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Completely Parametrized $\mathbf{A}^1_*$-fibrations on the Affine Plane

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§0. Introduction

Let k be an algebraically closed field of characteristic zero, which we fix as the ground field. In the present article we consider $\mathbf{A}^1_*$-fibrations on the affine plane $\mathbf{A}^2$, where $\mathbf{A}^1_*$ denotes the affine line $\mathbf{A}^1$ with one point deleted. Let X be a smooth affine surface with Pic $(X) = (0)$ and $\Gamma(X, \mathcal{O}_X)^* = k^*$. Let $\rho : X \to B$ be an $\mathbf{A}^1_*$-fibration, where B is a smooth algebraic curve. Then ρ is untwisted because Pic $(X) = (0)$ and B is isomorphic to $\mathbf{A}^1$ or $\mathbf{P}^1$ because $\Gamma(X, \mathcal{O}_X)^* = k^*$. We call ρ a *completely* (resp. *incompletely*) *parametrized* $\mathbf{A}^1_*$-fibration if B is isomorphic to $\mathbf{P}^1$ (resp. $\mathbf{A}^1$). See [6], [8] for the definitions and relevant results. If X is the affine plane and ρ is incompletely parametrized, then there exists an irreducible polynomial $f \in \Gamma(X, \mathcal{O}_X)$ such that the fibration ρ is given as $\{F_\lambda\}_{\lambda \in k}$, where F_λ is a curve defined by $f = \lambda$. Hence f is a generically rational polynomial with two places at infinity, and such polynomials are classified by H. Saito [10] (see [7]). On the other hand, there exist no references where the completely parametrized $\mathbf{A}^1_*$-fibrations on $\mathbf{A}^2$ are explicitly classified. The fibers of the given $\mathbf{A}^1_*$-fibration form a pencil of affine plane curves parametrized by $\mathbf{P}^1$. So, the classification is made by giving the defining equation of a general member of the pencil.

For this purpose, we make use of a description of $\mathbf{A}^2$ as a homology plane with $\mathbf{A}^1_*$-fibration over $\mathbf{P}^1$ as given in [6], [8]. Our results show that the pencil is given in the form

$$\Lambda = \left\{ \left(yx^{r+1} - p(x)\right)^{\mu_1} + \lambda x^{\mu_0} = 0; \lambda \in \mathbf{P}^1 \right\},$$

where $p(x) \in k[x], \deg p(x) \leq r$ and $p(0) \neq 0$.

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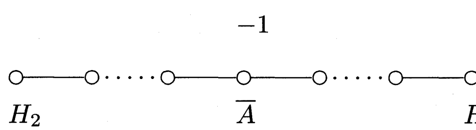
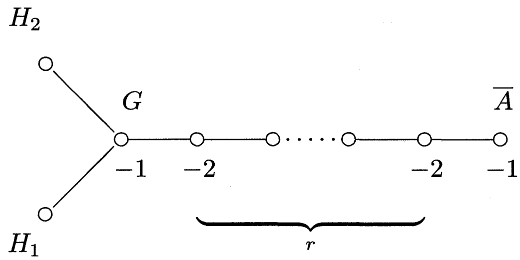
§1. $\mathbf{A}^1_*$-fibrations

Let X be a $\mathbf{Q}$-homology plane with an untwisted $\mathbf{A}^1_*$-fibration $\rho : X \longrightarrow B$, where B is isomorphic to $\mathbf{P}^1$. Then every fiber but one is isomorphic to $\mathbf{A}^1_*$ if taken with the reduced structure and the excepted fiber is isomorphic to $\mathbf{A}^1$. There exists a smooth projective surface V with a $\mathbf{P}^1$-fibration $p : V \longrightarrow B$ such that X is a Zariski open set of V, the boundary divisor $D := V - X$ is a divisor with simple normal crossings and p gives rise to the $\mathbf{A}^1_*$-fibration if restricted onto X. Since ρ is untwisted, there exist two cross-sections H_1 and H_2 of p, which are the loci of two points of the general fibers of ρ lying at infinity. Since the boundary divisor D has a tree as the dual graph, H_1 and H_2 meet each other at most in one point. If H_1 and H_2 meet each other, we blow up the point of intersection and its infinitely near points so that the proper transforms of H_1 and H_2 get separated from each other. Furthermore, if we assume that the embedding $X \hookrightarrow V$ is minimal in the sense that D contains no (-1) curves which are the fiber components of the $\mathbf{P}^1$-fibration p and that any contraction of such a (-1) curve makes the images of H_1 and H_2 meet each other, then it is known (cf. [6], [8]) that $\rho : X \longrightarrow B$ is obtained in the following fashion.

There exists a Hirzebruch surface F_a with a minimal section M_1 and a section M_2 with $(M_1 \cdot M_2) = 0$, and there exists a sequence of blowing-ups $\sigma : V \longrightarrow F_a$ such that H_1 and H_2 are the proper transforms of M_1 and M_2, respectively, and that $({H_1}^2) = ({M_1}^2) = -a$. Hence the blowing-ups σ starts with the blowing-ups of the points lying on M_2 and no points of M_1 are blown-up. The fibration $p : V \longrightarrow B$ is obtained from the $\mathbf{P}^1$-fibration on F_a. Let μA be a fiber of ρ with $A \cong \mathbf{A}^1_*$ and possibly $\mu > 1$ and let $\overline{A}$ be the closure of A in V. Then the fiber of p containing $\overline{A}$ has a linear chain as the dual graph:



On the other hand, if μA is a fiber of ρ with $A \cong \mathbf{A}^1$, the dual graph of the fiber containing $\overline{A}, H_1$ and H_2 looks like



Let μA be a singular fiber of ρ, i.e., either $\mu > 1$ or $A \cong \mathbf{A}^1$. Let $\overline{A}$ be the closure of A in V. Then μ is the multiplicity of $\overline{A}$ in the fiber $p^{-1}(\rho(A))$. Let δ be the contribution of $\overline{A}$ in the total transform $\sigma^*(M_2)$. It is known (cf. [6], [8]) that $0 \le \delta < \mu$ and $\delta > 0$ if $A \cong \mathbf{A}^1_*$. We begin with recalling the following structure theorem (cf. [6], [8]).

