $m \geq n + 2$ such that the metric g can be diagonalized near N. Choose local coordinates in a neighborhood of N so that*

$$g = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}$$

in this neighborhood. Let $\tilde{g}$ be another smooth regular metric on $M \setminus N$ so that

$$\tilde{g} = \begin{pmatrix} f^2 g_1 & 0 \\ 0 & g_2 \end{pmatrix}$$

in a neighborhood of N, where $f \in C^\infty(M \setminus N)$.

If $m = 2$, assume that $f \in L_g^{2+\epsilon}(M)$ for some $\epsilon \in (0, \infty)$.

If $m \geq 3$, assume that $\inf(f) > 0$ *and $f \in L_g^{(m(m-n)/2)+\epsilon}(M)$ for some $\epsilon \in (0, \infty)$.*

Then the Sobolev space $W_{\tilde{g}}^{1,2}(M \setminus N)$ with respect to $\tilde{g}$ is compactly included in $L_{\tilde{g}}^2(M \setminus N)$.

Proof. When $m \geq 3$ and $p \in (2, 2m/(m-2))$ (resp. $m = 2$ and $p \in (2, \infty)$), then the inclusion $W_g^{1,2}(M)$ into $L_g^p(M)$ is compact. When $m \geq 3$ and $f \in L^{(m(m-n)/2)+\epsilon}$ (resp. $m = 2$ and $f \in L^{2+\epsilon}$) for some positive ϵ, then the inclusion $L_g^p(M)$ into $L_{\tilde{g}}^2(M \setminus N)$ is continuous, by Hölder's inequality. For example, when $m \geq 3$, we can choose

$$p = \frac{m + (2\epsilon/(m-n))}{(m/2) + (\epsilon/(m-n)) - 1},$$

$$\begin{tikzcd}
 & L^2_g(M) & \\
W^{1,2}_g(M) \arrow[ur] \arrow[rr] & & L^p_g(M) \arrow[d] \\
W^{1,2}_{\tilde g}(M \setminus N) \arrow[u] \arrow[rr] & & L^2_{\tilde g}(M \setminus N)
\end{tikzcd}$$

Figure 1. The compact inclusion of $W^{1,2}_{\tilde g}(M \setminus N)$ into $L^2_{\tilde g}(M \setminus N)$.

and then the Hölder inequality implies

$$\|u\|_{L^2_{\tilde g}} = \sqrt{\int u^2 f^{m-n} dA} \le c \cdot \|u\|_{L^p_g}$$

for

$$c = \left(\int f^{(m(m-n)/2)+\epsilon} dA \right)^{((m/2)+(\epsilon/(m-n)))^{-1}/2} < \infty \, .$$

So we only need to show that $W^{1,2}_{\tilde g}(M \setminus N)$ is continuously contained in $W^{1,2}_g(M)$ to conclude $W^{1,2}_{\tilde g}(M \setminus N)$ is compactly contained in $L^2_{\tilde g}(M \setminus N)$. When $m \ge 3$, this is clear, since $\inf(f) > 0$. When $m = 2$, then $n = 0$, and g and $\tilde g$ are conformally equivalent on $M \setminus N$. Suppose by way of contradiction that the inclusion is not continuous, that is, that there exists a sequence of functions u_k such that $\|u_k\|_{W^{1,2}_g} = 1$ and $\|u_k\|_{W^{1,2}_{\tilde g}} < 1/k$. By choosing a subsequence if necessary, we may assume the following:

(1) there exists a function u such that $u_k \to u$, $W^{1,2}_g$-weakly,
(2) there exists a function v such that $u_k \to v$, L^p_g-strongly,
(3) $u_k \to v$, $L^2_{\tilde g}$-strongly,

(4) $u_k \to v$, L^2_g-strongly.

