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Either $71:35$ or $L_2(71)$ is a maximal subgroup of the Monster

Hiroyoshi Yamaki

§1. Introduction

Let $\mathbb{M}$ be the Monster simple group. Then

$$|\mathbb{M}| = 2^{46}.3^{20}.5^9.7^6.11^2.13^3.17.19.23.29.31.41.47.59.71.$$

By [2] $71:35$ is the normalizer of a Sylow 71-subgroup and $59:29$ is the normalizer of a Sylow 59-subgroup of $\mathbb{M}$.

The purpose of this note is to prove:

Theorem 1. *Either* $71:35$ *or* $L_2(71)$ *is a maximal subgroup of* $\mathbb{M}$.

Theorem 2. *Either* $59:29$ *or* $L_2(59)$ *is a maximal subgroup of* $\mathbb{M}$.

Remark. $71:35$ is a maximal subgroup of $L_2(71)$ and $59:29$ is a maximal subgroup of $L_2(59)$. However we do not know whether $L_2(71)$ or $L_2(59)$ is involved in $\mathbb{M}$ or not (See [6]). Since $|L_2(71)| = 72.71.35$ and $|L_2(59)| = 60.59.29$, these are surprisingly small groups in comparison with $\mathbb{M}$.

Theorems 1 and 2 are closely related to the prime graphs of finite groups. Let G be a finite group and $\Gamma(G)$ the prime graph of G. $\Gamma(G)$ is the graph such that the vertex set is the set of prime divisors of $|G|$, and two distinct vertices p and r are joined by an edge if and only if there exists an element of order pr in G. Let $n(\Gamma(G))$ be the number of connected components of $\Gamma(G)$. It has been proved that $n(\Gamma(G)) \leq 6$ in [7], [4], [5].

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§2. The proof of Theorems

We will give a proof of Theorem 1. Theorem 2 can be proved by the same way just replacing 71 by 59.

Lemma 1. *The* 71*-signalizer of* $\mathbb{M}$ *is trivial.*

Proof. The list of maximal p-local subgroups of $\mathbb{M}$ in [2] is complete if one adds $7^2 : SL(2,7)$ which is missing (See [6]). The result follows immediately. Q.E.D.

Lemma 2. $L_2(71)$ *is the only possible finite simple group involved in* $\mathbb{M}$ *whose order is divisible by* 71.

Proof. Lemma 2 can be proved using the classification of the prime graph components of finite simple groups in [7], [5], [4] since $\{71\}$ is a connected component of the prime graph of a simple group involved in $\mathbb{M}$ whose order is divisible by 71. Q.E.D.

Next important lemma was essentially proved by Gruenberg and Kegel (See [7]) before the classification of finite simple groups. Applying the classification of finite simple groups we have:

Lemma 3. *Let G be a finite group with $n(\Gamma(G)) \geq 2$. Then one of the following holds.*

1. *G is a Frobenius group or a 2-Frobenius group.*
2. *G has a chain of normal subgroups $G \triangleright L \triangleright N \triangleright 1$ such that N and G/L are nilpotent π-groups and L/N is a non abelian simple group where π is the connected component of $\Gamma(G)$ containing* 2.

Proof. See [1]. Q.E.D.

As is well known $\Gamma(\mathbb{M})$ has four connected components (See [3], [7]) and $\{71\}$ is a connected component of $\Gamma(\mathbb{M})$. Let G be a maximal subgroup of $\mathbb{M}$ whose order is divisible by 71. It follows that $n(\Gamma(G)) \geq 2$ and $\{71\}$ is a connected component of $\Gamma(G)$. We can apply Lemma 3.

Suppose that G is a Frobenius group. Then the Frobenius kernel is of order 71 and G is contained in $71 : 35$.

Suppose that G is a 2-Frobenius group. Then G has a chain of normal subgroups: $G \triangleright H \triangleright K \triangleright 1$ such that H is a Frobenius group with kernel K and G/K is also a Frobenius group with kernel H/K. It follows that $|K| = 71$. Since $71 : 35$ is the normalizer of a Sylow 71-subgroup in $\mathbb{M}$, G/K cannot be a Frobenius group, a contradiction.

Suppose that G has a chain of normal subgroups $G \triangleright L \triangleright N \triangleright 1$ such that N and G/L are nilpotent π-groups and L/N is a non-abelian simple group where π is the connected component of $\Gamma(G)$ containing 2. Since

π does not contain 71, $(L : N)$ is divisible by 71. Lemma 1 yields $N = 1$. It follows from Lemma 2 that L is $L_2(71)$ and $G = L$ or $G = PGL(2, 71)$. Since $\mathbb{M}$ does not contain 71 : 70, we have $G = L = L_2(71)$. The proof of Theorem 1 is complete.

Remark. The argument breaks down for the prime divisors of $|\mathbb{M}|$ less than 59 (See [2], [6]).

We have actually proved:

Theorem 3. *Let G be a maximal subgroup of $\mathbb{M}$ whose order is divisible by* 71. *Then G is isomorphic to* 71 : 35 *or* $L_2(71)$.

Theorem 4. *Let G be a maximal subgroup of $\mathbb{M}$ whose order is divisible by* 59. *Then G is isomorphic to* 59 : 29 *or* $L_2(59)$.

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Department of Mathematics
Faculty of Science, Kumamoto University
Kumamoto 860-8555 Japan
e-mail: yamaki@gpo.kumamoto-u.ac.jp