Lemma 1.1. *Let X be a $\mathbf{Q}$-homology plane with an $\mathbf{A}^1_*$-fibration $\rho : X \to B$. Suppose $B \cong \mathbf{P}^1$ and ρ is untwisted. Let $\mu_0 A_0, ..., \mu_n A_n$ be all singular fibers with respective multiplicities $\mu_0, ..., \mu_n$, where $A_0 \cong \mathbf{A}^1$ and $A_i \cong \mathbf{A}^1_*$ for $1 \le i \le n$. Then we have the following assertions:*

(1) *$\overline{\kappa}(X) = 1, 0$ or $-\infty$ if and only if*

$$(n-1) - \sum_{i=1}^{n} \frac{1}{\mu_i} > 0, = 0 \ \text{or} \ < 0, \ \text{respectively.}$$

(2) *$H_1(X;\mathbf{Z})$ is a torsion group of order equal to*

$$\left| \mu_0 \cdots \mu_n a - \sum_{i=0}^{n} \mu_0 \cdots \widehat{\mu_i} \cdots \mu_n \delta_i \right|.$$

(3) *There are no homology planes X with $\overline{\kappa}(X) = 0$ and an untwisted $\mathbf{A}^1_*$-fibration $\rho : X \to B \cong \mathbf{P}^1$.*

When X is isomorphic to $\mathbf{A}^2$ in Lemma 1.1, we can specify the data more precisely.

Lemma 1.2. *With the notations of Lemma 1.1, the following assertions hold:*

(1) *A smooth affine surface X is isomorphic to $\mathbf{A}^2$ if and only if $\overline{\kappa}(X) = -\infty, \mathrm{Pic}\,(X) = (0)$ and $\Gamma(X, \mathcal{O}_X) = k^*$. In particular, a $\mathbf{Q}$-homology plane X is isomorphic to $\mathbf{A}^2$ if and only if $\overline{\kappa}(X) = -\infty$ and $H_1(X;\mathbf{Z}) = (0)$.*
(2) *$n = 0$ or 1.*

(3) *If* $n=0$ *then either* $a=1, \mu_0=\delta_0+1$ *or* $a=0, \delta_0=1$.
(4) *If* $n=1$ *then either*

$$a=1, \quad \mu_0\mu_1-\mu_1\delta_0-\mu_0\delta_1=\pm 1$$

or

$$a=0, \quad \mu_0=\delta_1=1, \quad \delta_0=0.$$

(5) *If* $a=n=1$ *and* $\mu_0\mu_1-\mu_1\delta_0-\mu_0\delta_1=\pm 1$*, the pair* (δ_0,δ_1) *is uniquely determined by the pair* (μ_0,μ_1)*. Furthermore, if* $\mu_0\mu_1-\mu_1\delta_0-\mu_0\delta_1=1$*, then the pair* (δ_0',δ_1') *with* $\delta_i'=\mu_i-\delta_i$ $(i=0,1)$ *satisfies* $\mu_0\mu_1-\mu_1\delta_0'-\mu_0\delta_1'=-1$*, and vice versa.*

Proof. (1) We refer to [6].

(2) Note that $\mu_0\geq 1$ and $\mu_i\geq 2$ for $1\leq i\leq n$. Since $\overline{\kappa}(X)=-\infty$, it follows that

$$n-1-\frac{n}{2}\leq(n-1)-\sum_{i=1}^{n}\frac{1}{\mu_i}<0.$$

Hence $n=0$ or 1.

(3) Since $H_1(X;\mathbf{Z})=0$, we have

$$|H_1(X;\mathbf{Z})|=\left|\mu_0\cdots\mu_n a-\sum_{i=0}^{n}\mu_0\cdots\widehat{\mu_i}\cdots\mu_n\delta_i\right|=1.$$

If $n=0$ then this formula reads $\mu_0 a-\delta_0=\pm 1$, where $\mu_0>\delta_0$. Suppose $a\geq 2$. Then we have

$$(a-2)\mu_0+(\mu_0-\delta_0)+\mu_0\neq\pm 1.$$

Hence $a=0$ or 1. If $a=1$ then $\mu_0=\delta_0+1$. If $a=0$ then $\delta_0=1$.

(4) If $n=1$ then

$$a\mu_0\mu_1-\mu_1\delta_0-\mu_0\delta_1=\pm 1.$$

Suppose $a\geq 2$. Then we have

$$(a-2)\mu_0\mu_1+\mu_1(\mu_0-\delta_0)+\mu_0(\mu_1-\delta_1)\neq\pm 1.$$

Hence $a=0$ or 1. If $a=1$ then we have

$$\mu_0\mu_1-\mu_1\delta_0-\mu_0\delta_1=\pm 1.$$

If $a=0$ then $\mu_1\delta_0+\mu_0\delta_1=1$. Since $\mu_1\geq 2$, it follows that $\delta_0=0$. Then $\mu_0=\delta_1=1$.

(5) Suppose that $\mu_0\mu_1 - \mu_1\delta_0 - \mu_0\delta_1 = 1$ and $\mu_0\mu_1 - \mu_1\gamma_0 - \mu_0\gamma_1 = 1$ for the pairs (γ_0, γ_1) and (δ_0, δ_1) with $\mu_i > \gamma_i, \mu_i > \delta_i$ $(i = 0, 1)$. Then

$$\mu_1(\gamma_0 - \delta_0) = \mu_0(\delta_1 - \gamma_1).$$

Since $\gcd(\mu_0, \mu_1) = 1$, it follows that $\gamma_0 = \delta_0 + m\mu_0$ and $\delta_1 = \gamma_1 + m\mu_0$ for some integer m. If $m > 0$, then $\gamma_0 \geq \mu_0$, which is a contradiction. If $m < 0$ we obtain a contradiction in a similar fashion. So, $m = 0$. The rest is straightforward. Q.E.D.

