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Cyclic Hypersurfaces of Constant Curvature

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Abstract.

We study hypersurfaces in Euclidean, hyperbolic or Lorentz-Minkowski space with the property that it is foliated by a one-parameter family of round spheres. We describe partially such hypersurfaces with some assumption on its curvature. In general, we shall consider the situation that the mean curvature or the Gaussian curvature is constant.

§1. Introduction

A cyclic hypersurface in $(n+1)$-dimensional Euclidean space $\mathbf{E}^{n+1}$ is a hypersurface defined by a smooth one-parameter family of round $(n-1)$-spheres. We say then that M is foliated by spheres. The first example of cyclic hypersurfaces is a hypersurface of revolution, that is, a hypersurface which is stable under a group of rotations that leave a straight-line pointwise fixed. It has been known that the only minimal cyclic surfaces in Euclidean 3-space $\mathbf{E}^3$ are the catenoid (which it is rotational [15]) and the examples discovered by Enneper and Riemann in the nineteenth century [2], [3], [19]. Riemann's surface is a (non-rotational) surface constructed by circles in parallel planes with the exception of a discrete set of straight-lines. Moreover, each of these surfaces is invariant by a family of translations. In higher dimensions, Jagy proved that a cyclic minimal hypersurface in $\mathbf{E}^{n+1}$, $n \geq 3$, must be rotational, that is, it is the n-dimensional catenoid [6]. In contrast with the minimal case, the only cyclic surfaces in $\mathbf{E}^3$ with nonzero constant mean curvature are surfaces of revolution [17]. This note is motivated by these examples and the possible extensions of these results for other space forms. We are interested in studying cyclic hypersurfaces under some assumptions on their curvatures. One of our goal in this paper is to exhibit the existence of a family of maximal spacelike surfaces in the Lorentz-Minkowski

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3-space $\mathbf{L}^3$ that were announced in [9], [10], with similar properties as Riemann's examples in $\mathbf{E}^3$. These surfaces are foliated by circles in parallel planes with the exception of a discrete set of straight-lines and singularities. In this sense, we say that such surface is a 'Riemann type surface' in $\mathbf{L}^3$. See Figure 1 for an example.

We divide this paper into three parts:

1. Cyclic hypersurfaces of constant mean curvature in hyperbolic space.
2. Cyclic hypersurfaces of constant mean curvature in Lorentz-Minkowski space.
3. Cyclic surfaces of constant Gauss curvature in Euclidean space.

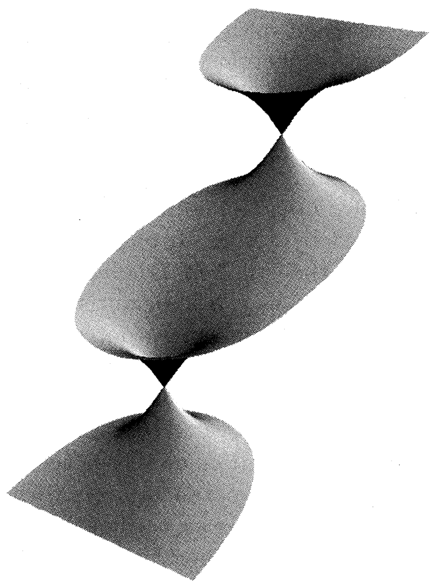

Fig. 1. A 'Riemann type surface' in $\mathbf{L}^3$.

§2. Cyclic hypersurfaces of constant mean curvature in hyperbolic space

In this section we study cyclic hypersurfaces in the $(n+1)$-dimensional hyperbolic space $\mathbf{H}^{n+1}$, for which the spheres that determine the hypersurfaces lie in parallel horospheres. Recall that horospheres are the umbilical hypersurfaces in $\mathbf{H}^{n+1}$ that are flat. Since there exist no intrinsic concept of parallelism in hyperbolic setting, we now give our precise definition.

Definition 2.1. A family of horospheres are called parallel if their asymptotic boundaries meet at exactly one point.

Since the asymptotic boundary of a horosphere is a point, two parallel horospheres meet at infinity in the same point. In the upper half-space model for $\mathbf{H}^{n+1}$, parallel horospheres can be viewed as horizontal parallel Euclidean hyperplanes or, after a rigid motion, Euclidean n-spheres tangent to the hyperplane $x_{n+1} = 0$ at the same point. On the other hand, note that in this model, $(n-1)$-spheres are Euclidean $(n-1)$-spheres.

