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Radical subgroups of the sporadic simple group of Suzuki

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Abstract.

For the sporadic Suzuki simple group, the radical p-subgroups for $p = 2$ and 3 are classified and the simplicial complex of their chains is shown to be homotopically equivalent to a p-local geometry. Further investigation of the related complexes for $p = 2$ gives a counterexample to Conjecture 1 in [4].

§1. Introduction and Notation

In this note, the poset $\mathcal{B}_p(Suz)$ of radical p-subgroups of the sporadic simple group of Suzuki, denoted Suz, is determined up to conjugacy for $p = 2$ and 3. Then we have a homotopy equivalence equivariant with group action between the simplicial complex $\Delta(\mathcal{B}_p^{cen}(Suz))$ of chains of centric radical p-subgroups of Suz and a p-local geometry of Suz. The latter is a well known complex of dimension 2: for $p = 2$ one of the remarkable examples of geometries which are almost buildings (GABs) arising from sporadic simple groups ([8] and §4); and for $p = 3$ the truncation at points of a flag-transitive extended generalized quadrangle (EGQs) which appears as the residue of the extended dual polar space of the Monster ([6] and §5). Further examination leads the author to modify the conjecture given in [4, §4, Conjecture 1].

To give precise expositions, recall the following terminologies for a finite group G and a prime p: a nontrivial p-subgroup P of G is called *radical* (resp. *centric*), if $O_p(N_G(P)) = P$ (resp. every p-element of $C_G(P)$ lies in $Z(P)$). The poset of nontrivial p-subgroups of G with respect to inclusion is denoted $\mathcal{S}_p(G)$. We use $\mathcal{B}_p(G)$, $\mathcal{B}_p^{con}(G)$ and $\mathcal{B}_p^{cen}(G)$ to denote the subposets of $\mathcal{S}_p(G)$ consisting of radical p-subgroups, radical p-subgroups P with *p-constrained* normalizer $N = N_G(P)$ (that

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is, $C_{\bar{N}}(O_p(\bar{N})) \le O_p(\bar{N})$ with $\bar{N} = N/O_{p'}(N)$) and centric radical p-subgroups, respectively. For a poset X, the *order complex*, denoted $\Delta(X)$, is the simplicial complex with X as the set of vertices and the chains of elements of X as the simplices.

The inclusion gives a G-homotopy equivalence of $\Delta(\mathcal{B}_p(G))$ with $\Delta(\mathcal{S}_p(G))$ (e.g. [2, 6.6.1]) and $\mathcal{B}_p(G) \supseteq \mathcal{B}_p^{con}(G) \supseteq \mathcal{B}_p^{cen}(G)$ for every G and p [4, §4]. If G is a finite group of Lie type in characteristic p, then $\mathcal{B}_p(G) = \mathcal{B}_p^{con}(G) = \mathcal{B}_p^{cen}(G)$ and $\Delta(\mathcal{B}_p(G))$ coincides with the barycentric subdivision of the *building* of G (e.g. [2, 6.6.1]). In [4], we verified analogues of these facts for some sporadic simple groups: if $\mathcal{B}_p(G) = \mathcal{B}_p^{con}(G) = \mathcal{B}_p^{cen}(G)$ then $\Delta(\mathcal{B}_p(G))$ is G-homotopically equivalent to a complex Δ, which is one of p-local geometries of G.

While buildings are defined in a unified way for groups of Lie type, there is no canonical definition of p-local geometry for sporadic G. It just means a G-simplicial complex in which some stabilizers of vertices are p-local subgroups. Some sporadic has no or more than one such complexes constructed in ad hoc manner, though, in general, there is the best one among them in the sense that it satisfies some local axioms similar to those for buildings.

Partially motivated by searching for a unified definition of "the best" p-local geometry of a sporadic simple group G, we gave the following conjecture [4, §4, Conjecture 1] as a generalization of the observations in [4]: for sporadic G having a p-local geometry Δ, $\mathcal{B}_p^{cen}(G) = \mathcal{B}_p^{con}(G)$ and $\Delta(\mathcal{B}_p^{cen}(G))$ (or $\Delta(\mathcal{B}_p^{con}(G))$) is G-homotopically equivalent to Δ.

However, further investigation of $\mathcal{B}_2(Suz)$ reveals that the former part of the above conjecture (and hence also Conjecture 2 in [4]) is *false*: in fact, $\mathcal{B}_2^{con}(Suz)$ consists of $\mathcal{B}_2^{cen}(Suz)$ *and* two conjugacy classes; moreover, the GAB of Suz is *not* even homotopically equivalent to $\Delta(\mathcal{B}_2^{con}(Suz))$, while it *is* to $\Delta(\mathcal{B}_2^{cen}(Suz))$. Thus the above conjecture should be modified as follows by ignoring the former part.

Conjecture: For sporadic G, $\Delta(\mathcal{B}_p^{cen}(G))$ is G-homotopically equivalent to a certain p-local geometry Δ.

The modified conjecture seems to hold for every finite simple group and every prime, if we take as Δ a variant of the best p-local geometry in the sense above. Thus the author dares to propose the following as a uniform definition of p-local geometries:

> For a finite group G and a prime divisor p of its order, a G-simplicial complex Δ is called a *p-local geometry*, if
> (a) it is G-homotopically equivalent to $\Delta(\mathcal{B}_p^{cen}(G))$, and
> (b) no proper subcomplex of Δ satisfies the condition (a).

