Advanced Studies in Pure Mathematics 34, 2002
Minimal Surfaces, Geometric Analysis and Symplectic Geometry
pp. 201–217

A Generalized Height Estimate for H-graphs, Serrin's Corner Lemma, and Applications to a Conjecture of Rosenberg

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Abstract.

In this note we give a generalized form of the well-known height estimate for constant mean curvature graphs due to J. Serrin. An application is also given that effectively relates the global rate of convergence of a family of constant mean curvature surfaces recently considered by A. Ros and H. Rosenberg to their convergence behavior on the boundary. Some information concerning boundary behavior is also obtained by applying reflection techniques and corner comparison in particular. This latter analysis allows one to construct a counterexample to one part of a conjecture of Rosenberg.

§ Introduction

Let Ω be a bounded domain in $\mathbf{R}^2$ and $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfy the equation of constant mean curvature (CMC)

$$-\operatorname{div} \mathbf{T}u = 2H \quad (= \text{constant}) \quad \text{on } \Omega, \tag{0}$$

where $\mathbf{T}u = \mathrm{D}u/\sqrt{1+|\mathrm{D}u|^2}$ and without loss of generality $H > 0$. If $u_{|\partial\Omega} \equiv 0$, then one has the well-known height estimate given by Serrin [S1]

$$\max_{x\in\Omega} u \leq \frac{1}{H}. \tag{1}$$

This bound can be obtained by noting that on $\mathcal{G} \equiv \operatorname{graph} u$, the function

$$\phi \equiv X_3 + \frac{1}{H} N_3 \tag{2}$$

2000 *Mathematics Subject Classification.* Primary 53C21; Secondary 58J70.

Key words and prases. constant mean curvature, height estimates, reflection.

Research at MSRI supported in part by NSF grant DMS-9022140.

is subharmonic, where $X_3 = u$ and N_3 is the third component of the (downward pointing) unit normal to $\mathcal{G}$. More precisely, one has for the intrinsic Laplacian of ϕ

$$\Delta_{\mathcal{G}}\phi = -\frac{2}{H}(H^2 - K)N_3 \geq 0,$$

where K is the Gaussian curvature of $\mathcal{G}$.

In the context of parametric CMC surfaces, the reflection technique of Alexandrov [A] was recently extended in [M] by noting that on a CMC graph $\mathcal{S}$ over a domain $\Omega \subset S^2$, the function

$$\Phi \equiv \left\| X + \frac{1}{H}N \right\|^2 \tag{3}$$

is subharmonic, where $X \in \mathbf{R}^3$ denotes a point on $\mathcal{S}$ and N is the unit normal to $\mathcal{S}$ at X. [1] In fact,

$$\Delta_{\mathcal{S}}\Phi = -\frac{4}{H}(H^2 - K)X \cdot N \geq 0.$$

While reading a recent paper of Ros and Rosenberg [R] in which the height estimate (1) is used extensively in conjunction with the Alexandrov technique, we guessed that (3) should be an appropriate generalization of (2) by which a "height" estimate can be obtained. We give such an estimate (6) for spherical graphs that yields (1) as a limiting special case.

The paper of Ros and Rosenberg was partially motivated by a conjecture of Rosenberg which we now describe. Let Γ be a convex planar curve that bounds a domain $\Omega \subset \{x_3 = 0\}$. For each volume $V > 0$, let M_V be an embedded CMC surface with boundary Γ such that $M_V \cup \Omega$ encloses a volume V. It is clear at least for small V, that M_V exists and is a graph which, without loss of generality, is above the plane $\{x_3 = 0\}$. As the volume increases, the mean curvature will increase until the surface ceases to be a graph, and sometime thereafter a critical volume is reached for which the mean curvature starts to decrease. Let us assume that the surfaces M_V continue to enclose a volume V and that the normal $(0, 0, 1)$ on Ω always points into V.

Conjecture 1 (Rosenberg). *If Γ is not a circle, then there is a second critical volume V_c for which the following hold:*

[1] Note that $\mathcal{S}$ is a graph over the *sphere* S^2 centered at $0 \in \mathbf{R}^3$ instead of the plane. The condition that $\mathcal{S}$ is a spherical graph can be expressed by requiring $X \cdot N < 0$ in the interior of $\mathcal{S}$.

(1) *For* $V \leq V_c$, M_V *lies in* $\{x_3 \geq 0\}$.
(2) M_{V_c} *is tangent to the plane* $\{x_3 = 0\}$.
(3) *The point of tangency occurs at a point of minimum curvature of* $\partial M_{V_c} = \Gamma$.
(4) *For* $V > V_c$, M_V *intersects* $\{x_3 < 0\}$.

If one assumes, by way of contradiction, that M_V lies in $\{x_3 \geq 0\}$ for all V, then the result of Ros and Rosenberg applies to show that the family $\{M_V\}$, suitably scaled, converges as $V \to \infty$ to a round sphere. We use a nominally more general form (5) of the height estimate, with some additional assumptions motivated by the conjecture, to describe the *rate* of convergence. If the additional assumptions can be justified, then the result presented here should be of use in proving the first, second, and fourth assertions of the conjecture. In an essentially independent part of the paper, we show that for certain domains the third assertion cannot hold.