The fourth item follows from the fact that $\|u_k - v\|_{L^2_g} \le \hat{c} \cdot \|u_k - v\|_{L^p_g}$, since (M, g) is smooth and compact. As u_k converges to both u and v L^2_g-weakly, $u = v$. Also,

$$1 = \liminf_{k \to \infty} \|u_k\|_{W^{1,2}_g} \ge \|u\|_{W^{1,2}_g} .$$

Let ∇ and dA (resp. $\tilde{\nabla}$ and $d\tilde{A}$) denote the gradient and area-form with respect to the metric g (resp. $\tilde{g}$). Then, using $\int_M |\nabla u_k|^2_g dA = \int_M |\tilde{\nabla} u_k|^2_{\tilde{g}} d\tilde{A}$, we have $\int_M u_k^2 dA \to 1$ and $\int_M u^2 dA = 1$ and $\int_M |\nabla u|^2_g dA = 0$, so u is a nonzero constant. Also, $\int_M u_k^2 d\tilde{A} \to \int_M u^2 d\tilde{A} = 0$, so $\int_M d\tilde{A} = 0$. This is a contradiction, since f is smooth on $M \setminus N$ and not identically zero. $\square$

Remark. For $m \ge 3$, the condition $\inf(f) > 0$ is a simple way to ensure $W^{1,2}_{\tilde{g}}$ is continuously contained in $W^{1,2}_g$, but it is necessary. This is not generally a continuous inclusion if $\inf(f) = 0$. For example, suppose $\inf(f) = 0$, and $n = 0$. Let $M_k = \{p \in M \setminus N \mid |f(p)| < 1/k\} \neq \emptyset$. Choose u_k so that $\mathrm{supp}(u_k) \subset M_k$ and $\|u_k\|^2_{W^{1,2}_g} = 1$. Then $\tilde{g} = f^2 g$ and g are conformally equivalent and

$$\|u_k\|^2_{W^{1,2}_{\tilde{g}}} = \int_{M_k} (u_k)^2 f^m dA + \int_{M_k} |\nabla u_k|^2_g f^{m-2} dA \le \frac{1}{k^{m-2}} \|u_k\|^2_{W^{1,2}_g}$$
$$= \frac{1}{k^{m-2}} \to 0$$

as $k \to \infty$. Hence, we do not have continuous inclusion.

Let $\overline{\triangle}^F_{\tilde{g}}$ denote the Freidrichs' self-adjoint extension of the Laplacian with domain $C^\infty_0(M \setminus N)$, and let $W^{1,2}_{0,\tilde{g}}(M \setminus N)$ be the closure of $C^\infty_0(M \setminus N)$ in the $W^{1,2}_{\tilde{g}}(M \setminus N)$ norm. Standard arguments give the following:

Corollary 1.1. *Let $(M \setminus N, \tilde{g})$ be as in Theorem 1.1. The operator $\overline{\triangle}^F_{\tilde{g}}$ on $(M \setminus N, \tilde{g})$ has discrete spectrum consisting of eigenvalues $0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_j \le \cdots \to +\infty$, each with multiplicity 1. The corresponding eigenfunctions $\phi_1, \phi_2, \ldots \in W^{1,2}_{\tilde{g}}(M \setminus N)$ can be chosen as an orthonormal basis for $L^2_{\tilde{g}}(M \setminus N)$. Furthermore, the variational characterization for the eigenvalues holds*:

$$\lambda_j = \inf_{V^j} \sup_{\phi \in V^j, \phi \neq 0} \frac{\|\nabla \phi\|^2_{L^2_{\tilde{g}}(M \setminus N)}}{\|\phi\|^2_{L^2_{\tilde{g}}(M \setminus N)}} ,$$

where V^j *represents an arbitrary* j*-dimensional subspace of* $W^{1,2}_{0,\tilde{g}}(M \setminus N)$.

Remark. When $W^{1,2}_{0,\tilde{g}}(M \setminus N) = W^{1,2}_{\tilde{g}}(M \setminus N)$, $(M \setminus N, \tilde{g})$ has negligible boundary, in Gaffney's sense [G]. Therefore, the Laplacian considered on Gaffney's domain of functions is essentially self-adjoint. Let $\overline{\triangle}^G_{\tilde{g}}$ denote the unique self-adjoint extension. One can show that the two self-adjoint operators $\overline{\triangle}^F_{\tilde{g}}$ and $\overline{\triangle}^G_{\tilde{g}}$ have equal domains, that is,

$$\triangle := \overline{\triangle}^G_{\tilde{g}} = \overline{\triangle}^F_{\tilde{g}} \ .$$

Since $(M \setminus N, \tilde{g})$ has negligible boundary, this operator's domain has no boundary conditions at N. So when $W^{1,2}_{0,\tilde{g}}(M \setminus N) = W^{1,2}_{\tilde{g}}(M \setminus N)$, this is the operator for which we will study the spectrum, and it is the same operator as that used in the study of Morse index of minimal surfaces in $\mathbb{R}^3$ and mean curvature 1 surfaces in $\mathbb{H}^3$.