§2. Maximal 2-local subgroups of Suz

We follow the notation in §1 as well as the standard terminologies on group theory (e.g. [1] and [5]). Throughout the note, set $G := Suz$. We only consider the primes $p = 2$ and 3, as $|G|_p \leq p^2$ for other primes.

There are two classes $2A$ and $2B$ of involutions of G. The product of commuting distinct two $2A$-involutions is a $2A$-involution [5, Table III]. Thus every maximal 2-local subgroup is contained in the normalizer of a $2A$ or $2B$-pure elementary abelian subgroup. It is shown in [5] that the normalizer of a $2A$-pure elementary abelian subgroup is conjugate to a subgroup of one of the following three groups.

$C_G(z) \cong 2_-^{1+6} \cdot U_4(2)$ (z, a $2A$-involution)
$N_G(F_2) \cong 2^{2+8} : (A_5 \times S_3)$ (F_2, a $2A$-pure 2^2-group)
$N_G(F_3) \cong 2^{4+6} : (3 \cdot A_6)$ (F_3, a $2A$-pure 2^4-group)

In [7], it is shown that a $2B$-pure elementary abelian subgroup is of order at most 4, and that its normalizer is conjugate to a subgroup of $N_G(F_2)$ or one of the following two groups:

$N_G(F_4) \cong (A_4 \times L_3(4)).2$ (F_4, a $2B$-pure 2^2-group)
$N_G(F_5) \cong (E_4 \times 3^2 : Q_8).S_3$ (F_5, a $2B$-pure 2^2-group)

Details of the structure $N_G(F_i)$ ($i = 4, 5$) are given below with the latter classification, because they are not contained in [5] but required later.

The centralizer of a $2B$-involution u has a subgroup $\langle u, v\rangle \times L$ of index 2, where $\langle u, v\rangle$ is a $2B$-pure 2^2-subgroup and $C_G(u)^\infty = L \cong L_3(4)$ [5, 2.5]. Every element of order 3 of L is a $3C$-element, as it commutes with the $2B$-involution u [5, Table V]. For a $3C$-element x of L, $C_G(x) = C_G(M) = M \times A$, where M is a $3C$-pure 3^2-subgroup containing x and $A \cong A_6$. As $\langle u, v\rangle \leq C_G(x)$, $\langle u, v\rangle \leq A$.

Let $D := \langle u, v, t\rangle \cong D_8$ be a Sylow 2-subgroup of A containing $\langle u, v\rangle$. We may take $t^2 = 1$, $v^t = uv$, $Z(D) = \langle u\rangle$. Two 2^2-subgroups $F_4 := \langle u, v\rangle$ and $F_5 := \langle u, t\rangle$ of D are $2B$-pure, as they commute with the $3C$-element x. The normalizers in $A \cong A_6$ of F_i ($i = 4, 5$) are S_4: we denote $N_A(F_4) = F_4\langle x, t\rangle$ and $N_A(F_5) = F_5\langle y, v\rangle$ with x and y elements of order 3 inverted by t and v, respectively. Thus $N_G(F_i)$ is a nontrivial split extension of $C_G(F_i)$ by S_3 ($i = 4, 5$). In particular, $O_2(N_G(F_i)) \leq C_G(F_i)$ ($i = 4, 5$). We have ($C_G(u) \geq$) $C_G(F_4) = F_4 \times L \cong 2^2 \times L_3(4)$. Thus $O_2(N_G(F_4)) = F_4 \in \mathcal{B}_2(G)$. However, $N_G(F_4)$ is not 2-constrained nor centric, because of $L \cong L_3(4)$.

To determine the structure of $C_G(F_5)$, we examine the action of t on L. As M is a 3-subgroup of $C_G(u)$, it lies in L, and so it is a Sylow 3-subgroup of $L \cong L_3(4)$. Since $[t, v] \neq 1$, we have $C_G(u) = (\langle u, v\rangle \times L)\langle t\rangle$. As $|C_G(g)|_2 = 2^2$ for an element g of order 7 of L, the involution t does not centralize L. Thus t induces a unitary automorphism of $L \cong L_3(4)$, as $C_L(t) \geq M \cong 3^2$. Then $C_L(t) = MQ \cong U_3(2^2)$ with $Q \cong Q_8$ acting fixed point freely on M. Inside $C_G(u)$, we have $C_G(F_5) = F_5 \times (M.Q)$. Then $O_2(N_G(F_5)) = F_5 \in \mathcal{B}_2(G)$, which is not 2-centric because of Q. The normalizer $N_G(F_5)$ is solvable, and so 2-constrained.

The complement $\langle x, t\rangle \cong S_3$ acts on L with a non-normal subgroup $\langle t\rangle$ inducing a unitary automorphism. Thus its normal subgroup $\langle x\rangle$ centralizes L. Then $N_G(F_4) = (F_4 \langle x\rangle \times L) \langle t\rangle$, where $F_4 \langle x\rangle \cong A_4$.