§ The estimate

Let $\mathcal{S}$ be a spherical graph as described above.[1] Since $H\Phi - 1/H$ is subharmonic, one has immediately from the maximum principle

$$\|X\|^2 H + 2X \cdot N \leq \max_{Y \in \partial \mathcal{S}} (\|Y\|^2 H + 2Y \cdot N). \tag{4}$$

Let the expression appearing on the right in this inequality be denoted by M. Using the fact that $X \cdot N \geq -\|X\|$, one has

$$H\|X\|^2 - 2\|X\| - M \leq 0.$$

This inequality is quadratic in $\|X\|$, and we easily obtain from it

$$\|X\| \leq \frac{1}{H}(1 + \sqrt{1 + HM}). \tag{5}$$

Note that $M \geq \max(\|Y\|^2 H - 2\|Y\|) = \max(\|Y\|\sqrt{H} - 1/\sqrt{H})^2 - 1/H \geq -1/H$. Thus, the expression on the right in (5) is real (and positive).

In order to complete our comparison to CMC graphs over the plane that satisfy the Dirichlet condition $u_{|\partial\Omega} \equiv 0$, let us assume that $\|X\|_{|\partial \mathcal{S}} \equiv \rho =$constant. In this case, $M \leq \rho^2 H$ and (5) yields

$$\|X\| \leq \frac{1}{H} + \sqrt{\frac{1}{H^2} + \rho^2}. \tag{6}$$

Just as equality in (1) holds for a hemisphere over a disc, we get equality in (6) when $\mathcal{S}$ is a portion of a sphere that meets the sphere of radius ρ

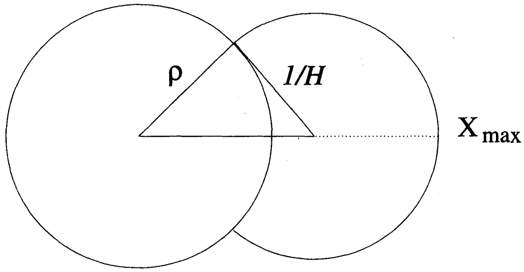

Fig. 1. Extremal spherical H-graph.

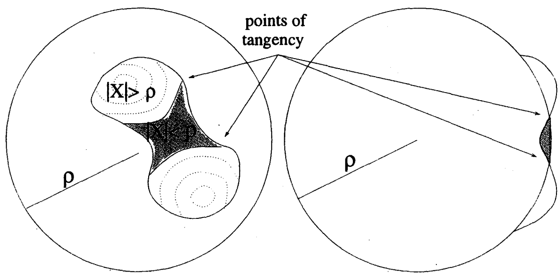

Fig. 2. Possible CMC graph over S_ρ with $H = 1/\rho$.

orthogonally along its boundary (See Fig. 1). Notice that (1) is obtained from (6) by subtracting ρ from each side and letting ρ tend to $+\infty$.

Before considering more general boundary conditions, for which the form (5) is useful, we note the curious fact that, by the maximum principle, the function u in (1) must satisfy $u > 0$ in $\Omega \subset \mathbf{R}^2$ and $\partial u/\partial n > 0$ on $\partial\Omega$, while one expects in general the spherical graph $\mathcal{S}$ in (6) to have interior points with $\|X\| < \rho$ and/or boundary points where $\mathcal{S}$ is tangent to the sphere of radius ρ (See Fig. 2). Notice that tangency of a spherical graph to the sphere of radius $\|X_{\min}\| < \rho$ does not in general contradict the maximum principle (e.g., if $\rho = 1/H$), nor does a tangency to the sphere of radius ρ on $\partial\mathcal{S}$ (i.e., $Y \cdot N = -\|Y\|$ for some $Y \in \partial\mathcal{S}$) imply that $\|X\| \leq \rho$. We use similar conditions below, however, to get some useful information.

§ An application

Some interesting estimates follow from (5) when the spherical graph $\mathcal{S}$ very nearly "encompasses" a sphere of radius $1/H$ (See Fig. 3). To

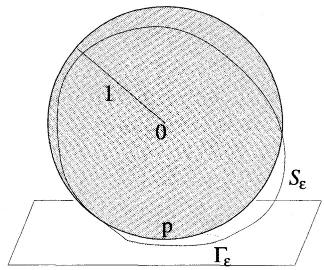

Fig. 3. A Rosenberg Bubble.

simplify notation, we specialize in this section to the case $H = 1$. To be even more specific, let $\mathcal{S}_\epsilon = \mathcal{S}_\epsilon(V)$ be the appropriate ϵ-homothety of M_V such that $\mathcal{S}_\epsilon$ has mean curvature 1. Recall that $\{M_V\}$ is the family of surfaces from Conjecture 1, and we are making the additional assumption that each surface lies in the half space $\{x_3 \geq 0\}$. We have then (as $V \to \infty$) that $\epsilon \to 0$ and the boundaries of the scaled surfaces $\partial\mathcal{S}_\epsilon = \Gamma_\epsilon$ converge as sets to a single point $p \in \mathbf{R}^3$.