As seen in the above remark, we would like to consider the cases where $W^{1,2}_{0,\tilde{g}}(M \setminus N) = W^{1,2}_{\tilde{g}}(M \setminus N)$. We will also need this property for establishing Weyl's asymptotic formula, so we now give a sufficient condition to imply this property [M2]. In order to state it, here we introduce the notion of capacity and Cauchy boundary [M1],[M2].

Definition 1.1. Let M be an arbitrary Riemannian manifold. We denote by $\mathcal{O}$, the family of all open subsets of the completion $\overline{M}$ of M. For $A \in \mathcal{O}$, we define the set of functions L_A by

$$L_A = \{f \in W^{1,2}(M) \mid f \geq 1 \text{ a.e. on } A\}.$$

We define the capacity of A, $\mathrm{Cap}(A)$, by

$$\mathrm{Cap}(A) = \begin{cases} \inf\limits_{f \in L_A} \|f\|_{W^{1,2}}, & L_A \neq \phi, \\ \infty, & L_A = \phi. \end{cases}$$

For a Borel set $B \subset \overline{M}$, we define the capacity $\mathrm{Cap}(B)$ by

$$\mathrm{Cap}(B) = \inf_{A \in \mathcal{O},\, B \subset A} \mathrm{Cap}(A).$$

We say that a subset B of $\overline{M}$ is *almost polar* if $\mathrm{Cap(B)} = 0$.

Definition 1.2. The Cauchy boundary ∂M of M is defined by

$$\partial M := \overline{M} \setminus M,$$

where $\overline{M}$ is the completion of M with respect to the Riemannian distance.

Lemma 1.1 ([M2]). *For an arbitrary Riemannian manifold M, let ∂M denote the Cauchy boundary of M. If the capacity of ∂M is finite, then the two Sobolev spaces $W_0^{1,2}(M)$ and $W^{1,2}(M)$ coincide if and only if ∂M is an almost polar set.*

In the case of Theorem 1.1, the Cauchy boundary of $M \setminus N$ is N. It is shown in [M1] that when the lower Minkowski codimension of the Cauchy boundary is not less than 2, then ∂M is almost polar, where the lower Minkowski codimension is defined as follows:

Definition 1.3. The lower Minkowski codimension of ∂M is defined to be

$$\underline{\mathrm{codim}}_{\mathcal{M}}(\partial M) := \liminf_{R \to 0} \frac{\log(\mathrm{vol}(\mathcal{N}_R))}{\log(R)} ,$$

where $\mathcal{N}_R$ is a radius R tubular neighborhood of ∂M.

We now consider some examples.

Example 1.1. Consider the "football". Set $M = \mathbb{C} \cup \{\infty\}$ and $N = \{0, \infty\}$ ($m = 2$ and $n = 0$) and set

$$g = \frac{4(dx^2 + dy^2)}{(1 + r^2)^2} , \quad f = \frac{\mu r^{\mu-1}(1 + r^2)}{1 + r^{2\mu}} , \quad \mu \in \mathbb{R}^+ , \quad \tilde{g} = f^2 g ,$$

where $r = \sqrt{x^2 + y^2}$. Note that $f \in L_g^{2+\epsilon}(M \setminus N)$ for some $\epsilon > 0$, and $\underline{\mathrm{codim}}_{\mathcal{M}}(N) = 2$ for any μ. When $\mu < 1$, the football is an Alexandrov space and $\triangle$ has discrete spectrum, by [KMS] or by Theorem 1.1 above. When $\mu > 1$, the football is not an Alexandrov space, but the spectrum is still discrete, by Theorem 1.1 (see also Lemma 4.3 of [LR]).

Example 1.2. Consider a compact m-dimensional manifold M with metric g, and suppose M contains the unit ball B^m so that g is the standard Euclidean metric on $B^m \subset M$. Let $N = \{\vec{o}\}$ be the center point of $B^m \subset M$. Let $f = r^\ell$ on B^m and $\tilde{g} = f^2 g$ with $\ell \in (-2/m, 0)$, and extend f to be positive and smooth on $M \setminus B^m$. Thus $(M \setminus N, \tilde{g})$ is not complete, and $\underline{\mathrm{codim}}_{\mathcal{M}}(N) = m$ for any ℓ. Also, as f satisfies the conditions of Theorem 1.1, $\triangle$ on $(M \setminus N, \tilde{g})$ has discrete spectrum.