The quotient group $\overline{N_G(F_5)} = N_G(F_5)/F_5$ is a direct product of $\overline{M.Q} \cong 3^2 Q_8$ with $\overline{\langle y, t\rangle} \cong S_3$, because $\overline{\langle y, t\rangle}$ acts on $\overline{M.Q}$, while t centralizes $M.Q$. However, $N_G(F_5)$ is not a direct product of $(F_5 \times M.Q)$ with $\langle y, t\rangle$ as we see below. The group $Q \cong Q_8$ acts on $A = C_G(M)' \cong A_6$, while it centralizes a Sylow 2-subgroup $\langle u, t, v\rangle$ of A. Note $[Q, A] \neq 1$: because $Q \in C_G(A)$ would imply that a Sylow 2-subgroup of the centralizer of an element of order 5 of A (which is a $5A$-element by [5, Table V]) is Q_8, while G has a subgroup $A_5 \times A_6$ and a Sylow 2-subgroup of A_6 is D_8. Thus $[Q : C_Q(A)] = 2$ and $Q/C_Q(A)$ corresponds to an odd permutation of S_6. Then $MC_Q(A)$ is normal in $N_G(F_5)$, and $N_G(F_5)/MC_Q(A)$ is a split extension by $\langle y, v\rangle \cong S_3$ of its permutation module $\mathbf{F}_2^3$.

The involution of $C_Q(A) \cong 4$ centralizes a $5A$-element of A, and so it lies in the class $2A$ [5, Table V]. As $(Q \leq)\ L \cong L_3(4)$ has a single class of involutions, every involution of L lies in the class $2A$. Involutions of $N_G(F_4) \setminus (F_4 \langle x\rangle \times L)$ are conjugate to t, and those of $(F_4 \langle x\rangle \times L) \setminus L$ are conjugate to u or ul with l an involution of L. They are $2B$-involutions, as the product of commuting distinct $2A$-involutions lies in $2A$. Hence L is a subgroup of $N_G(F_4)$ generated by $2A$-involutions.

Lemma 1. *If E is a $2B$-pure elementary abelian subgroup of G, then $N_G(E)$ is contained in $N_G(F_i)$ ($i = 4, 5$) or $N_G(F_2)$ up to conjugacy.*

Proof. Let u be the above $2B$-involution of F_i ($i = 4, 5$). We may assume $u \in E$ and $|E| = 4$, and hence $E \leq C_G(u) = (F_4 \times L)\langle t\rangle$. If $E \leq F_4 \times L$, then $E = F_4$ or $E = \langle u, gl\rangle$ for an involution $l \in L$ and $g \in F_4$. In the latter case, the subgroup of $C_G(E)$ generated by $2A$-involutions is $C_L(l)$, a Sylow 2-subgroup of L, and thus $Z(C_L(l)$ is conjugate to F_2 [5, 2.5]. Then $N_G(E) \leq N_G(F_2)$ up to conjugacy.

Note that if glt with $g \in F_4$, $l \in L$ is an involution, then $1 = (glt)^2 = gl.tgt.tlt = gg^t.ll^t \in F_4 \times L$, and hence $g \in \langle u\rangle$ and lt is an involution in Lt. Since $Aut(L_3(4)) \setminus L_3(4)$ has a single class of involutions, glt is conjugate to gt under L. Thus if $E \not\leq F_4 \times L$, then $E = \langle u, t\rangle = F_5$ up to conjugacy. Q.E.D.

§3. Radical 2-subgroups of Suz

We freely use the notation in §2. We also set $U_i := O_2(N_G(F_i))$ $(i = 1, \ldots, 5)$, where $F_1 = \langle z \rangle$. Then

$U_1 \cong 2_-^{1+6}$, $U_2 \cong 2^{2+8}$, $U_3 \cong 2^{4+6}$, $U_4 = F_4 \cong 2^2$ and $U_5 = F_5 \cong 2^2$;

and they are radical 2-subgroups of G, as we remarked. No two of them are conjugate in view of their normalizers. From the discussions of §2, we may take $F_1 \leq F_2 \leq F_3$ and $\langle F_4, F_5 \rangle = F_4F_5 \cong D_8$.

Proposition 2. *There are exactly* 10 *classes of radical* 2*-subgroups of* G *with the following representatives:*

R	$R \cong$	$Z(R)$	$N_G(R)$
U_5	2^2	2^2	$(U_5 \times 3^2 : Q_8) : S_3$
U_4	2^2	2^2	$(U_4 : 3 \times L_3(4)).2$
U_{45}	D_8	2	$U_4.2 \times 3^2 : Q_8$
U_3	2^{4+6}	2^4	$U_3 : (3 \cdot A_6)$
U_2	2^{2+8}	2^2	$U_2 : (A_5 \times S_3)$
U_{23}	$2^2[2^8 2^2]$	2^2	$U_2(2^2 : 3 \times S_3) = U_3(3.S_4)$
U_1	2_-^{1+6}	2	$U_1 \cdot U_4(2)$
U_{12}	$2^2[2^8.2^2]$	2	$U_1 \cdot 2^4 A_5 = U_2 : (A_5 \times 2)$
U_{13}	$2^3[2.2^6.2^2]$	2	$U_1 \cdot 2_+^{1+4}(3 \times S_3) = U_3 : (3 \times S_4)$
U_{123}	*Sylow*	2	$U_{123}.3$

Furthermore, we may take $U_{45} = U_4U_5$ *with* $N_G(U_{45}) = N_G(U_4) \cap N_G(U_5)$, $U_{ij} = U_iU_j$ $(i, j \in \{1, 2, 3\})$ *with* $N_G(U_{ij}) = N_G(U_i) \cap N_G(U_j)$, *and* $U_{123} = U_1U_2U_3$ *with* $N_G(U_{123}) = \cap_{i=1}^{3} N_G(U_i)$.
In particular, $\mathcal{B}_2^{cen}(G)$ *consists of* 7 *conjugacy classes of* $U_X (\emptyset \neq X \subseteq \{1, 2, 3\})$; *while* $\mathcal{B}_2^{con}(G)$ *consists of those classes together with* 2 *further classes of* U_5 *and* U_{45}.