Theorem 2.2 ([7], [10]). *Let M be a hypersurface in $\mathbf{H}^{n+1}$ of constant mean curvature which is foliated by spheres in parallel horospheres. Then M is a hypersurface of revolution.*

Proof. Consider the upper half-space model for $\mathbf{H}^{n+1}$, that is, $\mathbf{R}^{n+1}_+ = \{(x_1, \ldots, x_{n+1}) \in \mathbf{R}^{n+1}; x_{n+1} > 0\}$ endowed with the metric

$$\langle\, ,\, \rangle = \frac{dx_1^2 + \cdots + dx_{n+1}^2}{x_{n+1}^2}.$$

After a rigid motion in the ambient space, we may assume that the horospheres are Euclidean hyperplanes in $\mathbf{R}^{n+1}_+$ parallel to the hyperplane $x_{n+1} = 0$. We pick a piece M' of M bounded by two spheres $S_1 \cup S_2$. The proof consists of two parts.

The first part is done by a standard application of the Alexandrov reflection method [1]. We consider reflections across a family of vertical parallel geodesic hyperplanes (in Euclidean sense). These hyperplanes are also geodesic hyperplanes in $\mathbf{H}^{n+1}$. Reflections across vertical hyperplanes are isometries in $\mathbf{H}^{n+1}$, so the mean curvature remains unchanged. Consider a vertical hyperplane P disjoint from M' and move P parallel to itself (for example, to the right) until it touches M' at a first point. One continues to move P a little more to the right and considers the symmetry through P of the part of M' on the left-side of P. Now continue moving P to the right and reflecting the left-side of M' until this part touches the part of M' on the right-side of P. The strong maximum principle implies reflection symmetry if they contact and the Alexandrov reflection process yields that P is a hyperplane of symmetry of M'. Thus, for each vertical hyperplane P, some parallel translate of P is a hyperplane of symmetry of M' and M' inherits the symmetries of its boundary $S_1 \cup S_2$. So, the Euclidean centers of the spheres that define M' lies in a 2-plane. Without loss of generality, we may suppose that the curve of centers is parametrized by $(c(t), 0, \ldots, t)$. It then follows that M' is defined as the level hypersurface of the function

$$f(x_1, \ldots, x_n, t) = (x_1 - c(t))^2 + \sum_{i=2}^{n} x_i^2 - r(t)^2,$$

where $r(t) > 0$ denotes the Euclidean radius of each sphere $M' \cap \{x_{n+1} = t\}$.

The second part of the proof is done by computing of the mean curvature of M' in terms of the function f. For this, let $N = -\nabla f / |\nabla f|$ be a unit normal vector field of the immersion $M' \to \mathbf{E}_+^{n+1}$. Then the following formula is well-known:

$$n\, H_{\mathrm{e}} |\nabla f|^3 = \Delta f\, |\nabla f|^2 - \operatorname{Hess} f(\nabla f, \nabla f), \tag{1}$$

where H_{e} denotes the mean curvature of $M \subset \mathbf{E}_+^{n+1}$, and Δ and Hess are the Laplacian and Hessian operators in $\mathbf{E}^{n+1}$, respectively. Choose $x_{n+1} N$ as the Gauss map of $M' \subset \mathbf{H}^{n+1}$. Then its mean curvature H is related with H_{e} by the formula $H = x_{n+1} H_{\mathrm{e}} + N_{n+1}$, where $N = (N_1, \ldots, N_{n+1})$. Thus (1) yields

$$nH|\nabla f|^3 = nN_{n+1}|\nabla f|^3 + x_{n+1} \left(\Delta f |\nabla f|^2 - \operatorname{Hess} f(\nabla f, \nabla f) \right). \tag{2}$$

If the function $c(t)$ is constant, the curve of centers is a straight-line orthogonal to each hyperplanes of the foliation. Consequently, the spheres that define M' are coaxial and hence M' is a hypersurface of revolution.