Example 1.3. As it is known that Alexandrov spaces have discrete spectrum [KMS], we are interested in finding examples that are not Alexandrov spaces and for which Theorem 1.1 can be applied. The footballs with $\mu > 1$ provide such examples in two dimensions. The following example shows that one can easily find such examples in higher dimensions as well. (We choose a slightly complicated function f in order to easily verify it will not be an Alexandrov space.)

Consider the previous example with $m = 3$; that is, M is compact, 3-dimensional, $N = \{\vec{o}\} \subset B^3 \subset M$, and g is the Euclidean metric on B^3. Set $f = \cos^2(\phi)r^\ell + \sin^2(\phi)(1 + r^{3/2}) \in L_g^{(9/2)+\epsilon}(M)$ on $B^3 \subset M$, where (r, θ, ϕ) are the spherical coordinates of B^3, and $\ell \in (-2/3, 0)$. Extend f to be positive and smooth on $M \backslash B^3$, and let $\tilde{g} = f^2 g$. Then $(M \backslash N, \tilde{g})$ is not complete, and the conditions of Theorem 1.1 are satisfied. Hence the spectrum of $\overline{\triangle}_{\tilde{g}}^F$ is discrete. The ball B^3 is invariant under the isometry $(r, \theta, \phi) \to (r, \theta, \pi - \phi)$, thus the sectional curvature in the $\{\phi = \pi/2\}$-plane is $K_{\tilde{g}} = -(\triangle \ln(f))/f^2 \to -\infty$ near N, so it is not an Alexandrov space.

Example 1.4. Consider the 3-dimensional torus $M = T^3 = \mathbb{R}^3/\mathbb{Z}^3$ with the standard Euclidean metric g, and the 1-dimensional torus $N = S^1 = (\mathbb{R}/\mathbb{Z}, 0, 0) \subset M$. We will use cylindrical coordinates (x, r, θ), where r is the radial distance to N and x is the arc-length along N. Let $f = \cos^2(\theta) + \sin^2(\theta) r^\ell$ near N with $\ell \in (-2/3, 0)$, and extend f to be positive and smooth away from N. This manifold is incomplete, and $\overline{\triangle}_{\tilde{g}}^F$ has discrete spectrum, by Theorem 1.1 and Corollary 1.1.

Remark. Suppose M is 2-dimensional and contains B^2 so that g is the standard Euclidean metric when restricted to B^2. Suppose $N = \{\vec{o}\} \subset B^2 \subset M$ and $f = -1/(r \ln(r))$ near N. Then, with respect to $\tilde{g} = f^2 g$, we have a complete end at N that is a curvature -1 psuedosphere of finite area, so the spectrum is not discrete [D], [Mu]. Since $f \in L^2(M, g)$, but $f \notin L^{2+\epsilon}(M, g)$ for all positive ϵ, we know Theorem 1.1 is sharp when $m = 2$. (If we had chosen $f = 1/r \in L^{2-\epsilon}(M, g)$ for all small positive ϵ instead, we would have produced a round cylindrical end of radius 1 which does not have discrete spectrum and does not have finite area.)

Remark. Consider $M = T^2 \times T^{m-2}$ and $N = T^{m-2}$ and $f = -1/(r \ln(r))$ near N, where r is radial distance to N. Let the diagonalized coordinates near N be $(x_1, x_2, y_1, \ldots, y_{m-2})$, inherited from the standard rectangular Euclidean coordinates of $\mathbb{R}^m$. Then $(M \setminus N, \tilde{g})$ is complete, and the sectional curvatures are

$$K_{\tilde{g}}(\partial_{x_1}, \partial_{x_2}) = -1\ , \quad K_{\tilde{g}}(\partial_{x_i}, \partial_{y_j}) = 0\ , \quad K_{\tilde{g}}(\partial_{y_i}, \partial_{y_j}) = 0\ .$$

So the Ricci curvature is bounded below, and hence the essential spectrum is not empty [D, Theorem 3.1]. So Theorem 1.1 is not true for this $f \in L^p$, $p \leq 2$. Hence, for all m, the restriction on f in Theorem 1.1 cannot be weakened to $f \in L^p$ for some $p \leq 2$.