Proof. Let U be any radical 2-subgroup of G. If $N_G(U) \leq N_G(F_5)$ but $U \neq F_5 = U_5$, then $U/U_5 \in \mathcal{B}_2(N_G(F_5)/F_5)$ by [4, 1.9]. We saw $N_G(F_5)/F_5 \cong (3^2Q_8) \times S_3$ in §2. By Lemma [3, 3.2] $\mathcal{B}_2((3^2Q_8) \times S_3)$ consists of three conjugacy classes with representatives Q_8 in the first direct factor, 2 in the second direct factor, and $Q_8 \times 2$. In the first and the last cases, we may take $U = U_5 \times Q$ and $U = (U_5 \times Q)\langle v \rangle$ respectively. Then the central involution of Q is a unique $2A$-involution of U, because $L \cap U = Q$ and $2A$-involutions of $N_G(F_5)$ lie in L. Hence up to conjugacy $N_G(U) \leq C_G(z)$. In the second case, $U = U_5 \langle v \rangle \cong D_8$ with center $\langle u \rangle$, and hence $N_G(U) \leq C_G(u) \leq N_G(U_4)$.

Assume $N_G(U) \leq N_G(U_4)$ but $U \neq U_4$. Then U/U_4 is a radical 2-subgroup of $N_G(U_4)/U_4 \cong (Z_3 \times L_3(4)).2$. If $U \cap L \neq 1$, $N_G(U)$

($\leq N_G(U_4) \leq N_G(L)$) normalizes a $2A$-pure subgroup $\Omega_1(Z(U \cap L)) \neq 1$. If $U \cap L = 1$, we may take $U = U_4 \langle t \rangle$, and hence $U = U_4 \langle t \rangle = \langle F_4, F_5 \rangle \cong D_8$. Conversely, the normalizer of $U_{45} := \langle F_4, F_5 \rangle$ lies in the centralizer of a $2B$-involution u of $Z(U_{45})$. Inside $C_G(u) = (U_4 \times L)\langle t \rangle$, we see $C_G(U_{45}) = \langle u \rangle \times C_L(t)$ and $N_G(U_{45}) = U_{45} \times C_L(t) \cong D_8 \times (3^2 : Q_8)$, and hence U_{45} is a radical subgroup but not centric. The normalizer $N_G(U_{45})$ is solvable and so 2-constrained.

Assume now that $N_G(U) \leq N_G(U_3)$ but $U \neq U_3$. Then $U/U_3 \in \mathcal{B}_2(N_G(U_3)/U_3) = \mathcal{B}_2(3A_6)$. The action of $N_G(U_3)/O_{2,3}(N_G(U_3)) \cong A_6$ on $Z(U_3) = F_3 \cong 2^4$ is equivalent to the restriction to $Sp_4(2)'$ of the action of $Sp_4(2) \cong S_6$ on its natural module. Then the subspace $C_{F_3}(U) = Z(U)$ fixed by a unipotent radical U/U_3 is a totally isotropic 1 or 2-subspace. In the former case, $N_G(U) \leq C_G(z)$ up to conjugacy. In the latter case, $Z(U)$ is conjugate to F_2 by [5, 2.4], and thus $N_G(U) \leq N_G(U_2)$ up to conjugacy.

Assume that $N_G(U) \leq N_G(U_2)$. As $\mathcal{B}_2(N_G(U_2)/U_2) = \mathcal{B}_2(A_5 \times S_3)$, U/U_2 is one of the following by [3, 3.2]: the trivial group, a 2^2-subgroup of A_5, 2 of S_3, or $2^2 \times 2$. In the latter two cases, as S_3 is faithful on $Z(U_2) = F_2 \cong 2^2$, U contains an involution which flips two involutions in F_2. Thus $Z(U)$ has a unique involution, and $N_G(U) \leq C_G(z)$ up to conjugacy. In the second case, $Z(U) = F_2$ and $N_G(U) \leq N_G(F_2)$. Then $N_G(U)/U_2 \cong A_4 \times S_3$ and $O_2(N_G(U)) = U \in \mathcal{B}_2(G)$. Hence we have one new radical group U_{23} ($\leq C_G(F_2) \cong 2^{2+8} : A_5$) with $U_{23}/U_2 \cong 2^2$. In fact $U_{23} = \langle U_2, U_3 \rangle$.

Finally assume that $N_G(U) \leq C_G(z)$ but $U \neq U_1$. Since U_1 is an extraspecial group, $Z(U) = Z(U_1) = \langle z \rangle$ for every $U/U_1 \in \mathcal{B}_2(C_G(z)/U_1)$. Then $N_G(U) \leq C_G(z)$ and $U \in \mathcal{B}_2(G)$. As $C_G(z)/U_1 \cong \Omega_6^-(2)$ is of Lie rank 2, including U_1, there are exactly 4 classes of radical 2-subgroups of G with centers conjugate to $\langle z \rangle$. The subgroups U_1, U_1U_2, U_1U_3 and $U_1U_2U_3$ are their representatives, because the images of the last three in $C_G(z)/U_1$ are the unipotent radicals fixing a singular 1, 2-subspaces and its flag, respectively, of the natural module $U_1/\langle z \rangle$ for $\Omega_6^-(2)$.