Assume, on the contrary, that M' is not a hypersurface of revolution, that is, $c' \neq 0$. It is computed that

$$\begin{aligned}
&\nabla f = 2(x_1 - c, x_2, \ldots, x_n, -r\lambda), \quad |\nabla f|^2 = 4r^2(1+\lambda)^2, \\
&\Delta f = 2\left(n - r'\lambda - r\frac{\partial \lambda}{\partial r} \right), \\
&\operatorname{Hess} f(\nabla f, \nabla f) = 8r^2 \left\{ 1 + 2\lambda(\lambda - r') - \lambda^2 \left(r'\lambda + r\frac{\partial \lambda}{\partial t} \right) \right\},
\end{aligned}$$

where $\lambda = \lambda(x_1, t) = ((x_1 - c)c' + rr')/r$.

On the other hand,

$$\frac{\partial \lambda}{\partial t} = \frac{1}{c'\lambda} \{ (\lambda - r')(c''r - c'r') - (c')^3 + c'rr'' \}.$$

Let fix a level $x_{n+1} = t_0$. We introduce a new variable $\lambda = \lambda(x_1, t_0)$ instead of x_1. Then (1) and (2) are written respectively as

$$nrH(1+\lambda^2)^{3/2} = a_0 + a_1\lambda + a_2\lambda^2, \tag{3}$$

$$nrH(1+\lambda^2)^{3/2} = nr\lambda(1+\lambda^2) + t_0(a_0 + a_1\lambda + a_2\lambda^2), \tag{4}$$

where the coefficients a_i are independent of λ. We take the square of both sides of the equation (4) and compare terms by terms. The term of the highest degree corresponds to λ^6. Then $n^2r^2H^2 = n^2r^2$ and this yields $H^2 = 1$. Since the square of the left-hand side of (4) is a polynomial with only terms of even degree in λ, the coefficients of λ^5 and λ^3 vanish on the right-hand side. This yields $t_0a_2 = 0$ and $t_0a_0 = 0$, respectively. However the constant term on the left-hand side of (4) is $n^2r^2H^2 = n^2r^2 \neq 0$, obtaining a contradiction. $\square$

In this context, we recall a theorem of Hsiang [5], which is proved by using the Alexandrov reflection principle, stating that a complete embedded hypersurface $M \subset \mathbf{H}^{n+1}$ that remains within a uniform distance from a geodesic is a hypersurface of revolution.

Remark 1. The same reasoning can be carried over to the case of Euclidean space $\mathbf{E}^{n+1}$. Indeed, squaring (3), the coefficient of λ^6 on the rights-hand side is 0. As a consequence, we obtain that *'the only hypersurfaces in $\mathbf{E}^{n+1}$ with nonzero constant mean curvature which are foliated by $(n-1)$-spheres in parallel hyperplanes are the hypersurfaces of revolution'.*

§3. Cyclic hypersurfaces of constant mean curvature in Lorentz-Minkowski space

Let $\mathbf{L}^{n+1}$ be the $(n+1)$-dimensional Lorentz-Minkowski space, that is, $\mathbf{R}^{n+1}$ equipped with the metric $ds^2 = dx_1^2 + \ldots + dx_n^2 - dx_{n+1}^2$. We study cyclic hypersurfaces of constant mean curvature in $\mathbf{L}^{n+1}$. First, we prove the Lorentzian counterpart of the previous section. Then we investigate the 3-dimensional case, that is, constant mean curvature surfaces of $\mathbf{L}^3$ foliated by circles.

3.1. Cyclic hypersurfaces of constant mean curvature in $\mathbf{L}^{n+1}$

By a similar reasoning as in Theorem 2.2, we obtain the next result which is analogous to what occurs in Euclidean space $\mathbf{E}^{n+1}$ (see Introduction and Remark 1).

Theorem 3.1 ([10]). *Let M^n be a spacelike hypersurface in $\mathbf{L}^{n+1}$ of constant mean curvature H which is foliated by $(n-1)$-spheres in parallel spacelike hyperplanes. Then the following hold.*

1. *If $H \neq 0$, then M is a hypersurface of revolution.*
2. *If $H = 0$ and*
   (a) *$n \geq 3$, then M is a hypersurface of revolution.*
   (b) *$n = 2$, then M is a surface of revolution or is a 'Riemann type surface'.*