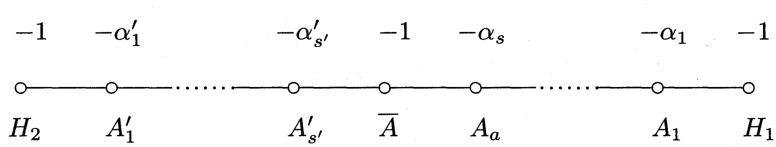
Given a pair (μ, δ) of positive integers μ, δ with $\mu > \delta$ and $\gcd(\mu, \delta) = 1$, we define integers $\alpha_1, \alpha_2, \ldots, \alpha_s$ by expanding μ/δ in a form of continued fraction

$$\frac{\mu}{\delta} = \alpha_1 - \cfrac{1}{\alpha_1 - \cfrac{1}{\alpha_3 - \cfrac{1}{\ddots - \cfrac{1}{\alpha_s}}}}$$

where $\alpha_i \geq 2$ for $1 \leq i \leq s$. We denote this fractional expansion by $\mu/\delta = [\alpha_1, \ldots, \alpha_s]$.

Given such a pair (μ, δ), the geometric meaning of fractional expansion of μ/δ in the setting leading to Lemma 1.1 is given in th following Lemma 1.3 which is well-known (cf. [9] and [4, pp. 75–78]).

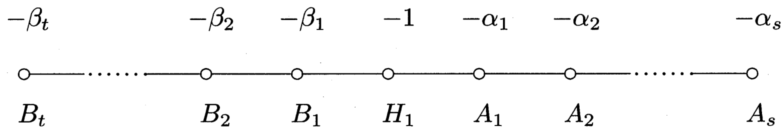
Lemma 1.3. *Let (μ, δ) be a pair of positive integers such that $\mu > \delta$ and $\gcd(\mu, \delta) = 1$. Let μA be a multiple fiber of $\rho : X \to B$ with the contribution δ of $\overline{A}$ in $\sigma^*(M_2)$. Let $\mu/\delta = [\alpha_1, \ldots, \alpha_s]$ and $\mu/(\mu - \delta) = [\alpha'_1, \ldots, \alpha'_{s'}]$ be the fractional expansions. Then the fiber $p^*(\rho(A))$ has the following dual graph:*



where $({H_1}^2) = ({H_2}^2) = -1$ if $n = a = 1$.

The next result will clarify the geometric meaning of the condition $\mu_0\mu_1 - \mu_1\delta_0 - \mu_0\delta_1 = \pm 1$.

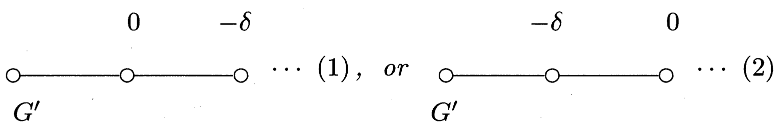
Lemma 1.4. *Let (μ_0, δ_0) and (μ_1, δ_1) be pairs as in Lemma 1.2 satisfying the condition $\mu_0\mu_1 - \mu_1\delta_0 - \mu_0\delta_1 = \pm 1$. Suppose that $\delta_0 > 0$ and $\delta_1 > 0$. Let $\mu_1/\delta_1 = [\alpha_1, \ldots, \alpha_s]$ and $\mu_0/\delta_0 = [\beta_1, \ldots, \beta_t]$ be the* *fractional expansions. Let E be a union of smooth rational curves with simple normal crossings on a smooth projective surface whose dual graph is given as below:*



Then the following assertions hold.

(1) *Suppose $\mu_0\mu_1 - \mu_1\delta_0 - \mu_0\delta_1 = 1$. Then E is contractible to a smooth point.*

(2) *Suppose $\mu_0\mu_1 - \mu_1\delta_0 - \mu_0\delta_1 = -1$. Then E contracts to a union of two smooth rational curves with one of the following dual graphes:*



where G' denotes the proper transform of the component G in the fiber $p^*(\rho(\mu_0 A_0))$ and $(G'^2) = \delta - 1$ (resp. $(G'^2) = -1$) in the case (1) *(resp.* (2)*).*

Proof. First of all, we shall show that either $\alpha_1 = 2$ or $\beta_1 = 2$. Write the condition $\mu_0\mu_1 - \mu_1\delta_0 - \mu_0\delta_1 = \pm 1$ as

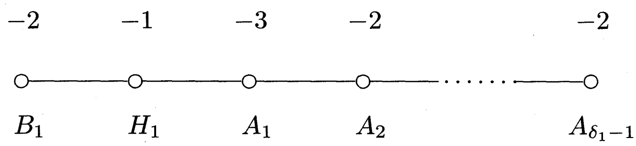
$$\left(\frac{\mu_0}{\delta_0} - 1\right)\left(\frac{\mu_1}{\delta_1} - 1\right) = 1 \pm \frac{1}{\delta_0\delta_1}.$$

Suppose $\alpha_1 \geq 3$ and $\beta_1 \geq 3$. Write $\mu_1 = \alpha_1\delta_1 - \delta_1'$ and $\mu_0 = \beta_1\delta_0 - \delta_0'$ with $0 \leq \delta_1' < \delta_1$ and $0 \leq \delta_0' < \delta_0$. Then we have

$$\begin{aligned}\left(\frac{\mu_0}{\delta_0} - 1\right)\left(\frac{\mu_1}{\delta_1} - 1\right) &= \left(\beta_1 - 1 - \frac{\delta_0'}{\delta_0}\right)\left(\alpha_1 - 1 - \frac{\delta_1'}{\delta_1}\right) \\ &\geq \left(\beta_1 - 2 + \frac{1}{\delta_0}\right)\left(\alpha_1 - 2 + \frac{1}{\delta_1}\right) \\ &\geq \left(1 + \frac{1}{\delta_0}\right)\left(1 + \frac{1}{\delta_1}\right) > \left(1 + \frac{1}{\delta_0\delta_1}\right)\end{aligned}$$

which is a contradiction.