We obtained 10 classes of radical subgroups. No two of them are conjugate, in view of their structures. The normalizers and generators of representatives are calculated inside maximal 2-locals containing them.

Q.E.D.

§4. Homotopy equivalence for $p = 2$

Let $\mathcal{F}$ be the poset of the conjugates of F_i ($i = 1, 2, 3$) with respect to inclusion $\leq$. The order complex $\Delta(\mathcal{F})$ is called *the GAB of* $G = Suz$ (which is the most famous 2-local geometry of G, and so is denoted by

$\mathcal{L}_2(G)$ later). This looks different from that given in [8, Def.6.2] (denoted $(\mathcal{G}, I)$ there), but using the flag-transitivity of $\mathcal{G}$ it is immediate to verify that the following map gives an isomorphism of $(\mathcal{G}, I)$ with $(\mathcal{F}, \leq)$: $\tau : \mathcal{G} \ni x \mapsto Z(O_2(K_x)) \in \mathcal{F}$, where K_x denotes the kernel of the action of the stabilizer of x in G on the residue at x. More precisely, τ sends a 'point', 'line' or 'cross' [8, §6] to a conjugate of F_1, F_2 or F_3, respectively, and a flag is mapped to a chain.

In the following, we sometimes identify $\mathcal{G}$ with $\mathcal{F}$ via the map τ, and use terms, points, lines and crosses. For each nonempty subset X of $\{1,2,3\}$, $F_X := \{F_i | i \in X\}$ is a flag of type X, because we take $F_1 < F_2 < F_3$. We may verify that the stabilizer of F_X in G is $N_G(U_X)$ and that U_X is the O_2-part of the kernel of the action of $N_G(U_X)$ on the residue of the flag F_X.

Recall that the *barycentric subdivision* $\tilde{\Delta}$ of a simplicial complex Δ is a complex with the simplices of Δ as its vertices and the chain $(\sigma_1 \subset \sigma_2 \subset \ldots \subset \sigma_n)$ of simplices of Δ as the simplices. The geometric realization of Δ and its barycentric subdivision $\tilde{\Delta}$ are the same. In particular, they are homotopically equivalent.

From the above remarks and Proposition 2, we have:

Proposition 3. *The order complex $\Delta(\mathcal{B}_2^{cen}(G))$ of the poset of centric radical 2-subgroups of G is isomorphic to the barycentric subdivision of the GAB $\mathcal{L}_2(G)$ of G. Consequently $\Delta(\mathcal{B}_2^{cen}(G))$ is G-homotopically equivalent to the 2-local geometry $\mathcal{L}_2(G)$.*

Since $\mathcal{B}_2^{cen}(G)$ is a proper subset of $\mathcal{B}^{con}(G)$ (Proposition 2), the former part of Conjecture 1 in [4] (see also introduction) does not hold, but from Proposition 3 the latter part holds for $\Delta(\mathcal{B}_2^{cen}(G))$. To examine the homotopy equivalence of $\Delta(\mathcal{B}_2^{con}(G))$ with the 2-local geometry of G, we calculate their Euler characteristics.

A radical subgroup conjugate to U_X ($X \subseteq \{1,2,3\}$ or $X \subseteq \{4,5\}$) is called to be *of type* X. The sequence of types of terms in a chain is called the *type* of that chain. We have $U_X < U_Y$ for $X \subset Y \subseteq \{1,2,3\}$ or $X \subset Y \subseteq \{4,5\}$ by Proposition 2. Furthermore,

Lemma 4. *Up to conjugacy the following inclusions hold:*

$U_4 < U_2$ and $U_4 < U_3$, but $U_4 \not\leq U_1$;
$U_5 \not\leq U_i$ for every $i = 1,2,3$;
$U_5 < U_{13}$ but $U_5 \not\leq U_{12}$ and $U_5 \not\leq U_{23}$;
$U_{45} \not\leq U_X$ for every proper subset X of $\{1,2,3\}$.

Proof. The group $C_G(U_1)/U_1 \cong O_6^-(2) \cong SU_4(2)$ has two classes of involutions, called $2A$ and $2B$ (see e.g.[1]). In the natural unitary module $GF(4)^4$ for $SU_4(2)$, the subspace fixed by a $2A$ (resp. $2B$)-involution of $SU_4(2)$ is of dimension 3 (resp. 2). It is straightforward to verify that there are exactly two conjugacy classes of $2B$-pure 2^2-subgroups: a representative of one class fixes 1 isotropic point p and 3 isotropic lines though p; that of the other class fixes 5 isotropic points p_i $(= 1, \dots, 5)$ on an isotropic line l_1 and 5 isotropic lines l_i $(i = 1, \dots, 5)$, where l_2 and l_3 (resp. l_4 and l_5) though p_2 (resp. p_3).

We may assume that $F_1 = \langle z \rangle$ with a $2A$-involution of $C_Q(A) \leq L \cong L_3(4)$ (with notation in Section 2) centralized by $U_{45} \cong D_8$. Then U_Y lies in $C_G(F_1)$ for every $Y \subseteq \{4,5\}$. Now $O_2(C_G(F_1)) = U_1 \cong 2_-^{1+6}$ contains $F_2 > Z(U_1) = F_1$, and hence every involution of U_1 is a $2A$-involution. In particular, $2B$-pure 2^2-subgroups U_i $(i = 4, 5)$ and so U_{45} of G intersect trivially with U_1, and each of them is isomorphic to its image in $C_G(F_1)/U_1 \cong SU_4(2)$. For a $2B$-involution $u \in Z(U_{45})$, we may verify that its image in $C_G(F_1)/F_1$ does not centralize a subgroup of order 9, and hence it is a $2B$-involution of $SU_4(2)$. Thus the images $\overline{U_i}$ $(i = 4, 5)$ are $2B$-pure 2^2-subgroups of $SU_4(2)$.