Ros and Rosenberg show, moreover, that this particular family of surfaces, converges smoothly to a unit sphere on any compact set in $\mathbf{R}^3$ that does not contain p. Next, we slide our coordinates up one unit so that $p = (0, 0, -1)$ and the boundaries Γ_ϵ lie in $\{x_3 = -1\}$. The convergence of the surfaces is then to the unit sphere $S(0)$ centered at the origin, and we may assume (just by scaling the original curve) that $\max_{\Gamma_\epsilon} \|p - X\| \leq \epsilon$.

We would like to say something about the *rate* of convergence of $\mathcal{S}_\epsilon$ to $S(0)$. In order to do this, we will make an assumption about the behavior of $X \cdot N$ on Γ_ϵ. In fact, Ros and Rosenberg show that

$$\max_{\Gamma_\epsilon} |-1 - X \cdot N| \to 0.$$

We will need to assume more detailed information. As motivation, let $S(t)$ denote the unit sphere with center $(0, 0, t)$, and consider the family of spheres $S(-1 + \sqrt{1 - \epsilon^2})$ which translate up to the unit sphere $S(0)$ and intersect the plane $x_3 = -1$ in a circle of radius ϵ. On the boundary of one of these spheres, explicit calculation yields $X \cdot N = -(\sqrt{1 - \epsilon^2} + \epsilon^2) = -(1 + \epsilon^2/2 - \epsilon^4/8 - \cdots)$. In the result below, we require the boundary behavior to be bounded by that of these translated spheres up to second order.

Finally, before stating the main result we mention that in the case considered in [R] the method of [M] applies to show that $\mathcal{S}_\epsilon$ is a graph over $S(0)$ when ϵ is sufficiently small.

Theorem 1. *If $\mathcal{S}_\epsilon$ is a family of spherical graphs over $S(0)$ of CMC $H = 1$, and*

(1) $\max_{\partial\mathcal{S}_\epsilon} \|X - p\| \leq \epsilon$, *where* $p = (0, 0, -1)$, *while*
(2) $\max_{\partial\mathcal{S}_\epsilon} X \cdot N \leq -(1 + \epsilon^2/2 - \alpha(\epsilon))$, *where* $0 < \alpha(\epsilon) = o(\epsilon^2)$,

then $\max_{\mathcal{S}_\epsilon} \|X\| \leq 1 + \sqrt{2\alpha}$.

Proof. From the first bound in the statement of the theorem, it follows that $\|Y\| \leq \sqrt{1+\epsilon^2}$ for each $Y \in \partial\mathcal{S}_\epsilon$. Using this and the second bound to estimate $M = \max_{Y\in\Gamma_\epsilon}(\|Y\|^2 + 2Y \cdot N)$, we get $M \leq -1 + 2\alpha(\epsilon)$. Thus, (5) gives the conclusion of the theorem. $\square$

§ Generalities

It should be remarked that Serrin's result was actually stated in [S1] as follows.

Theorem S. *If $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ is a solution of (0) on the bounded plane domain Ω, then*

$$\min_{y\in\partial\Omega} u(y) < u \leq \max_{y\in\partial\Omega} u(y) + \frac{1}{H} \quad \textit{in } \Omega \tag{7}$$

with the equality being attained only if graph u is a hemisphere.

We now discuss how this theorem can be generalized to spherical graphs. The proofs of this section are straightforward applications of the reasoning above and that of [S1], and we omit them. Our main result along these lines is the following.

Theorem 2. *If X parameterizes a spherical H-graph $\mathcal{S}$ over a domain $\Omega \subset S^2$ and $\mathcal{S}$ extends continuously to $\bar{\Omega}$, then for each $X \in \operatorname{int}\mathcal{S}$*

$$\|X\| \leq \frac{1}{H} + \sqrt{\frac{1}{H^2} + \max_{Y\in\partial\mathcal{S}} \|Y\|^2}. \tag{8}$$

Moreover, the equality is obtained only if $\mathcal{S}$ is a portion of a sphere.

By a limiting argument, (8) can be used to obtain the upper bound in (7). The equality condition, however, does not survive the limiting procedure.

A more serious omission is the lower bound, and one sees immediately that a spherical graph $\mathcal{S}$ can come arbitrarily close to the origin, while $\partial\mathcal{S}$ lies on $S(0)$; consider spheres of radius $1/H$ that intersect $S(0)$ at its equator and have $H \searrow 0$. Nevertheless, the following result shows that Fig. 2 should not be taken too literally.

Theorem 3. *If S is a spherical H-graph over a domain in a* hemisphere *of S^2 and $\min_{Y\in\partial S}|Y| \geq 1/H$, then*

$$\min_{Y\in\partial S}\|Y\| \leq \|X\|$$

for every $X \in \operatorname{int} S$ with equality only if S is a portion of a sphere.

Finally, Serrin noted that for a *minimal* graph u

$$u(x) \leq \max_{y\in\partial\Omega} u(y), \tag{9}$$

while the upper bound in (7) tends to $+\infty$ as $H \to 0$. Thus, he gives a complementary result. [2]

Theorem S′. *If u is described by* Theorem S, *and Ω is contained in a disc of radius $a \leq 1/H$, then*

$$u(x) \leq \max_{y\in\partial\Omega} u(y) + \frac{1}{H} - \sqrt{\frac{1}{H^2} - a^2} \tag{10}$$

with equality only if graph u is a hemisphere.