(1) We shall prove the first assertion. Suppose $\beta_1 = 2$. Write $\mu_0 = 2\delta_0 - \delta'_0$ with $0 \le \delta'_0 < \delta_0$. Suppose further that $t = 1$, i.e., $\mu_0 = 2, \delta_0 = 1, \delta'_0 = 0$. Then $\mu_1 = 2\delta_1 + 1$ and the dual graph becomes

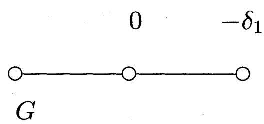


Hence it contracts to a smooth point. Suppose that $t \ge 2$. Let $\mu'_0 = \delta_0, \mu'_1 = \mu_1 - \delta_1$ and $\delta'_1 = \delta_1$. Then the pairs (μ'_0, δ'_0) and (μ'_1, δ'_1) satisfy

$$\mu'_0\mu'_1 - \mu'_1\delta'_0 - \mu'_0\delta'_1 = 1.$$

If $\alpha_1 = 2$ we can argue in a similar fashion. Hence we are done by induction. The first assertion is verified.

(2) Next we shall verify the second assertion. Suppose $\beta_1 = 2$ and $t = 1$. Then $\mu_1 = 2\delta_1 - 1$ and $\mu_1/\delta_1 = [2, \delta_1]$. Hence E contracts to a union of smooth rational curves with the dual graph:

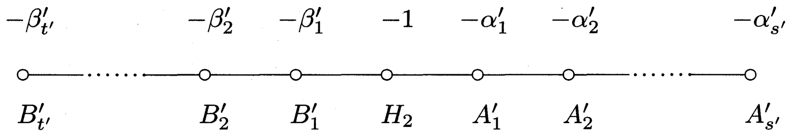


where $\delta_1 \ge 2$. Note that $\delta_1 \ne 1$. If $\alpha_1 = 2$ and $s = 1$, we have a similar conclusion as above with the second dual graph in the statement. Suppose that $\alpha_1 = \beta_1 = 2, s \ge 2$ and $t \ge 2$. We shall show that this case does not occur. Write $\mu_i = 2\delta_i - \delta'_i$ with $\delta'_i \ge 1$ for $i = 0, 1$. Then the condition $\mu_0\mu_1 - \mu_1\delta_0 - \mu_0\delta_1 = -1$ reads as $\delta_1\delta'_0 + \delta_0\delta'_1 = \delta'_0\delta'_1 + 1$. This is a contradiction since $\delta_0 > \delta'_0$ and $\delta_1 > \delta'_1$. So, $\alpha_1 \ge 3$ if $\beta_1 = 2, s \ge 2$ and $t \ge 2$. As in the proof of the assertion (1), let $\mu'_0 = \delta_0, \mu'_1 = \mu_1 - \delta_1$ and $\delta'_1 = \delta_1$. Then the pairs (μ'_0, δ'_0) and (μ'_1, δ'_1) satisfy

$$\mu'_0\mu'_1 - \mu'_1\delta'_0 - \mu'_0\delta'_1 = -1.$$

Hence we are done by induction.

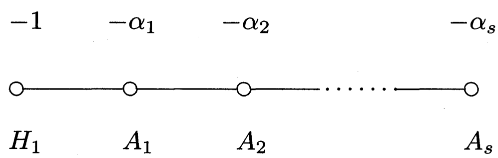
In the graph, call the component with self-intersection number 0 (resp. $-\delta$) L (resp. S). In view of Lemma 1.2, if E contracts to a union of two rational curves $L+S$, the linear chain E' contracts to a smooth point, where E' has the following dual graph with $\mu_0/(\mu_0 - \delta_0) = [\beta'_1, \dots, \beta'_{t'}]$ and $\mu_1/(\mu_1 - \delta_1) = [\alpha'_1, \dots, \alpha'_{s'}]$.



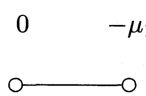
Let W be the surface obtained from V by the contractions of E and E' as described above. Then W has a $\mathbf{P}^1$-fibration $p' : W \to \mathbf{P}^1$ given by the pencil $|L|$ and S is a cross-section of p'. In the first case, the count of the Picard number of W shows that G' is a cross-section of p' with $({G'}^2) = \delta - 1$. In the second case, the count of the Picard number shows again that $({G'}^2) = -1$ and p' has a unique singular fiber which contains G' and $\overline{A}$ as the terminal (-1) components and the (-2) components in between (see the dual graph of the fiber $p^{-1}(\rho(\mu_0 A_0))$). Q.E.D.

Consider the case where $\mu_0 = 1$ and $\delta_0 = 0$.

Lemma 1.5. *Suppose $\mu_0 = 1$ and $\delta_0 = 0$. Then $\delta_1 = 1$ if $a = 0$ and $\mu_1 = \delta_1 + 1$ if $a = 1$. Let $\mu_1/\delta_1 = [\alpha_1, \dots, \alpha_s]$ be the fractional expansion. Let E be a union of smooth rational curves on a smooth projective surface V with the dual graph:*



Then either E contracts to a smooth point (case $a = 1$) or E is a union of two smooth rational curves with the dual graph (case $a = 0$) :



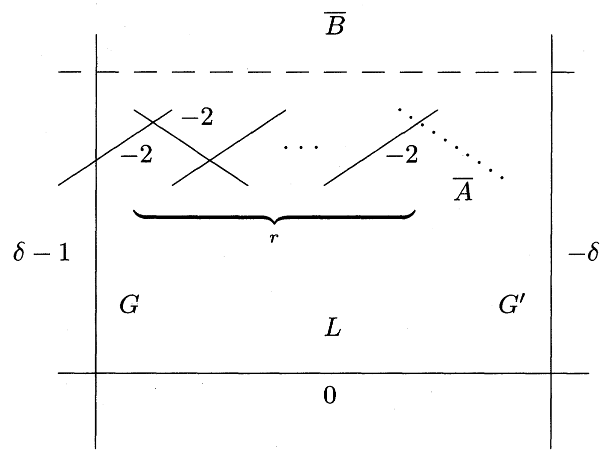
Proof. If $a = 0$ then $({H_1}^2) = 0, s = 1$ and $({A_1}^2) = -\mu_1$. If $a = 1$, then $[\alpha_1, \dots, \alpha_s] = [2, \dots, 2]$. It is clear that E contracts to a smooth point. Q.E.D.