Since $C_G(U_5)$ does not contain a $2A$-pure 2^2-subgroup (see Section 2), U_5 is not contained in U_2, U_3 nor U_{23} up to conjugacy. On the other hand, from the explicit shapes of U_2 and U_3 (e. g. see W_2 and W_3 in [8, §3,4]), there is a $2B$-pure 2^2-subgroup contained in both of them. In view of $C_G(u)$, this should be U_5. Since the sets of isotropic points and lines correspond respectively to the 'crosses' and 'lines' incident to the 'point' F_1 [8, §6], the image $\overline{U_5} \leq U_2/F_1$ corresponds to a $2B$-pure subgroup of $SU_4(2)$ fixing 5 points and 5 lines. On the other hand, the image $\overline{U_4}$ corresponds to a $2B$-pure subgroup of $SU_4(2)$ fixing 1 point and 3 lines. Interpreting these informations in terms of inclusions of the corresponding stabilizers (subgroups of type X with $1 \in X \subseteq \{1,2,3\}$), we conclude the above inclusions. Q.E.D.

Proposition 5. (1) *The Euler characteristic of $\mathcal{B}_2(G)$ (that is, the alternating sum of numbers of m-chains for $m = -1, \dots, 3$) is $2^{13}.514507 = 2^{13} \cdot 7 \cdot 31 \cdot 2371$.*

(2) *The Euler characteristic of $\Delta(\mathcal{B}_2^{con}(G))$ is $-2^{11}.823.1229$, while that of $\Delta(\mathcal{B}_2^{cen}(G))$ (or $\mathcal{L}_2(G)$) is $2^{10}.4091$. Thus $\Delta(\mathcal{B}_2^{con}(G))$ is not homotopically equivalent to the 2-local geometry $\mathcal{L}_2(G)$.*

Proof. From the above lemma, the possible types of chains are: Sequences of properly increasing nonempty subsets of $\{1,2,3\}$; 45, $(4,45)$, $(5,45)$, $(45,123)$, $(4,45,123)$, $(5,45,123)$; 5, $(5,13)$, $(5,123)$; $(5,13,123)$; 4, $(4,2)$, $(4,3)$, $(4,12)$, $(4,23)$, $(4,123)$, $(4,2,12)$, $(4,2,23)$, $(4,2,123)$, $(4,3,23)$, $(4,3,13)$, $(4,3,123)$; $(4,2,12,123)$, $(4,2,23,123)$, $(4,3,13,123)$, $(4,3,23,123)$. Denote by $n(X_1, \dots, X_m)$ the number of chains of type $(X_1, \dots, X_m)$.

Let χ be the Euler characteristic of $\Delta(\mathcal{B}_2^{cen}(G))$, that is, the alternating sum of $n(X_1, \ldots, X_m)$'s with sign $(-1)^{m-1}$ for all nonempty sequences $X_1 \subset \cdots \subset X_m \subseteq \{1,2,3\}$ together with the additional term -1. Moreover, let $\chi(4)$ (resp. $\chi(5)$) be the alternating sum of the numbers of chains of type containing 4 (resp. 5) but not 45, and $\chi(45)$ be the alternating sum of the numbers of type containing a term of type 45:

Note that for $m > 1$, $n(X_1, \ldots, X_m)$ equals $n(X_1, \ldots, X_{m-1})$ times the number of subgroups of type X_m containing $U_{X_{m-1}}$. The latter number is easy to find if $X_{m-1} \subset \{1,2,3\}$, because the 2-local geometry $\mathcal{L}_2(G)$ has orders 2, 2 and 4. Then it is straightforward to verify: $\chi = 2^{10}.4091$, $\chi(5) = n(5)(1-9-9.3+9.3) = (-8)|G|/|N_G(F_5)| = -2^{10}.3^4.5^2.7.11.13$ and $\chi(45) = n(45)(1-1-1-9+9+9) = 8|G|/|N_G(U_{45})| = 2^{10}.3^5.5^2.7.11.13$. As for $\chi(4)$, the above remark yields the sums of terms $n(4,2,\ldots) = n(4,2)(1-3-5+15-15-15) = (-8)n(4,2)$, $-n(4,12) + n(4,12,123) = 4n(4,12)$, $-n(4,23)+n(4,23,123) = 2n(4,23)$, and $-n(4,13)+n(4,13,123) = 2n(4,13)$.