Remark. Donnelly and Li [DL] have found complete examples $(M \setminus N, \tilde{g})$ where $M = \mathbb{R}^m \cup \{\infty\}$ and $N = \infty$ $(n = 0)$ and $\tilde{g}$ is rotationally invariant, so that sectional curvature converges to $-\infty$ at N

and $(M \setminus N, \tilde{g})$ has pure point spectrum. For example, let $m = 2$ and $\tilde{g} = dr^2 + \exp(-r^k)d\theta^2$, $k > 1$, in radial coordinates (r, θ) of $\mathbb{R}^2$. It is complete and its single end is conformally a punctured disk, and since the curvature converges to $-\infty$ at the end, it has pure point spectrum [DL]. Theorem 1.1 does not apply to such examples.

§2. Weyl's formula

In this section, let M be an m-dimensional Riemannian manifold with finite volume and finite diameter. M can be noncompact and incomplete.

Remark. The manifolds $(M \setminus N, \tilde{g})$ in Theorem 1.1 have finite volume, since $f \in L_g^{(m(m-n)/2)+\epsilon} \subseteq L_g^{m-n}$ implies $\mathrm{vol}(M \setminus N, \tilde{g}) = \int_M d\tilde{A} = \int_M f^{m-n} dA < \infty$.

Before stating and proving Weyl's asymptotic formula, we establish some notation. Let $\mathcal{N}_R$ be a radius R tubular neighborhood of the Cauchy boundary ∂M of M. Note that $\mathrm{vol}(M \setminus \mathcal{N}_R) + \mathrm{vol}(\mathcal{N}_R) = \mathrm{vol}(M)$ and $\mathrm{vol}(\mathcal{N}_R) \to 0$ as $R \to 0$. Define the Neumann isoparimetric constant of $\mathcal{N}_R$ by

$$C_R := \inf_{\gamma} \frac{\mathrm{vol}(\gamma)}{\min\{\mathrm{vol}(M_1), \mathrm{vol}(M_2)\}^{(m-1)/m}} ,$$

where the infimum is taken over all hypersurfaces γ of $\mathcal{N}_R$ which divide $\mathcal{N}_R$ into two parts M_1 and M_2, and where $\mathrm{vol}(\gamma)$ represents the $m-1$ dimensional volume of γ and $\mathrm{vol}(M_j)$ represents the m-dimensional volume of M_j.

Here, we will assume that

$$C := \inf_{R>0} C_R > 0 .$$

Then, since M has finite volume, one can see that M, N_R and $M \setminus N_R$ all have pure point spectra. Let $\lambda_j^{1,N}$ (resp. $\lambda_j^{2,N}$) be the Neumann eigenvalues on $\mathrm{Int}(M \setminus \mathcal{N}_R)$ (resp. $\mathrm{Int}(\mathcal{N}_R)$) counted with their multiplicities (i.e. listed in nondecreasing order, and the number of times that any eigenvalue appears in the list equals its multiplicity). Let $\lambda_j^{3,N}$ be the Neumann eigenvalues of $\mathrm{Int}(M \setminus \mathcal{N}_R) \cup \mathrm{Int}(\mathcal{N}_R)$ counted with their multiplicities. Let λ_j^D be the Dirichlet eigenvalues on $\mathrm{Int}(M \setminus \mathcal{N}_R)$ counted with their multiplicities. Here, we state Weyl's asymptotic formula for M.

Theorem 2.1. *Let M be an m-dimensional Riemannian manifold with finite volume and finite diameter. If the Cauchy boundary of M is* *an almost polar set and $C > 0$, then the eigenvalues λ_j of the Laplacian $\triangle$ satisfy Weyl's asymptotic formula*

$$\lim_{j\to\infty} \frac{\lambda_j^{m/2}\mathrm{vol}(M)}{j} = \frac{(2\pi)^m}{\mathrm{vol}(B^m)} .$$