§2. Explicit equations

First of all, consider the case $n = 1$. We only consider the case $a = 1$ and $\delta_0 \neq 0$. The case $a = 0$ and $\delta_0 = 0$ can be treated in a similar fashion. Furthermore, we assume that $\mu_0\mu_1 - \mu_1\delta_0 - \mu_0\delta_1 = -1$. The $\mathbf{P}^1$-fibration $p : V \to \mathbf{P}^1$, which extends the given $\mathbf{A}^1_*$-fibration $\rho : X \to \mathbf{P}^1$, has two degenerate fibers S_0 and S_1 and two sections H_1 and H_2. We assume that $S_0 \cap X = \mu_0 A$ and $S_1 \cap X = \mu_1 B$, where $A \cong \mathbf{A}^1$ and $B \cong \mathbf{A}^1_*$. Let E (resp. E') be the connected component of

$$D - G \cup \{\text{the side linear chain between } G \text{ and } \overline{A}\}$$

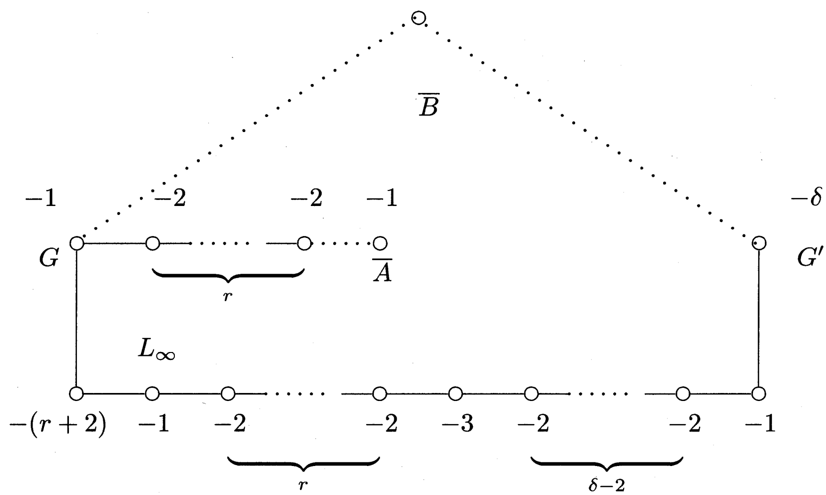
which contains H_1 (resp. H_2) (see the notations at the beginning of the section 1). By Lemma 1.4, E (resp. E') contracts to a union of two curves of the form (1) or (2) (resp. a smooth point). Suppose first that E contracts to a union of two curves of the form (1). By the contaction of E and E', we obtain a smooth projective surface W with the boundary divisor Δ such that $W - \Delta$ is isomorphic to X and Δ has the following configuration (Figure 1):



(Figure 1.)

where $\overline{A}$ (resp. $\overline{B}$) denotes, by abuse of notations, the image of $\overline{A}$ (resp. $\overline{B}$) under the contraction.

We blow up the intersection point $G \cap L$ and its infinitely near points to produce a configuration with the following dual graph (Figure 2):



(Figure 2.)

In the configuration, all curves but $\overline{A}, \overline{B}$ and L_∞ are contracted to two points, say P and Q, on the image of L_∞ (which we denote by the same symbol L_∞). In fact, the obtained surface is the projective plane $\mathbf{P}^2$ and $\mathbf{P}^2 - L_\infty$ is isomorphic to X. The image $\widetilde{B}$ of $\overline{B}$ is a curve of degree $r+2$ having a cuspidal singularity at P of multiplicity $r+1$ and passing through Q smoothly, and the image $\widetilde{A}$ of $\overline{A}$ is a line meeting $\widetilde{B}$ at P with order of contact $r+2$.

Choose a system of homogeneous coordinates (X, Y, Z) on $\mathbf{P}^2$ so that L_∞ and $\widetilde{A}$ are defined by $Z = 0$ and $X = 0$, respectively. Then $\widetilde{B}$ is defined by an equation

$$YX^{r+1} - P(X, Z) = 0,$$

where

$$P(X, Z) = a_1 X^{r+1} Z + a_2 X^r Z^2 + \cdots + a_{r+2} Z^{r+2}$$

with $a_{r+2} \neq 0$. We may assume $a_1 = 0$ by replacing Y by $Y - a_1 Z$. Let Λ be the pencil on $\mathbf{P}^2$ consisting of the closures of fibers of the given $\mathbf{A}^1_*$-fibration $\rho : X \to \mathbf{P}^1$. Since $\mu_1 B$ is a multiple fiber, we have