Proof. After a rigid motion of $\mathbf{L}^{n+1}$, we may assume that the parallel spacelike hyperplanes are parallel to $x_{n+1} = 0$ (in this case, 'spheres' are 'Euclidean spheres'). The proof is similar to that for Theorem 2.2 and so we only give an outline. Note that Alexandrov reflection method works as in $\mathbf{E}^{n+1}$ and $\mathbf{H}^{n+1}$: indeed, a spacelike hypersurface in $\mathbf{L}^{n+1}$ of constant mean curvature locally satisfies an elliptic partial differential equation for which we can use the standard maximum principle. We compute H through the identity:

$$nH|\nabla f|^3 = \langle \nabla f, \nabla f\rangle \Delta f - \operatorname{Hess} f(\nabla f, \nabla f), \tag{5}$$

where in this case

$$\nabla f = (f_1, \ldots, f_n, -f_{n+1}), \quad \Delta f = \sum_{i=1}^{n} f_{i,i} - f_{n+1,n+1},$$

$$\operatorname{Hess} f(\nabla f, \nabla f) = \sum_{i,j} f_i f_j f_{i,j},$$

$$f = f(x_1, \ldots, x_{n+1}), \quad f_i = \frac{\partial f}{\partial x_i}, \quad f_{i,j} = \frac{\partial^2 f}{\partial x_i \partial x_j}.$$

With the same variable λ defined in Section 2, the identity (5) reads as

$$nrH(-1+\lambda^2)^{3/2} = g_0 + g_1\lambda + g_2\lambda^2,$$

where the coefficients g_i do not depend on λ. We follow the same argument by squaring the above equation. Special attention should be paid to the case $H = 0$ and $n = 2$. In this situation, $g_1 = g_0 = 0$, which yields the next two ordinary differential equations:

$$\frac{rc''}{c'} - 2r' = 0,$$

$$1 + r'^2 - c'^2 + rr'' - \frac{rr'c''}{c'} = 0.$$

A first integration of both equations can do as in [16, p. 87]. So, the first equation leads $c' = ar^2$, for a constant a. This yields in the second equation $1 - r'^2 + rr'' - a^2r^4 = 0$. Consider $x = r^2$ and $y = (r^2)'$ as the new dependent and independent variables. Thus

$$\frac{dt}{dr} = \frac{1}{\sqrt{a^2r^4 + 2br^2 + 1}}.$$

Then M is parametrized by $\mathbf{X}_{a,b}(u,\theta) = (x(u,\theta), y(u,\theta), z(u,\theta))$, where

$$\begin{aligned} x(u,\theta) &= a\int^u \frac{u^2}{\sqrt{a^2u^4 + 2bu^2 + 1}}du + u\cos\theta,\\ y(u,\theta) &= u\sin\theta,\\ z(u,\theta) &= \int^u \frac{du}{\sqrt{a^2u^4 + 2bu^2 + 1}}, \end{aligned}$$

and $a, b \in \mathbf{R}$. □

The integrals that appear in this parametrization are of elliptic type (as it occurs with Riemann's examples in $\mathbf{E}^3$). We illustrate Theorem 3.1 by presenting two examples.

Example 1. Let $a = 0$. In this case, $c' = 0$ and the surface is rotational. This surface is the Lorentzian catenoid:

$$\mathbf{X}_{0,b}(u,\theta) = \left(u\cos\theta, u\sin\theta, \frac{1}{\sqrt{2b}}\operatorname{arc\,sinh}(\sqrt{2b}u)\right),$$

which is generated by the rotation of the curve $((1/\sqrt{2b})\sinh(\sqrt{2b}u), 0, u)$ with respect to the x_3-axis. The Lorentzian catenoid is the only maximal spacelike surface of revolution in $\mathbf{L}^3$ with respect to a timelike rotation axis.

Example 2. Let $a = b = 1$. The integrals that define M can be explicitly calculated. Then M is given by

$$\mathbf{X}_{1,1}(u,\theta) = (u - \arctan u + u\cos\theta, u\sin\theta, \arctan u).$$

This surface has a singularity of cone type at the origin. Moreover it is asymptotic to the planes $x_3 = \pm\pi/2$ and at these heights, M has two flat ends. The circles that define M converge to straight-lines as $u \to \pm\infty$. Thus, we can reflect M across them to obtain a simply periodic embedded maximal surface invariant by a family of translations. Figure 1 in Introduction represents precisely this surface. Up to homotheties in $\mathbf{L}^3$, the immersions $\mathbf{X}_{a,b}$ define a one-parameter family of maximal spacelike surfaces in $\mathbf{L}^3$ that, in a sense, correspond with Riemann's examples in $\mathbf{E}^3$.