Notice in particular that, for a fixed domain Ω, (9) follows from (10) by letting H tend to 0.

For a spherical graph, the disc with boundary on the equator of $S(0)$ displays the futility of trying to bound $\|X\|$ away from 0. For an outer bound we have the following.

Theorem 2′. *Let S be a spherical H-graph described by* Theorem 2 *and let B be the expression on the right in* (8)*. Let $D = \{\xi \in S^2 : 0 \leq \phi < \arctan(H^2B^2 - 1)^{-1/2}\}$, where $\phi = \phi(\xi)$ is the angle that ξ makes with the positive x_3-axis. If $\Omega \subset D$, then*

$$\|X\| \leq \max\left\{ \max_{Y\in\partial S}\|Y\|, \right.$$
$$\left. \max_{Y\in\partial S}\|Y\|\sqrt{1 - \frac{1}{H^2B^2}} + \frac{1}{H} - \sqrt{\frac{1}{H^2} - \frac{\max_{Y\in\partial S}\|Y\|^2}{H^2B^2}} \right\}.$$

Note that $HB \to 2$ as $H \to 0$. In particular, as a set D converges to $D_0 \equiv \{\xi \in S^2 : 0 \leq \phi < \pi/6\}$. Thus, if Ω is compactly contained in D_0, then the theorem can be used to conclude that $\|X\| \leq \max_{Y\in\partial S}\|Y\|$ when S is a minimal surface. While the bound in Theorem 2′ looks complicated, it is completely elementary and can be read off from Fig. 4 and a straightforward application of Serrin's argument.

[2] The lower bound, of course, is no problem.

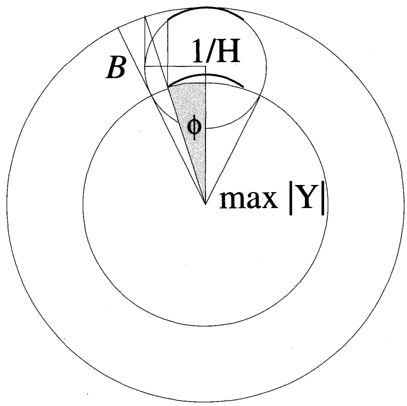

Fig. 4. Diagram for proof of Theorem 2′.

§ Serrin's corner lemma

For this section we assume Γ is a non-circular convex curve and that $\{M_V\}$ is a family of surfaces as in Conjecture 1. We assume furthermore that the family $\{M_V\}$ satisfies conclusions (1) and (2) of Conjecture 1.

Now, let $M = M_{V_c}$ be the bubble of critical volume which is tangent to the plane Π containing $\Gamma = \partial M$. We apply the Alexandrov reflection procedure (see, e.g., [W]) to M with planes that are orthogonal to Π. Recall that as the reflection plane P begins to cut M, the small cap M^- that is cut off reflects across P into V. We can continue to push P parallel to itself farther through M until the procedure terminates (with the reflection $\hat{M}$ of M^- about to exit V). At this terminal point, there are several possibilities.

It may be that the reflected surface $\hat{M}$ is tangent to M at a point interior to $\hat{M}$. In this case, it follows that the terminal plane P is a symmetry plane for M. Another possibility is that $\hat{M}$ is tangent to M at a point on ∂M^- but away from Π. In this case, P is again a plane of symmetry (which follows from the Hopf boundary point lemma). In both of these first two cases, the curve Γ, in particular, must possess a symmetry, and we say that this is a symmetry *in the direction of* P.

Of course, it may be the case that Γ does not have a symmetry in a certain direction. In this case, the touching described above cannot occur. One alternative is that $\hat{M}$ touches M along Γ but away from P. This touching may be tangential (in which case there must be a symmetry) or non-tangential (which case we ignore).

The last possibility is that $\hat{M}$ is tangent to M at a point p in $\Pi \cap P$. Each of the two surfaces may be expressed locally as a graph over a corner domain in the common tangent plane at p. It is customary at this point to apply a version of the Hopf boundary point lemma at a corner due in various forms to Serrin [S2] and Gidas, Ni, and Nirenberg [G]. Let u be the nonnegative difference of the two graphs over the corner domain K. A simple version of the corner lemma is as follows.

Lemma S. *If u satisfies*

(1) $u \geq 0$ *in* K,
(2) $u(p) = 0$ (*where p is the corner point*),
(3) $Du(p) = 0$, *and*
(4) $D_{\eta\eta}u(p) = 0$ *for every non-tangential direction η pointing into K from p*,

then $u \equiv 0$ *in* K.

If the hypotheses (1)–(4) hold, then the conclusion yields that M, and hence Γ, has a symmetry. Furthermore, note that if the $\Pi \cap P$ touching occurs, then conditions (1)–(3) automatically hold. Also, recall that the function to which we apply Lemma S is a difference $u = f(x, y) - \hat{f}(x, y)$, and one of the functions, $\hat{f}$, is obtained from the other by reflection so that in the appropriate coordinates $\hat{f}(x, y) = f(-x, y)$. It follows that $u_{xx}(p) = u_{yy}(p) = 0$ and $u_{xy}(p) = 2f_{xy}(p)$.