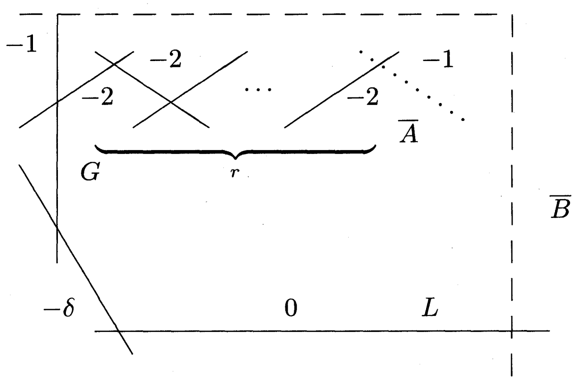
$$\Lambda = \{(YX^{r+1} - P(X, Z))^{\mu_1} + \lambda X^{\mu_0} Z^{\mu_1(r+1)+\mu_1-\mu_0} = 0;\ \lambda \in \mathbf{P}^1\} \cdots (1)$$

where we consider

$$(YX^{r+1} - P(X, Z))^{\mu_1} Z^{\mu_0 - \mu_1(r+2)} + \lambda X^{\mu_0} = 0$$

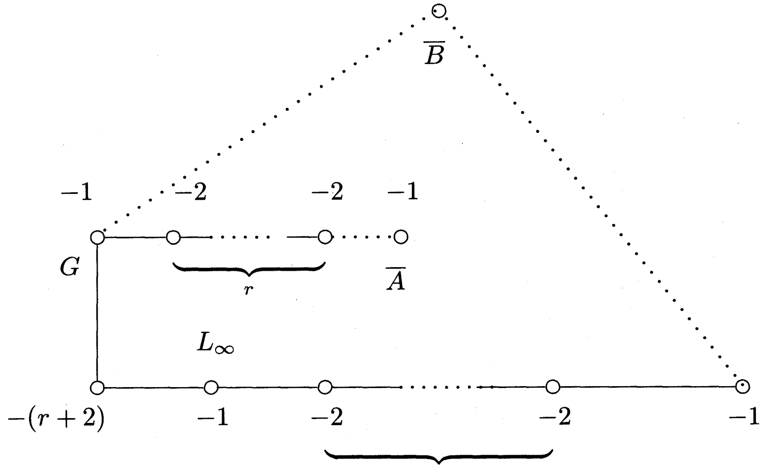
instead of the given equation if $\mu_0 > \mu_1(r+2)$.

Suppose next that E contracts to a union of two curves of the form (2). Then, with the above notation, Δ has the following configuration (Figure 3):



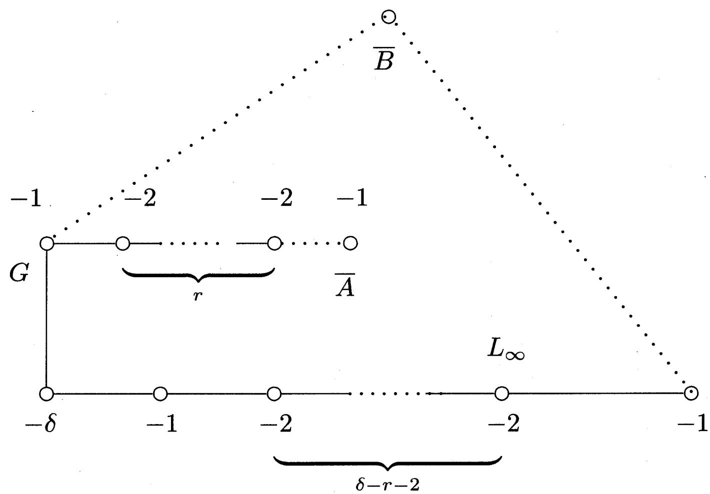
(Figure 3.)

We consider two cases according as $-\delta+r+1 \geq 0$ or $-\delta+r+1 < 0$. Suppose first $-\delta+r+1 \geq 0$. Then we obtain the following dual graph after a suitable blowing-up of the above configuration (Figure 4):



(Figure 4.)

Again, all curves but $\overline{A}, \overline{B}$ and L_∞ are contracted to two points, say P and Q, on the image of L_∞. The surface obtained by this contraction is $\mathbf{P}^2$ and L_∞ is the line at infinity, i.e., $\mathbf{P}^2 - L_\infty \cong X$. The image $\widetilde{B}$ of $\overline{B}$ is a curve of degree $r+2$ having a cuspidal singurarity at P of multiplicity $r+1$ and passing through Q, and the image $\widetilde{A}$ of $\overline{A}$ is a line meeting $\widetilde{B}$ at P with order of contact $r+2$. Then we reach to the expression (1) of the pencil Λ. Consider next the case $-\delta + r + 1 < 0$. Then we blow up the intersection point $L \cap \overline{B}$ and its $(\delta - r - 2)$ infinitely near points lying on the curve $\overline{B}$ (Figure 4):



(Figure 5.)

Then all curves but $\overline{A}, \overline{B}$ and L_∞ are contracted to two points on the image of L_∞, and the surface obtained by this contraction is $\mathbf{P}^2$ with L_∞ as a line at infinity. The same argument as in the previous cases gives the expression (1) of the pencil Λ.

Consider the case $\mu_0 = 1$ and $\delta_0 = 0$. Turning the configuration upside down if necessary, we have only to consider the case $a = 0, \mu_0 = \delta_1 = 1$ and $\delta_0 = 0$. Then one can easily show that we have the same configuration as in Figure 1 with $\delta = \mu_1$ after a suitable contraction of the components of D. So, we have the same expression of Λ as given in (1).

Consider finally the case $n = 0$. The case $a = 1$ and $\mu_0 = \delta_0 + 1$ is obtained from the case $a = 0$ and $\delta = 1$ by turning the graph upside down, i.e., changing the roles of H_1 and H_2. So, we treat only the case $a = 0$ and $\delta_0 = 1$. Then we have the form (2) in the case $n = 1$. So, the argument is a complete repetition in the case $n = 1$ with the form (2). We have thus the same expression as (1) with $\mu_1 = 1$.

Hence we obtain the following result.

Theorem 2.1. *Let $\rho : X \to \mathbf{P}^1$ be an $\mathbf{A}^1_*$-fibration parametrized by $\mathbf{P}^1$. Then, with the above notations, the pencil associated to ρ is given as follows:*

$$\Lambda = \left\{ \left(yx^{r+1} - p(x)\right)^{\mu_1} + \lambda x^{\mu_0} = 0; \quad \lambda \in \mathbf{P}^1 \right\},$$

where $p(x) \in k[x], \deg p(x) \le r$ and $p(0) \neq 0$. Furthermore, we understand that $\mu_1 = 1$ when there is no multiple fiber whose reduced form is isomorphic to $\mathbf{A}^1_*$.