Proof. Let $W = (2\pi)^m/\mathrm{vol}(B^m)$. Note that $\lambda_j \leq \lambda_j^D$ by Dirichlet-Neumann bracketing techniques (see, for example, volume 4 of [RS]). Note also that, since M has finite diameter and therefore $M \setminus \mathcal{N}_R$ is relatively compact, the λ_j^D satisfy Weyl's asymptotic formula on $M\setminus\mathcal{N}_R$. So

$$\lambda_j \leq \lambda_j^D \approx W\mathrm{vol}(M \setminus \mathcal{N}_R)^{-2/m}j^{2/m} \to W\mathrm{vol}(M)^{-2/m}j^{2/m}$$

for large j, as $R \to 0$. This implies

$$\limsup_{j\to\infty} \frac{\lambda_j^{m/2}\mathrm{vol}(M)}{j} \leq W .$$

Consider the Neumann heat kernel

$$H_R(x,y,t) = \sum_{i=1}^{\infty} e^{-\lambda_i^{2,N}t}\phi_i^{2,N}(x)\phi_i^{2,N}(y)$$

on $\mathcal{N}_R$, where $\{\phi_i^{2,N}\}_{i=1}^{\infty}$ is an orthonormal basis of eigenfunctions in $L^2(\mathcal{N}_R)$ associated to the eigenvalues $\lambda_i^{2,N}$. Using the method of [LT], we know that the Neumann heat kernel on $\mathcal{N}_R$ belongs to the Sobolev space $W^{1,2}(\mathcal{N}_R)$ and has the above form. As the isoperimetric constant C_R of $\mathcal{N}_R$ is positive and the coarea formula on $\mathcal{N}_R$ holds for nonnegative functions, the associated Neumann Sobolev constant of $\mathcal{N}_R$ is also positive. Additionally, we have $H_R(x,y,t)$ in the above form, so the methods in [CL] can be applied to show

$$\lambda_j^{2,N} \geq \alpha(m)C_R^2\left(\frac{j}{\mathrm{vol}(\mathcal{N}_R)}\right)^{2/m} \geq \alpha(m)C^2\left(\frac{j}{\mathrm{vol}(\mathcal{N}_R)}\right)^{2/m} ,$$

where $\alpha(m)$ is a positive constant depending only on m.

Note that the list $\{\lambda_j^{3,N}\}$ is equal to the disjoint union of the lists $\{\lambda_j^{1,N}\}$ and $\{\lambda_j^{2,N}\}$ rearranged in increasing order. Note also that $\lambda_j \geq \lambda_j^{3,N}$, by Dirichlet-Neumann bracketing. Since $\lambda_j^{2,N} \geq \alpha(m)C^2(j/\mathrm{vol}(\mathcal{N}_R))^{2/m}$ and $\lambda_j^{1,N} \approx W\mathrm{vol}(M\setminus\mathcal{N}_R)^{-2/m}j^{2/m}$ for large j, and since $\mathrm{vol}(\mathcal{N}_R) \to 0$ and $\mathrm{vol}(M \setminus \mathcal{N}_R) \to \mathrm{vol}(M)$ as $R \to 0$, we have

$$\liminf_{j\to\infty} \frac{\lambda_j^{m/2}\mathrm{vol}(M)}{j} \geq W \ .$$

□

Example 2.1. Examples 1.1 and 1.2 satisfy the conditions of Theorem 2.1, hence their eigenvalues satisfy Weyl's asymptotic formula.

Remark. Using the methods of [CL], one can additionally conclude that $\lambda_j^{m/2} \geq \alpha(m)\hat{C}^{m/2}j/\mathrm{vol}(M)$ for some positive constant $\hat{C}$ depending only on the lower bound of the Sobolev constants of $\mathcal{N}_R$ for all $R > 0$.

Remark. Because the "football" in Example 1.1 satisfies the conditions of Theorem 2.1, it is clear that all Met_1 surfaces also satisfy the conditions of Theorem 2.1. The authors hope to consider the more general case where the conical singularities form a fractal set, and hope that Theorem 2.1 can be applied to such cases. As an example of such a case, since Minkowski dimension and Hausdorff dimension coincide on self-similar fractals, the Cauchy boundary of $(S^3 \setminus \mathcal{C}, g_{S^3})$ is almost polar, where $\mathcal{C}$ is a Cantor set.

Remark. The results here bear some relation to the work [KS], in which Kuwae and Shioya have recently studied the convergence of the spectra of a sequence of Riemannian manifolds (they do not assume completeness of the manifolds). Some of the results in [KS] involve the almost polarity condition.

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