On the other hand, if $\eta = (\eta_1, \eta_2)$ is a non-tangential direction, then since $u \geq 0$ and $Du(p) = 0$, one has

$$0 \leq D_{\eta\eta}u(p) = 4f_{xy}(p)\eta_1\eta_2.$$

We conclude that either (1) there is a symmetry *or* (2) $f_{xy}(p) > 0$.

Notice that the x-direction in this discussion is the direction of reflection and is tangent to $\Gamma = \partial M$ at p. Let us assume that M is a graph over Ω locally near p. Let $\phi = \phi(x, n)$ represent the graph where x is measured in the tangent direction to Γ as before and n is measured in the inward normal direction to Ω. An elementary computation shows that

$$f_{xy}(p) = -\frac{1}{1+\phi_n^2}(\phi_n)_x,$$

where ϕ_n is the usual inward normal derivative of ϕ. This means that *either there is a symmetry in the x direction or the angle between M and Π measured within V is strictly increasing along Γ in the direction of reflection at p.*

Similarly, and more interesting for what we say below, the same conclusion holds when M is locally a graph over $\mathbf{R}^2 \backslash \Omega$ near p.

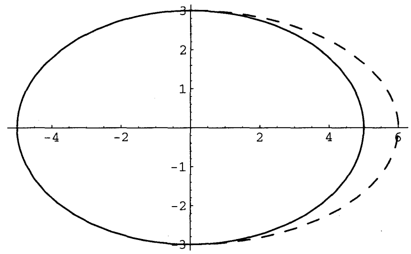

Fig. 5. Modified ellipse.

§ Counterexample

Consider what happens if conclusions (1) and (2) hold when $\Gamma = \mathcal{E}$ is an ellipse. In this case, there are only two symmetries of Γ and hence of $M = M_{V_c}$. Consequently, in every other direction of reflection, the reflection procedure described above terminates due to a boundary touching. Furthermore, due to the monotonicity properties possessed by the curvature of an ellipse as the boundary is traversed, one can check (as we do in the next section for all curves with two symmetries and four vertices) that the procedure must end in a $\Pi \cap P$ touching without symmetry. One concludes that the angle between M and Π is increasing with decreasing curvature along the boundary of the ellipse and, in accord with the conjecture, the maximum angle which occurs at the tangency must be at the minimum point of curvature. We have shown that conclusions (1) and (2) of Conjecture 1 imply (3) when Γ is an ellipse.

It is possible, however, to construct a smooth convex curve Γ with the following properties; see Fig. 5.

(1) $\Gamma \cap \{x_1 \leq 0\}$ is half of an ellipse $\mathcal{E}$ given by $x_1^2/a^2 + x_2^2/b^2 = 1$ with $b < a$.
(2) $\Gamma \cap \{x_1 \geq 0\}$ is a curve that is symmetric with respect to the x_1-axis and has monotone curvature which increases with x_1 to a vertex $(a', 0)$ with $a' > a$.
(3) The domain Ω bounded by Γ contains the half of $\mathcal{E}$ with $x_1 > 0$.

It is not at all difficult to believe that such a modification of an ellipse exists (though Fig. 5 actually shows two ellipses and hence the resulting modification is not smooth). We give an explicit construction in the next section. For now, observe that, given such a curve, the points of minimum curvature still occur at $(0, \pm b)$ and have the same value b/a^2 as $\mathcal{E}$. Reflection in the positive x_1 direction, however, does not result in

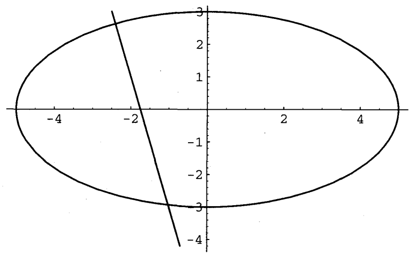

Fig. 6. The line l normal to $\mathcal{E}$ at p.

a symmetry, but in a $\Pi \cap P$ corner touching without symmetry. Thus, the angle between M and Π is increasing along Γ at $(0, \pm b)$; it cannot be a maximum there as required by Conjecture 1 part (3).

§ Details

We first show that any curve $\mathcal{E}$ with the symmetries of an ellipse and four vertices must reflect strictly inside the domain it bounds until a tangency occurs on the plane of reflection P.

Suppose that $\mathcal{E}$ is symmetric with respect to the x_1 and x_2-axes with minimum curvature at $(0, \pm b)$ and maximum curvature at $(\pm a, 0)$. of $\mathcal{E}$ in the j-th quadrant. Let l be a line of negative slope whose normal points into the first quadrant. We move l parallel to itself in the direction of its normal through the domain Ω bounded by $\mathcal{E}$. (The line l may be thought of as $\Pi \cap P$ in our previous discussion.) There are two points of $\mathcal{E}$ where the normal to $\mathcal{E}$ is parallel to l. There is one, p_2, in the second quadrant and another, p_4, in the fourth quadrant.