§3. Complements to the previous results

(I) Let C be an irreducible curve of $\mathbf{A}^2$ and let X be anew the complement $\mathbf{A}^2 - C$. In Aoki [1], it is observed whether or not X has an étale non-finite endomorphism which is not an automorphism. In the case where X has an $\mathbf{A}^1_*$-fibration $\rho : X \to B$ and ρ extends to an $\mathbf{A}^1_*$-fibration $\widetilde{\rho} : \mathbf{A}^2 \to \widetilde{B}$, i.e., a general fiber of ρ is closed in $\mathbf{A}^2$, the case $\widetilde{B} \cong \mathbf{P}^1$ is missing in the observation. We shall consider here this case by applying Theorem 2.1. Note then that C is a fiber of $\widetilde{\rho}$ taken with the reduced structure. We consider the following three cases separately:

(1) C is a multiple fiber $\mu_0 A_0$, where $A_0 \cong \mathbf{A}^1$.
(2) C is a multiple fiber $\mu_1 A_1$, where $A_1 \cong \mathbf{A}^1_*$.
(3) C is a general fiber of ρ.

In the case (1), X has logarithmic Kodaira dimension $\overline{\kappa}(X) = -\infty$ and this case is treated in [1]. In the case (2), it follows from Theorem 2.1 and the arguments leading to its proof that C is defined by an equation of the form $yx^{r+1} - p(x) = 0$, where $p(x) \in k[x], \deg p(x) \le r$ and $p(0) \neq 0$. The polynomial $yx^{r+1} - p(x)$ is then a generically rational polynomial, and this case is also treated in [1]. So, consider the case (3). By the arguments in [6] to prove the first assertion of Lemma 1.1, we know that

$$\overline{\kappa}(X) = 1 \quad \text{(resp. 0)} \quad \text{if and only if} \quad n - \sum_{i=1}^{n} \frac{1}{n_i} > 0 \quad (\text{resp.} = 0),$$

where $n = 0, 1$. If $n = 1$ (resp. 0) then $\overline{\kappa}(X) = 1$ (resp. 0). If $n = 0$ (hence $\mu_1 = 1$) then the general fiber C is defined by $f = 0$ with $f = yx^{r+1} - p(x) + x^{\mu_0}$, and f is a generically rational polynomial. So we may assume that $n = 1$. Hence $\overline{\kappa}(X) = 1$.

Let $\alpha : X_1 \to X_2$ be an étale endomorphism, where we denote the source (resp. target) X by X_1 (resp. X_2). Accordingly, we denote by $\rho_i : X_i \to B_i$ $(i = 1, 2)$ the same $\mathbf{A}^1_*$-fibration $\rho : X \to B$, where $B_1 \cong B_2 \cong \mathbf{A}^1$. By [1, Lemma 3.2], there exists an endomorphism $\beta : B_1 \to B_2$ such that $\rho_2 \cdot \alpha = \beta \cdot \rho_2$.

We shall show that β is the identity automorphism. In fact, β extends to an endomorphism $\widetilde{\beta} : \widetilde{B}_1 \to \widetilde{B}_2$, where $\widetilde{B}_i \cong \mathbf{P}^1$ and $\widetilde{B}_i = B_i \cup \{P\}$ for $i = 1, 2$ with $P := \widetilde{\rho}(C)$. It is clear that $\widetilde{\beta}^{-1}(P) = P$. Let $P_i := \widetilde{\rho}(A_i)$ for $i = 0, 1$. By [3, Lemma 3.1], it follows that $\widetilde{\beta}(P_i) = P_i$ for $i = 0, 1$ because $\gcd(\mu_0, \mu_1) = 1$. Note that $\widetilde{\beta}$ is unramified at P_0 and P_1. By the same lemma, it follows that if $\widetilde{\beta}(Q) = P_i$ $(i = 0, 1)$ for $Q \neq P_i$, then the ramification index of $\widetilde{\beta}$ at Q equals to μ_i. Let $d := \deg \widetilde{\beta}$. Suppose that r (resp. s) points of $\widetilde{B}_1$ other than P_1 (resp. P_0) are mapped to P_1 (resp. P_0) under $\widetilde{\beta}$. By the Riemann-Hurwitz theorem, we have

$$\begin{aligned} -2 &= -2d + (d-1) + r(\mu_1 - 1) + s(\mu_0 - 1) \\ &= d - r - s - 3 \end{aligned}$$

where $d = \mu_1 r + 1 = \mu_0 s + 1$. Hence we obtain

$$d = r + s + 1 = \mu_1 r + 1 = \mu_0 s + 1. \tag{1}$$

If $d \neq 1$ then $r > 0$ and $s > 0$. It is then easy to derive a contradiction from (1) because $\gcd(\mu_0, \mu_1) = 1$. Hence $d = 1$. Since β is an automorphism of $\mathbf{P}^1$ fixing three points P, P_0, P_1, it follows that β is the identity automorphism.