3.2. Cyclic maximal surfaces in $\mathbf{L}^3$

In this and the next subsection we focus on cyclic surfaces with constant mean curvature in $\mathbf{L}^3$. Recall that a surface in $\mathbf{L}^3$ is called nondegenerate if the induced metric on it is nondegenerate. In $\mathbf{L}^3$, we have two possibilities: the induced metric is Riemannian and the surface is called spacelike; or the induced metric is Lorentzian and the surface is called timelike. In Theorem 3.1 we studied the case that the spheres that form the hypersurface are contained in parallel spacelike hyperplanes. We want to consider a more general situation on the hyperplanes that determine the foliation. First, we give the following definition.

Definition 3.2. A circle in $\mathbf{L}^3$ is the orbit of a point p under the action of a group of rotations in $\mathbf{L}^3$.

There exist three families of rotations in $\mathbf{L}^3$ according to the causal character of the line L that define each family (see for example [4]). For an easy description of the circles obtained in each case, let $B = (e_1, e_2, e_3)$ be the standard basis in $\mathbf{L}^3$. Then the following hold:

1. (timelike axis) If $L = \text{span}(e_3)$, then the circles are Euclidean circles in horizontal planes.
2. (spacelike axis) If $L = \text{span}(e_1)$, then the circles are hyperbolas in vertical planes.
3. (lightlike axis) If $L = \text{span}(e_2 + e_3)$, then the circles are parabolas in planes parallel to $x_2 - x_3 = 0$.

Surfaces of revolution in $\mathbf{L}^3$ of constant mean curvature have been studied in [4], [8], [20]. In the Lorentzian case, a surface with $H = 0$ everywhere is called maximal. Now we are in a position to give the following two results for (spacelike or timelike) surfaces (see [9]):

Theorem 3.3. *Let M be a nondegenerate maximal cyclic surface in $\mathbf{L}^3$. Then the planes containing pieces of circles must be parallel.*

Theorem 3.4. *Let M be a nondegenerate maximal surface in $\mathbf{L}^3$ foliated by pieces of circles in parallel planes. Then M is a surface of revolution or it is contained in a 'Riemann type surface'.*

Proof. [Sketch] For simplicity of the proof of Theorem 3.3, we consider the case where the planes containing the circles are spacelike. The proof is done by contradiction. Assume that these planes are not parallel. Let $\Gamma(u)$ be an orthogonal curve to each u-plane of the foliation. Since Γ is not a straight-line, we can consider the Frenet frame of Γ. Remark that the unit tangent vector field $\mathbf{t}(u)$ to Γ has a timelike causal character. Then $\mathbf{t}'$ is a spacelike vector field. Let $\mathbf{n}(u)$ be the unit spacelike vector field such that $\mathbf{t}' = \kappa\mathbf{n}$, for some function $\kappa \neq 0$. Put

$\mathbf{b} = \mathbf{t} \wedge \mathbf{n}$. Then Frenet basis associated to Γ is given by $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$. Hence M can be parametrized by

$$\mathbf{X}(u, v) = \mathbf{c}(u) + r(u)(\cos v\, \mathbf{n}(u) + \sin v\, \mathbf{b}(u)) \quad r > 0, \mathbf{c} \in \mathbf{L}^3.$$

Let us compute the mean curvature H of $\mathbf{X}$ by the classical local theory (see [18]). Let $I = (E, F, G)$ and $II = (e, f, g)$ be the coefficients of the first and the second fundamental forms respectively, and set $W = EG - F^2$ (W is positive if M is spacelike and negative if M is timelike). Then the mean curvature H is given by

$$H = \frac{eG - 2fF + gE}{2W}. \tag{6}$$

Put $\mathbf{c}' = \alpha \mathbf{t} + \beta \mathbf{n} + \gamma \mathbf{b}$, where α, β, γ are smooth functions on u. Let us use the corresponding Frenet equations of Γ:

$$\begin{aligned} \mathbf{t}' &= \kappa \mathbf{n}, \\ \mathbf{n}' &= \kappa \mathbf{t} + \sigma \mathbf{b}, \\ \mathbf{b}' &= -\sigma \mathbf{n}. \end{aligned}$$