To determine the numbers $n(4,2)$ etc., we need the following facts (the proofs are omitted, as they are straightforward): there are exactly 9 $2A$-involutions in $C_G(F_5)$: there are exactly 21×2 crosses (subgroups conjugate to F_3) in $L \cong L_3(4)$: there are two classes of lines (subgroups conjugate to F_2) in L, each line l of a class of length 3.5.7 is contained in exactly 5 crosses π with $U_4 \leq U_{l,\pi}$, and each line m of the other class of length $2^2.3.5.7$ is contained in exactly one cross μ with $U_4 \leq U_{m,\mu}$, where for example $U_{m,\pi}$ denotes the kernel of the action of the stabilizer of a flag (m,π) on the set of three points incident with (m,π). From these facts and the substructure of the residue at a point F_1 fixed by $\overline{U_i}$ $(i = 4,5)$ (see the proof of the previous lemma), we have: $n(4,2)/n(4) = 3.5.7$, $n(4,12)/n(4) = 3^2.5.7$, $n(4,23)/n(4) = 3.5.7 \times 5 + 2^2.3.5.7 \times 1$, $n(4,13)/n(4) = 3^2.5.7 \times 2$, $n(4,123)/n(4) = 3^2.5.7 \times (5+4)$. As $n(4) = [G : N_G(F_4)]$, we then calculate $\chi(4) = n(4)(1 - 3^4.5.7 + 2.3^4.5.7 - 8.3.5.7 + 8.3^2.5.7 - 2^5.3.7) = 2^{10}.3^4.5.11.13$. This yields the Euler characteristic of $\Delta(\mathcal{B}_2(G))$, because it is $\chi + \chi(4) + \chi(5) + \chi(45)$. Moreover, the Euler characteristic of $\Delta(\mathcal{B}_2^{con}(G))$ is given by $\chi + \chi(5) + \chi'(45)$, where $\chi'(45)$ is the alternating sum of numbers of chains of type containing 45 but not 5, so it is $n(45)(1-1+9-9) = 0$. Q.E.D.

§5. Radical 3-subgroups of G

There are three classes of elements of order 3 in G. It follows from [5, 2.2, 2.3] that a maximal 3-local subgroup of G is conjugate to one of the following four groups:

$N_G(T_1) \cong$ $3 \cdot U_4(3).2$, where T_1 is generated by a $3A$-element, and there is an involution inverting T_1 which induces a field automorphism on $C_G(T_1)/T_1 \cong U_4(3)$.
$N_G(T_2) \cong$ $3^{2+4} : 2(A_4 \times 2^2).2$, where T_2 is a 3^2-subgroup with two (resp. two) cyclic subgroups of type $3A$ (resp. $3B$).
$N_G(T_3) \cong$ $3^5 : M_{11}$, where $T_3 \cong 3^5$ consists of 11 (resp. 110) cyclic subgroups of type $3A$ (resp. $3B$).
$N_G(T_4) \cong$ $(3^2 : 4 \times A_6).2$, where T_4 is a $3C$-pure 3^2-subgroup.

Note that with notation in Section 2, $T_4 = M$ and $(M : C_Q(A)) \times A$ is a subgroup of $N_G(F_4)$ isomorphic to $3^2 : 4 \times A_6$. As the involutions of $C_Q(A)$ lie in the class $2A$, elements of order 3 of A are $3A$ or $3B$-elements. As the product of commuting distinct two $3A$-elements is either a $3A$ or $3B$-element [5, Table III], every $3A$-element of $N_G(F_4)$ lies in A. We also note that every 3^2-subgroup generated by $3A$-elements is conjugate to T_2. Thus we may assume $T_1 < T_2 < T_3$.

We set $V_i := O_3(N_G(T_i))$ $(i = 1, \dots, 4)$. Then $T_1 = V_1 \cong 3$, $V_2 \cong 3^{2+4}$, $T_3 = V_3 \cong 3^5$ and $T_4 = V_4 \cong 3^2$, $T_i = Z(V_i)$ and $N_G(T_i) = N_G(V_i)$ for $i = 1, \dots, 4$. Clearly, V_i is a radical 3-subgroup for every $i = 1, \dots, 4$.

Proposition 6. *There are exactly* 5 *classes of radical* 3*-subgroups of* G *with representatives* V_i $(i = 1, \dots, 4)$ *and a Sylow* 3*-subgroup* S*. In particular,* $\mathcal{B}_3^{cen}(G) = \mathcal{B}_3^{con}(G)$ *consists of* 3 *classes of subgroups* V_i $(i = 2, 3)$ *and* S*.*

Proof. Let V be any radical 3-subgroup of G. If $N_G(V) \leq N_G(T_4)$ but $V \neq T_4$, it follows from [4, Lemma 1.9] that V/T is a radical 3-subgroup of $N_G(T_4)/T_4 \cong (4 \times A_6).2$, which implies that $V/T \cong 3^2$. Then we have $V = T_4 \times T$, where T is a Sylow 3-subgroup of the A_6-subgroup $A = C_G(T_4)'$. It follows from the previous remarks that $T \cong 3^2$ is the subgroup of V generated by $3A$-elements in V. Thus $N_G(V) \leq N_G(T)$ and T is conjugate to T_2. Then up to conjugacy $N_G(V) \leq N_G(T_2)$. But then $V \cong 3^4$ would contain $O_3(N_G(T_2)) = V_2 \cong 3^{2+4}$ by [4, Lemma 1.9], which is a contradiction.

Assume that $N_G(V) \leq N_G(T_i)$ but $V \neq V_i$ for $i = 2$ or 3. Observe that the radical 3-subgroups of $N_G(T_2)/V_2 \cong 2(A_4 \times 2^2).2$ are Sylow 3-subgroups, and similarly those of $N_G(T_3)/V_3 \cong M_{11}$. Thus V is a Sylow 3-subgroup of G in these cases.