Since α satisfies now $\rho \cdot \alpha = \rho$, the étale endomorphism α induces an endomorphism $\alpha_K : X_{1,K} \to X_{2,K}$ of the generic fiber X_K of ρ, where K is the function field of B. Since ρ is an untwisted $\mathbf{A}^1_*$-fibration, we know that $X_K = \operatorname{Spec} K[u, u^{-1}]$. Hence $\alpha_K^*(u) = au^{\pm n}$ with $a \in K^*$ and $n = \deg \alpha$. Let G be the group of the n-th roots of unity in k. Then G acts on $X_{1,K}$ and $X_{2,K}$ is the quotient curve $X_{1,K}/G$. Hence the function field $k(X_1)$ is a Galois extension of $k(X_2)$ with Galois group G. Let $\widetilde{X}_2$ (resp. W) be the normalization of X_2 (resp. $\mathbf{A}^2$) in $k(X_1)$, where X_2 is the open set $\mathbf{A}^2 - C$ of $\mathbf{A}^2$, and let $\nu : \widetilde{X}_2 \to X_2$ (resp. $\widehat{\nu} : W \to \mathbf{A}^2$) be the normalization morphism. By [5, Lemma 5], $\nu : \widetilde{X}_2 \to X_2$ is an étale Galois covering with group G with $\widetilde{X}_2$ containing X_1 as an open set, the composite $\rho_2 \cdot \nu : \widetilde{X}_2 \to B$ is an $\mathbf{A}^1_*$-fibration such that $\rho_2 \cdot \nu|_{X_1} = \rho_1$, and $(\rho_2 \cdot \nu)^{-1}(P_0)$ with $P_0 = \rho(A_0)$ is a disjoint union of n copies of the affine lines ${}^gA_0$ $(g \in G)$ so that $\widetilde{X}_2 - X_1 = \coprod_{g \in G, g \neq 1} {}^gA_0$, where

$A_0 \cong \mathbf{A}^1$. The surface W is a normal affine surface with a G-action, and $\mathbf{A}^2$ is the quotient surface W/G. Furthermore, $\widetilde{X}_2$ is a Zariski open set of W. Note that $\widetilde{\rho}\cdot\widehat{\nu} : W \to \widetilde{B}$ is an $\mathbf{A}^1_*$-fibration. Let $Z = (\widetilde{\rho}\cdot\widehat{\nu})^{-1}(P)$, where $P = \widetilde{\rho}(C)$. Then $\widehat{\nu}$ induces a finite morphism $\overline{\nu} : Z \to C$. Since the $\mathbf{A}^1_*$-fibration $\widetilde{\rho}\cdot\widehat{\nu} : W \to \widetilde{B}$ is extended to a $\mathbf{P}^1$-fibration with two cross-sections at infinity and since every irreducible component of Z has at least two places at infinity (for otherwise it cannot dominate C which is isomorphic to $\mathbf{A}^1_*$), it follows that

(1) Z is irreducible,
(2) W has no singular points along Z,
(3) Z is isomorphic to $\mathbf{A}^1_*$.

In fact, let V be a completion of W such that V is smooth along $V - W$, the complement $V - W$ supports a divisor with simple normal crossings and the $\mathbf{A}^1_*$-fibration $\widetilde{\rho}\cdot\widehat{\nu}$ extends to a $\mathbf{P}^1$-fibration $q : V \to \widetilde{B}$. If Z is reducible, the fiber $q^{-1}(P)$ must contain a loop of the irreducible components because each irreducible component of Z has at least two places at infinity. So, Z is irreducible. We may assume that $q^{-1}(P)$ contains no (-1) curves lying in $V - W$. If W has singular points on Z, the proper transform $\widehat{Z}$ of Z by a minimal resolution of singularities of W is a unique (-1) curve in the fiber meeting three or more components of the fiber. This is a contradiction. So, W is smooth along W. Now it is clear that Z is isomorphic to $\mathbf{A}^1_*$. This implies that $\widehat{\nu} : W \to \mathbf{A}^2$ is an étale finite Galois covering. Hence $\widehat{\nu}$ is an isomorphism. In particular, $\alpha : X_1 \to X_2$ is an automorphism. Thus we obtain the following:

Theorem 3.1. *Let C be an irreducible curve in $\mathbf{A}^2 := \operatorname{Spec} k[x,y]$ defined by*

$$\left(yx^{r+1} - p(x)\right)^{\mu_1} + \lambda x^{\mu_0} = 0,$$

where $\mu_0 \geq 1, \mu_1 > 1$ and $\lambda \neq 0$ and let $X := \mathbf{A}^2 - C$. Then $\overline{\kappa}(X) = 1$ and every étale endomorphism of X is an automorphism.

(II) In [2], we considered an automorphism of infinite order of $\mathbf{A}^2$ which stabilizes an irreducible curve C. In [2, Lemma 1.4], the case where the curve C has a defining equation

$$f := \left(yx^{r+1} - p(x)\right)^{\mu_1} + \lambda x^{\mu_0} = 0,$$

is missing. We shall complete the result by treating here the missing case. If $\mu_1 = 1$, i.e., $n = 0$, then f is a generically rational polynomial, and this case is treated in [2]. So, we assume that $\mu_1 > 1$. As in the proof of Theorem 3.1, $\overline{\kappa}(X) = 1$ and any automorphism α of X preserves the

$\mathbf{A}^1_*$-fibration ρ, i.e., $\rho \cdot \alpha = \rho$. Then $\alpha^{-1}(A_0) = A_0$ and $\alpha^{-1}(A_1) = A_1$. Namely, we have $\alpha(x) = cx$ and $\alpha(yx^{r+1} + p(x)) = d\left(yx^{r+1} + p(x)\right)$ with $c, d \in k^*$. Here note that A_0 (resp. A_1) is defined by $x = 0$ (resp. $yx^{r+1} + p(x) = 0$). Since $p(0) \neq 0$, it follows that $d = 1$. Then we have

$$\alpha(y) = c^{-(r+1)}y + \frac{p(x) - p(cx)}{c^{r+1}x^{r+1}}.$$

Hence $p(x) = p(cx)$, and c is an m-th root of unity for some m with $0 < m < r+1$ because $\deg p(x) \leq r$. So, we obtain the following:

Theorem 3.2. *Let C be an irreducible curve in $\mathbf{A}^2 := \mathrm{Spec}\, k[x, y]$ defined by*

$$\left(yx^{r+1} - p(x)\right)^{\mu_1} + \lambda x^{\mu_0} = 0,$$

where $\mu_0 \geq 1, \mu_1 > 1$ and $\lambda \neq 0$ and let $X := \mathbf{A}^2 - C$. Then every automorphism of $\mathbf{A}^2$ which stabilizes the curve C is of finite order.

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