Lemma 1. *The portion $\mathcal{E}^-$ of $\mathcal{E}$ through which l has passed when l contains $p = p_2$ is the graph of a function $u(x)$ over l.*

Proof. Since there are only two points where l can be normal to $\mathcal{E}$, the statement is equivalent to saying p_2 is reached first—before p_4. When the slope of l is very close to $-\infty$, this is clear because the radius of curvature near $(0, b)$ is greater than b. We conclude by continuity that the only way for the assertion to fail is if, for some slope, p_2 and p_4 are reached at the same time.

In this case, it follows from symmetry that the circle centered at the origin and passing through p_2 and p_4 is tangent to $\mathcal{E}$ at both of these points. A comparison of the curvature of $\mathcal{E}$ with the curvature of this circle yields a contradiction. If, for example, the curvature of $\mathcal{E}$ at p_2 is less than that of the circle, then since the curvature of $\mathcal{E}$ decreases up to $(0, b)$, one sees that the slope of $\mathcal{E}$ at $(0, b)$ must be less than that of the circle when it crosses the x_2-axis (which is zero). This contradicts the fact that $\mathcal{E}$ is smooth at $(0, b)$ and completes the proof. □

We position l as indicated in Fig. 6, passing through the point $p = p_2$. Denote by q the other point of intersection of l with $\mathcal{E}$. We need to prove the following.

Theorem 4. *The reflection $\hat{\mathcal{E}}$ of $\mathcal{E}^-$ lies strictly inside the domain Ω except at the endpoints p and q.*

Let us slightly abuse notation by letting l be the first axis of a plane coordinate system with real coordinates p and q corresponding to the points p and q. The entire curve $\mathcal{E}$ is then the union of the graphs of two smooth functions u and v defined on some interval $p \leq x \leq q'$, where $(q', u(q')) = (q', v(q')) = p_4$. In harmony with the notation of Lemma 1, we require that $\mathcal{E}^- \subset \text{graph}(u)$. In order to prove the theorem, it is enough to show that $\hat{u}(x) = -u(x) < v(x)$ for each x between p and q.

Let $\kappa_u(x)$ be the curvature of graph (u) at $(x, u(x))$ and $\kappa_v(x)$ that of graph (v) at $(x, v(x))$. Due to the simple nature of the curve $\mathcal{E}$ the curvatures satisfy some simple relations.

Lemma 2. *Let $x_0 = (p + q')/2$. Then the following hold:*

(1) $\kappa_u(p) = \kappa_v(p)$.
(2) $\kappa_u(x) > \kappa_v(x)$ *for* $p < x < x_0$.
(3) $\kappa_u(x_0) = \kappa_v(x_0)$.
(4) $\kappa_u(x) < \kappa_v(x)$ *for* $x_0 < x < q'$.

Proof. For x near p, it follows from the monotonicity of curvature that $\kappa_u(x) > \kappa_v(x)$. Since κ_u is increasing and κ_v decreasing initially, this inequality will be maintained at least until $(x, u(x))$ or $(x, v(x))$ is a vertex—and for some time thereafter. There are three cases determined by whether a vertex of minimum curvature is reached first, one of maximum curvature, or both at the same time. In all cases, one sees that the assertions of the lemma hold. □

The key points from Lemma 2 are (1) κ_u is initially greater than κ_v, and (2) κ_u may become smaller than κ_v, but once the curvatures are equal at x_0, κ_u will always be less from there on.

In the event that $q \leq x_0$, we can easily finish the proof of Theorem 4 since the reflection $\hat{\mathcal{E}}$ of $\mathcal{E}^-$ is given by the graph of $\hat{u}(x) = -u(x)$ and the curvature $\kappa_{\hat{u}} = \kappa_u \geq \kappa_v$. The two graphs are tangent at p and that of $\hat{u}$ has greater curvature at each point. Hence, $\hat{u}(x) \leq v(x)$ for $p \leq x \leq q$ with equality only at p as required.

In the other case ($x_0 < q$) we proceed by contradiction and use Lemma 3 below. If Theorem 4 is to fail, then there must be a first value x_1 with $p < x_1 < q$ for which $\hat{u}(x_1) = v(x_1)$. Moreover, we have that $x_0 < x_1 < q$ and $\hat{u}'(x_1) \geq v'(x_1)$. This is enough.

Lemma 3. *If $\hat{u}(x_1) = v(x_1)$ and $\hat{u}'(x_1) \geq v'(x_1)$, but the curvature of graph $(\hat{u})$ is less than the curvature of graph (v) on some interval $x_1 < x \leq q$, then $\hat{u}(x) > v(x)$ for $x_1 < x \leq q$.*

Since we know that $\hat{u}(q) < v(q)$, we have a contradiction—and a theorem. □

We now turn our attention to constructing a modification Γ of an ellipse as described above.