Finally, consider the case $N_G(V) \leq N_G(T_1)$ but $V \neq T_1$. Then V/T_1 is a radical 3-subgroup of $N_G(T_1)/T_1 \cong U_4(3).2$. Thus V/T_1 is the unipotent radical of the stabilizer of an isotropic point, line or a flag of the 3-dimensional unitary projective space over $GF(9)$. In the last case, V is a Sylow 3-subgroup of G. Then $(N_G(T_2) \cap N_G(T_1))/T_1 \cong 3^{1+4}2.(A_4 \times 2).2$ (resp. $(N_G(T_3) \cap N_G(T_1))/T_1 \cong 3^4 : M_{10}$) is a parabolic subgroup of

$U_4(3).2$ ($U_4(3)$ extended by a field automorphism) corresponding to an isotropic point (resp. line) with unipotent radical $T_2/T_1 \cong 3^{1+4}$ (resp. $T_3/T_1 \cong 3^4$). Thus in the former two cases, V is conjugate to T_2 or T_3. Q.E.D.

The EGQ $\mathcal{L}_3(G)$ of $G = Suz$ (which is a 3-local geometry) is usually defined as the order complex of the poset $\mathcal{T}$, the union of the conjugates of T_i $(i = 1, 2, 3)$ under G. Let $\mathcal{T}'$ be the subposet of $\mathcal{T}$ of conjugates of T_2 and T_3. Then $\Delta(\mathcal{T}')$ is the truncation of $\mathcal{L}_3(G)$ at the 'points'.

Proposition 7. *The complex $\Delta(\mathcal{B}_3^{cen}(G))$ is G-homotopically equivalent to the truncation of the EGQ $\mathcal{L}_3(G)$ at the conjugates of T_1.*

Proof. Let $\mathcal{P}$ be the union of $\mathcal{B}_3^{cen}(G)$ (consisting of conjugates of V_2, $V_3 = T_3$ and Sylow 3-subgroups) with the conjugates of $T_2 = Z(V_2)$. As a 3^2-subgroup of $M_{11} \cong N_G(T_3)/T_3$ fixes exactly two of 11 subgroups of type $3A$ of T_3, we may take $T_2 = Z(S)$ with S a Sylow 3-subgroup of G containing T_3.

We examine the subposet $\mathcal{P}_{>T_2} := \{X \in \mathcal{P} | T_2 < X\}$. Observe that V_2 has exactly two subgroups of type $3A$, which lie in $Z(V_2) = T_2$. Thus if $T_2 \le V_2^g$ for $g \in G$, then the subgroup T_2^g generated by $3A$-elements of V_2^g coincides with T_2, and hence $g \in N_G(T_2) = N_G(V_2)$. As T_3 is generated by all $3A$-elements in S, we have $T_2 < S^g$ iff $T_2 < T_3^g$ for $g \in G$. The 5-transitivity of $N_G(T_3)/T_3 \cong M_{11}$ on the 11 subgroups of T_3 of type $3A$ implies that $T_2 < T_3^g$ iff $T_2^{g^{-1}} = T_2^h$ for some $h \in N_G(T_3)$, and thus $T_3^g = T_3^k$ for $k = hg \in N_G(T_2)$. Hence $\mathcal{P}_{>T_2}$ consists of S^k, T_3^k $(k \in N_G(T_2))$ together with V_2. For $k \in N_G(T_2) = N_G(V_2)$, there is no inclusion relation between V_2 and T_3^k; S^k contains V_2, as $S > V_2$; while S^k is the unique conjugate of S containing T_3^k. Thus the complex $\Delta(\mathcal{P}_{>T_2})$ is contractible to V_2. Then it follows from Theorem of Bouc [2, 6.6.5] that the inclusion of $\mathcal{B}_3^{cen}(G)$ to $\mathcal{P}$ gives a G-homotopy equivalence between their order complexes.

We will verify that the inclusion of $\mathcal{T}'$ into $\mathcal{P}$ is also a G-homotopy equivalence of $\Delta(\mathcal{T}')$ with $\Delta(\mathcal{P})$ by the same theorem. The subposet $\mathcal{P}_{<V_2} = \{X \in \mathcal{P} | X < V_2\}$ consists of a single element T_2 and so is contractible. The subposet $\mathcal{P}_{<S} = \{X \in \mathcal{P} | X < S\}$ consists of T_3 (the unique subgroup generated by 11 subgroups of type $3A$ in S), the $\binom{11}{2}$ conjugates of T_2 under $N_G(T_3)$, and V_2. (We see V_2 is the unique conjugate of V_2 contained in S as follows. Assume $S \ge V_2^g$ for some $g \in G$. Then $V_2^g \trianglelefteq S$ as $[S : V_2^g] = 3$, and then S acts on the set of two subgroups of V_2^g of type $3A$. Then the Sylow 3-subgroup S centralizes T_2^g generated by those subgroups, and thus $T_2 = Z(S)$ coincides with T_2^g. Then $g \in N_G(T_2) = N_G(V_2)$ and $V_2^g = V_2$.) We may collapse

$\Delta(\mathcal{P}_{<S})$ to the 1-simplex with vertices T_2, T_3 by simultaneously deleting T_2^g and (T_2^g, T_3) for all $g \in N_G(T_3)$ with $T_2^g \neq T_2$ as well as V_2 and (T_2, V_2). Thus $\Delta(\mathcal{P}_{<S})$ is contractible. Q.E.D.

Remark. Using similar methods to those in Section 3, the Euler characteristic of $\Delta(\mathcal{B}_3(G))$ (resp. $\Delta(\mathcal{B}_3^{cen}(G))$ and $\Delta(\mathcal{L}_3(G))$) is calculated to $-3^6.67843$ (resp. $-3^5.38587$ and $3^5.41.733$).

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