Remark that these equations are slightly different from the Euclidean case. It follows from (6) that $H = 0$ is written as

$$\sum_{n=0}^{3} A_n(u) \cos nv + \sum_{n=1}^{3} B_n(u) \sin nv = 0$$

for some functions A_n and B_n. This is a linear combination of the independent functions $\sin nv$ and $\cos nv$. Thus $A_n = B_n = 0$ for all n. A hard work to obtain explicit expressions of the coefficients A_n and B_n together with the fact that $W, r \neq 0$ gives a contradiction. Therefore $\kappa = 0$ and Γ is a straight-line. When the circles are contained in timelike or lightlike planes, the reasoning is analogous, with the observation that in each case the Frenet frame of Γ changes as well as the corresponding Frenet equations. A more explicit example of the reasoning of this kind can be seen in Theorem 3.5 below.

The proof of Theorem 3.4 is easier. Now, after a rigid motion in $\mathbf{L}^3$ we may assume that the circles of M are Euclidean circles, hyperbolas or parabolas, depending on the causal character of the planes containing the circles. For example, in the case where the planes of the foliation are spacelike, we assume without loss of generality that the surface is given by

$$\mathbf{X}(u, v) = (a(u) + r(u) \cos v, b(u) + r(u) \sin v, u),$$

where a and b are smooth functions on u. If we compute the mean curvature, then the similar reasoning to the proof of Theorem 3.1 for $n = 2$ applies. $\square$

Remark 2. Maximal spacelike surfaces in $\mathbf{L}^3$ can be described in terms of complex data. More exactly, there exists a Weierstrass representation as in the case of minimal surfaces in $\mathbf{E}^3$. Let M be a Riemann surface and $X : M \to \mathbf{E}^3$ a conformal minimal immersion. If $(M, (\phi_1, \phi_2, \phi_3))$ is the corresponding Weierstrass representation, then it is easy to prove that $(M, (i\phi_1, i\phi_2, \phi_3))$ defines a maximal spacelike immersion of M in $\mathbf{L}^3$. This process allows us to obtain a correspondence between minimal surfaces in $\mathbf{E}^3$ and maximal spacelike surfaces in $\mathbf{L}^3$. Therefore it is possible to use the complex analysis machinery in the study of maximal spacelike surfaces and, in particular, of cyclic surfaces. This point of view is developed in [9].

3.3. Cyclic surfaces of nonzero constant mean curvature in $\mathbf{L}^3$

The case $H \neq 0$ in $\mathbf{L}^3$ is different from the maximal one, as it is the case in the Euclidean ambient (see Introduction and [17]).

Theorem 3.5 ([11], [12]). *Let M be a nondegenerate cyclic surface in $\mathbf{L}^3$ with nonzero constant mean curvature. Then either the planes containing the circles must be parallel or M is a subset of a pseudohyperbolic surface or a pseudosphere.*

Comparing with Theorem 3.3, let us first observe that possibly the planes are not parallel. But in this case, the surface is contained in a surface of revolution. This phenomenon also occurs in $\mathbf{E}^3$: the intersection between any smooth one-parameter family of (not necessarily parallel) planes with a sphere produces circles. In the Lorentzian space, the role of spheres is played by the pseudohyperbolic surfaces $\mathbf{H}^{2,1}(r)$ and the pseudospheres $\mathbf{S}^{2,1}(r)$:

$$\begin{aligned}\mathbf{H}^{2,1}(r) &= \{p \in \mathbf{L}^3; \langle p, p\rangle = -r^2\},\\ \mathbf{S}^{2,1}(r) &= \{p \in \mathbf{L}^3; \langle p, p\rangle = r^2\}.\end{aligned}$$

The surfaces $\mathbf{H}^{2,1}(r)$ and $\mathbf{S}^{2,1}(r)$ are spacelike and timelike, respectively. Moreover, both surfaces have nonzero constant mean curvature $|H| = 1/r$. Theorem 3.5 is a revised and corrected version of [11, Th. 1] and examples therein: although the examples exhibited in [11] are spacelike surfaces with nonzero constant mean curvature and foliated by pieces of circles in non-parallel planes, these surfaces are subsets of $\mathbf{S}^{2,1}(r)$ or $\mathbf{H}^{2,1}(r)$.