By symmetry it is enough to construct the portion of Γ in the fourth quadrant. We recall furthermore that a curve can be constructed from its curvature. In our case, if θ denotes the angle made by the tangent to the curve with the positive x_1-axis and we specify a positive function $\kappa(\theta)$ on $[0, \pi/2]$, then the formula

$$X(\theta) = (0, -b) + \int_0^\theta \frac{1}{\kappa(\vartheta)}(\cos\vartheta, \sin\vartheta)d\vartheta$$

defines a convex curve that extends $\mathcal{E}$ from the third quadrant into the fourth; see Fig. 7. Various other conditions must by satisfied by κ as described below. In order to introduce these conditions, we consider the curvature κ_0 and parameterization X_0 of the corresponding portion of the ellipse $\mathcal{E}$.

In order for Γ to be a smooth extension of $\mathcal{E}$ at $(0, -b)$, we need

$$\kappa(0) = \kappa_0(0) \quad \text{and} \quad \kappa'(0) = 0. \tag{1a}$$

The condition on monotone curvature is

$$\kappa'(\theta) > 0 \quad \text{for} \quad 0 < \theta < \frac{\pi}{2}. \tag{1b}$$

In order for Γ to close smoothly with it's reflection at a point $(a', 0)$, we need

$$x_2(\pi/2) = -b + \int_0^{\pi/2} \frac{1}{\kappa(\vartheta)} \sin\vartheta \, d\vartheta = 0, \tag{2a}$$

and

$$\kappa'\left(\frac{\pi}{2}\right) = 0. \tag{2b}$$

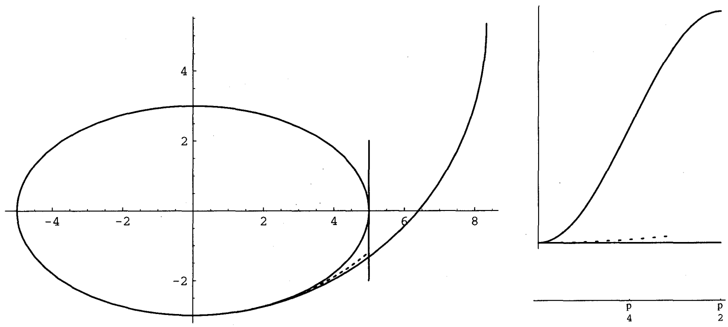

Fig. 7. Shown on the left are $\mathcal{E}$ and a quarter of the osculating circle at $(0,-b)$. On the right are the corresponding curvature functions. A possible modification is shown (dashed) which stays close to the circle for $0 \le \theta \le \theta_c(a)$.

In order to describe a sufficient condition that $\mathcal{E}$ be enclosed by Γ, we note that given a positive function κ, the function

$$\xi(\theta) = \int_0^\theta \frac{1}{\kappa(\vartheta)} \cos\vartheta \, d\vartheta$$

has an inverse. In particular, this is the case for κ_0, and we denote the corresponding inverse, defined on the interval $[0, a]$, by θ_0. Let the inverse of the function ξ defined by κ be denoted by θ, then we will require

$$\kappa(\theta(x_1)) < \kappa_0(\theta_0(x_1)) \quad \text{for} \quad 0 < x_1 < a. \tag{3}$$

Thus, we are done, if we can find a positive function κ defined on $[0, \pi/2]$ such that (1a), (1b), (2a), (2b), and (3) are satisfied.

We begin by considering the circle shown in Fig. 7. It has parameterization

$$X_c(\theta) = (0, -b) + \int_0^\theta \frac{1}{\kappa_0(0)} (\cos\vartheta, \sin\vartheta) \, d\vartheta.$$

This curve does have the virtue that it satisfies (3), i.e., it encloses the part of $\mathcal{E}$ in the fourth quadrant. We have already considered the inverse θ of the x_1-coordinate ξ. Similarly, the x_2-coordinate

$$\eta(\theta) = -b + \int_0^\theta \frac{1}{\kappa(\vartheta)} \sin\vartheta \, d\vartheta$$

has an inverse, which we denote by $\bar{\theta}$. We denote quantities corresponding to the circle by a subscript c. Note that $\theta_c(a) < \bar{\theta}_c(0)$. Let $\theta_1 = (\theta_c(a) + \bar{\theta}_c(0))/2 > \theta_c(a)$.

In order to make sure that (3) is satisfied, we will require that κ stay close to κ_c for $0 \le \theta \le \theta_1$.

To be precise, we choose a function κ of the form

$$\kappa(\theta) = \kappa_0(0) + \epsilon A(\theta) + hB(\alpha(\theta - \beta)).$$

In this definition $\epsilon < 1$, h, α, and β are positive numbers to be chosen. $A(\theta)$ is a fixed smooth increasing function that satisfies $A(\theta) \le (\kappa_0(\theta) - \kappa_0(0))/2$ for $0 \le \theta \le \theta_1$ and $A'(\theta) \equiv A'(\theta_1)$ for $\theta_1 \le \theta \le \pi/2$. $B(\mu)$ is a bump function to be chosen with the standard properties ($B \ge 0$, $\operatorname{supp} B = [-1, 1]$, symmetric, one inflection on $[0, 1)$).