Proof. The case that the planes of the foliation are spacelike is studied in [11]. It remains the cases that they are timelike or lightlike. In order to simplify the presentation, we explicitly discuss the case that the planes are lightlike. By contradiction, we assume that the planes are not parallel and that, after a homothety in $\mathbf{L}^3$, the mean curvature of M is $H = 1/2$. In each u-plane of the foliation that defines M, let $\mathbf{e}_1(u)$ and $\mathbf{e}_2(u)$ be vector fields such $\langle \mathbf{e}_1, \mathbf{e}_1 \rangle = 1$ and $\langle \mathbf{e}_1, \mathbf{e}_2 \rangle = \langle \mathbf{e}_2, \mathbf{e}_2 \rangle = 0$. Then M is parametrized as

$$\mathbf{X}(u,v) = \mathbf{c}(u) + v\mathbf{e}_1(u) + r(u)v^2\mathbf{e}_2(u), \quad r \neq 0.$$

Denote $\mathbf{n} = \mathbf{e}_1$ and $\mathbf{t} = \mathbf{e}_2$, and use null coordinates: for each u, let $\mathbf{b}(u)$ be the unique lightlike vector orthogonal to $\mathbf{n}(u)$ such that

$$\langle \mathbf{t}, \mathbf{b} \rangle = 1, \quad \text{determinant}(\mathbf{t}, \mathbf{n}, \mathbf{b}) = 1.$$

With a change on the variables u and v, we assume that $\mathbf{t}' = \kappa\mathbf{n}$ for some function κ (see discussion in [9]). Remark that $\kappa \neq 0$ because the planes are not parallel. The Frenet equations are

$$\begin{aligned} \mathbf{t}' &= \kappa\mathbf{n}, \\ \mathbf{n}' &= \sigma\mathbf{t} - \kappa\mathbf{b}, \\ \mathbf{b}' &= -\sigma\mathbf{n}. \end{aligned}$$

In the above notation, the surface is parametrized as

$$\mathbf{X}(u,v) = \mathbf{c} + v\mathbf{n} + rv^2\mathbf{t}.$$

Put $\mathbf{c}' = \alpha\mathbf{t} + \beta\mathbf{n} + \gamma\mathbf{b}$, for smooth functions α, β, γ. Squaring the identity (6), we obtain

$$\sum_{n=0}^{9} A_n(u)v^n = 0.$$

This is a polynomial equation on the variable v and thus the coefficients A_n vanish everywhere. The coefficient A_9 is given by $A_9 = 8\kappa^3(2\gamma r^2 - r')^3$. Then $r' = 2\gamma r^2$. Hence $A_7 = A_8 = 0$ and

$$A_6 = 8\kappa^3(2\beta r - 8\kappa r^2 - \sigma)(-2\beta r + \sigma)^2.$$

We have two possibilities:

1. $\sigma = 2\beta r$. Then $A_4 = -81\alpha^2\kappa^4 r^2$. In particular, $\alpha = 0$ and this implies $W = 0$, a contradiction because M is a nondegenerate surface.

2. $\sigma = 2\beta r - 8\kappa r^2$. The computation of A_4 leads

$$A_4 = 384\kappa^5 r^4(-\alpha + 8\gamma r^2).$$

Then $\alpha = 8\gamma r^2 = 4r'$. By using the Frenet equations, we obtain

$$\mathbf{c}' = 4r'\mathbf{t} + \beta\mathbf{n} + \gamma b = -\left(\frac{\mathbf{b}}{2r} - 4r\mathbf{t}\right)'.$$

Therefore there exists a point $\mathbf{c}_0 \in \mathbf{L}^3$ such that $\mathbf{c} = \mathbf{c}_0 - \mathbf{b}/(2r) + 4r\mathbf{t}$ and the parametrization of M is

$$\mathbf{X}(u,v) = \mathbf{c}_0 + r(4 + v^2)\mathbf{t} + v\mathbf{n} - \frac{1}{2r}\mathbf{b}.$$

Thus

$$\langle \mathbf{X}(u,v) - \mathbf{c}_0, \mathbf{X}(u,v) - \mathbf{c}_0 \rangle = -4,$$

and M is contained in the pseudohyperbolic surface $\mathbf{H}^{2,1}(2)$.

□

Let us study the case where the planes containing the circles are parallel. As in Theorem 3.4, an easy reasoning leads to

Theorem 3.6. *Let M be a nondegenerate surface in $\mathbf{L}^3$ with nonzero constant mean curvature which are foliated by pieces of circles in parallel planes. Then M is a surface of revolution.*

From Theorems 3.5 and 3.6, we have

Corollary 3.7 ([11], [12]). *All cyclic nondegenerate surfaces in $\mathbf{L}^3$ with nonzero constant mean curvature are surfaces of revolution.*

This result claims that there exist no 'Riemann type surfaces' in $\mathbf{L}^3$ with nonzero constant mean curvature.

§4. Cyclic surfaces of constant Gauss curvature in Euclidean space

We close this paper with a study of cyclic surfaces in $\mathbf{E}^3$ with constant Gaussian curvature. We have the following two results:

Theorem 4.1. *Let M be a surface in $\mathbf{E}^3$ with constant Gauss curvature which is foliated by pieces of circles. Then M is contained in a sphere or, in the non-spherical case, the planes containing the circles of the foliation must be parallel.*

Theorem 4.2. *Let M be a surface in $\mathbf{E}^3$ with constant Gauss curvature K which is foliated by pieces of circles in parallel planes.*

1. *If $K \neq 0$, then M is a surface of revolution.*
2. *If $K = 0$, then the surface is not necessarily rotational. However, the curve of centers is a straight-line and the radius of the circles is given by a linear function on the parameter of the foliation.*

As a consequence of these theorems, we have:

Corollary 4.3 ([13]). *All cyclic surfaces in $\mathbf{E}^3$ with nonzero constant Gauss curvature are surfaces of revolution.*

In a sense, this corollary is the analogue of Nitsche's theorem for nonzero constant mean curvature cyclic surfaces in $\mathbf{E}^3$.

Proof. The proof of Theorem 4.1 is similar to Theorem 3.3. By contradiction, assume that the u-planes containing the circles are not parallel. Consider a curve $\Gamma(u)$ orthogonal to each u-plane. Then M can be parametrized in the form

$$\mathbf{X}(u,v) = \mathbf{c}(u) + r(u)(\cos v\, \mathbf{n}(u) + \sin v\, \mathbf{b}(u)),$$

where $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ denotes the Frenet frame of Γ. The formula for the Gaussian curvature in local coordinates with respect to $\mathbf{X}$ is:

$$K = \frac{eg - f^2}{EG - F^2}.$$

By using the Frenet equations as in Theorem 3.3, the above equation implies that

$$\sum_{n=0}^{4} A_n(u) \cos nv + \sum_{n=1}^{4} B_n(u) \sin nv = 0.$$

This is a linear combination of the independent functions $\cos nv$ and $\sin nv$. Thus $A_n = B_n = 0$ for all n. A delicate study with the coefficients A_n, B_n concludes that M is contained in a sphere in the case that $K > 0$ or a contradiction. Theorem 4.2 is proved by considering a more explicit parametrization of the surface. After a rigid motion in $\mathbf{E}^3$, we may assume that the planes containing the circles are parallel to the plane $x_3 = 0$. Then the parametrization of M is in the form

$$\mathbf{X}(u,v) = (a(u) + r(u)\cos v, b(u) + r(u)\sin v, u),$$

where $a, b, r > 0$ are smooth functions on u. Then we compute the Gaussian curvature K. If $K \neq 0$, we conclude that $a' = b' = 0$, that is, the curve of centers of the circles is a vertical straight-line orthogonal to each u-plane of the foliation. Thus M is a surface of revolution. In the case $K = 0$, we obtain $a'' = b'' = r'' = 0$. $\square$

Remark 3. Recently the present author has extended Theorems 4.1 and 4.2 to the case of the Lorentz-Minkowski space $\mathbf{L}^3$ [14]: a non-degenerate cyclic surface in $\mathbf{L}^3$ with nonzero constant Gauss curvature is a surface of revolution. The result is divided into two parts. First, it is proved that the planes of the foliation are parallel and secondly, we prove that the surface is rotational. The proof follows the same steps as in Theorems 4.1 and 4.2, but needs to take care of extra complication that there are three cases to distinguish according to the causal character of the planes that define the surface (see Theorems 3.3 and 3.5.)

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