If we stipulate that $\beta - 1/\alpha = \theta_1$, then $\kappa(\theta) = \kappa_0(0) + \epsilon A(\theta)$ for $0 \le \theta \le \theta_1$ and, for ϵ small enough, conditions (1a) and (3) will be satisfied. Condition (1a) follows immediately from the properties of A and κ_0. To see that (3) holds, note first that the restriction of ξ to the interval $[0, \theta_1]$ has a well-defined inverse θ. Secondly, $\xi(\theta)$ converges uniformly to $\xi_c(\theta)$ on $[0, \theta_1]$ as $\epsilon \to 0$. In particular, $\xi_c(\theta_1) > a$ so that for ϵ small $\theta(x_1)$ is well-defined on $[0, a]$. Thirdly, $\kappa_0(\theta) \ge \kappa(\theta) = \kappa_0(0) + \epsilon A(\theta)$ for $0 \le \theta \le \theta_1$. In fact, $\kappa_0(\theta) - \kappa(\theta) \ge (\kappa_0(\theta) - \kappa_0(0))(1 - \epsilon/2) = \theta \kappa_0'(\theta_*)(1 - \epsilon/2) \ge 0$ for some $\theta_* \in (0, \theta)$, and equality holds only for $\theta = 0$. It follows that whenever $\xi(\vartheta) = \xi_0(\varphi)$, one must have $\vartheta < \varphi$, i.e., $\theta(x_1) < \theta_0(x_1)$. Finally then, we see that $\kappa(\theta(x_1)) < \kappa_0(\theta(x_1)) < \kappa_0(\theta_0(x_1))$. Let $\epsilon_1 > 0$ be such that (1a) and (3) hold whenever $\epsilon < \epsilon_1$.

Let μ_1 be the unique value in $[0, 1)$ such that $B''(\mu_1) = 0$. We then have $B'(\mu_1) < B'(\mu) < 0$ for $0 < \mu < \mu_1$. For any $\mu \in (0, \mu_1]$, we can define $\alpha = \alpha(\mu)$ and $\beta = \beta(\mu)$ by the equations

$$\frac{\mu}{\alpha} + \beta = \frac{\pi}{2} \quad \text{and} \quad \beta - \frac{1}{\alpha} = \theta_1.$$

Consider

$$\kappa = \kappa_1(\theta) = \kappa_0(0) + \epsilon A(\theta) + hB(\alpha(\theta - \beta)),$$

as before except that we have substituted the values $\alpha(\mu)$ and $\beta(\mu)$. Consider conditions (1b) and (2b). If $0 \le \theta \le \theta_1$, then $\kappa_1' = \epsilon A'(\theta) \ge 0$ with equality only at $\theta = 0$. For $\theta_1 \le \theta \le \pi/2$,

$$\kappa_1'(\theta) = \epsilon A'(\theta_1) + h\alpha B'(\alpha(\theta - \beta)) > \epsilon A'(\theta_1) + h\alpha B'\left(\alpha\left(\frac{\pi}{2} - \beta\right)\right).$$

Thus, there is a value

$$h = h(\epsilon, \mu) = -\frac{\epsilon A'(\theta_1)}{\alpha B'(\alpha(\pi/2 - \beta))}$$

for which the last expression is zero and, hence, (1b) and (2b) are satisfied.

The only remaining condition is (2a). Notice that for fixed μ we have $h(\epsilon, \mu) \to 0$ as $\epsilon \to 0$. It follows that $\kappa_1(\theta)$ converges uniformly to $\kappa_0(0)$ as $\epsilon \to 0$. Hence,

$$\eta_1\left(\frac{\pi}{2}\right) = -b + \int_0^{\pi/2} \frac{1}{\kappa_1(\vartheta)} \sin\vartheta \, d\vartheta \to \eta_c\left(\frac{\pi}{2}\right) > 0.$$

Consequently, there is a pair $(\epsilon_2, \mu_2) \in (0, \epsilon_1) \times (0, \mu_1)$ for which (1a), (1b), (2b), (3), and $\eta_1(\pi/2) > 0$ are satisfied.

On the other hand, the same reasoning concerning convergence of κ_1 to κ_0 works for $0 \le \theta \le \theta_1$ to show that for some $\epsilon = \epsilon_3$ small enough one has $\eta_1(\theta_1) < 0$. But when $\epsilon = \epsilon_3$ is fixed and μ is small, then α is small, β is close to $\pi/2$, and h gets monotonically *large*. More precisely, $\kappa_1(\theta)$ is a decreasing family of functions in μ, and we can make $\kappa_1(\theta)$ uniformly large on any interval $[\theta_1 + \delta, \pi/2]$. If we choose δ small enough so that $\eta_1(\theta_1 + \delta) < 0$, then it follows that

$$\eta_1\left(\frac{\pi}{2}\right) = \eta_1(\theta_1 + \delta) + \int_{\theta_1+\delta}^{\pi/2} \frac{1}{\kappa_1(\vartheta)} \sin\vartheta \, d\vartheta \to \eta_1(\theta_1) < 0$$

as $\mu \to 0$. Thus, for some (ϵ_3, μ_3), we have (1a), (1b), (2b), (3), and $\eta_1(\pi/2) < 0$.

By continuity, there is a point (ϵ_*, μ_*) on the line segment between (ϵ_2, μ_2) and (ϵ_3, μ_3) for which all the conditions (1a)-(3) hold. □

Acknowledgment. I would like to express my thanks to David Hoffman for several comments that improved the presentation above.

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