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Operator Algebras, Topology and Subgroups of Quantum Symmetry
– Construction of Subgroups of Quantum Groups –

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Abstract.

In this article, we will discuss several interactions between the non commutative Galois problems, i.e., inclusions of operator algebras and topological quantum field theory in three dimensions. Those interactions are shown to be concretely solved and are related to both the quantum subgroups of the quantum group $SU(2)_N$ at the deformation parameter $q = \exp(2\pi i/N)$.

§1. Basics on operator algebras

1.1. C^*-algebras and von Neumann algebras

Let H be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$, $B(H)$ be a set of bounded linear operators on H and M be a $*$-subalgebra of $B(H)$. The commutant of M in $B(H)$, $M' \cap B(H)$, is defined to be the set $\{x \in B(H) : xm = mx \ \forall m \in M\}$. The algebra M is said to be a *C^*-algebra* if it is closed with respect to the operator norm $\|\cdot\|$. The algebra M is said to be a *von Neumann algebra* if it is closed with respect to the strong operator topology. It is necessary and sufficient for M to be a von Neumann algebra that $M = M''$, which reads an algebraic condition is equivalent to a topological condition. For example, let G be a locally compact group represented on H with a unitary representation ρ. Then the intertwiner space $\rho(G)' \cap B(H)$ is a von Neumann algebra. From this example, one considers that the trivial intertwiner space is meaningful to analyze von Neumann algebras. M is said to be a *factor* when its center is trivial, i.e., $M' \cap M = \mathbb{C}1_M$, where 1_M is the identity element of M. A factor is classified to be either of type I (when values of the trace of projections in M are in $\{1, 2, \ldots, n\}$ or $\mathbb{N}$), type II_1 (when

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values of the trace of projections in M are in $[0,1]$), type II_∞ (when values of the trace of projections in M are in $[0,+\infty]$) or type III (when there is no trace, i.e., when all non zero projections are equivalent in the sense of Murray-von Neumann).

1.2. Murray-von Neumann's continuous geometry (The hyperfinite II_1 factor)

We will see how a von Neumann algebra R with the continuous dimension of subspaces in the ambient space where R acts is constructed. Namely, find an algebra R with a trace $\mathrm{tr} : R \to \mathbb{C}$ such that $\mathrm{tr}(p) \in [0,1]$, where p is a projection in R.

First, let A_1 be the matrix algebra $M_2(\mathbb{C})$. Then, the value of the projections in A_1 is either 0, 1/2 or 1. Embed A_1 into $A_2 = M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ so that x in A_1 is embedded in A_2 as $x \otimes 1$. Then, the value of the projections in A_1 is either 0, 1/4, 1/2, 3/4 or 1. Continue this procedure, then we have an increasing sequence of finite dimensional C^*-algebras A_n. Each A_n has an inner product defined by $\langle x, y\rangle = \mathrm{tr}(xy^*)$. Hence we can make the L^2-completion $L^2(A_n)$ and make an infinite dimensional Hilbert space H which is a completion of $\bigcup L^2(A_n)$. Let R be a weak closure of $\bigcup A_n$ represented on H. This R turns out to be a II_1 factor which has an increasing sequence of finite dimensional algebras approximating R. Such a von Neumann algebra is called *hyperfinite* or *approximately finite dimensional.* It is obvious that the values of the projections in R are in $[0,1]$ from the construction. Hyperfinite II_1 factors are isomorphic to each other due to a theorem of Murray-von Neumann. Hence, we can take R to be a completion of a Clifford algebra of a real separable Hilbert space for example, which reads that R has many symmetries. Moreover, any subfactor of R is automatically isomorphic to either $M_n(\mathbb{C})$ or R due to a deep theorem of A. Connes, which reads R is good for Galois theory.

§2. Subfactors and bimodules

Let R be the hyperfinite II_1 factor. Then, any left module H over R, i.e., any normal representation of R on H, is projective. Hence we have the spatial isomorphism ${}_RH \cong_R L^2(R)^{\oplus n}P$ which commutes with the left actions of R, where P is a projection in $B(L^2(R)^{\oplus n})$. (We denote the completion of R with respect to the inner product $\langle x, y\rangle = \mathrm{tr}(xy^*)$ by $L^2(R)$.) The "rank" $\dim({}_RH)$ of H over R is defined to be $\mathrm{tr}(P)$, which takes its value in $[0,+\infty]$. Let us assume we have an inclusion of II_1 factors $N \subset M$. V. Jones defined the index $[M : N]$ of the inclusion $N \subset M$ to be $\dim_N L^2(M)$, which surprisingly takes its value in

$\{4\cos^2 \frac{\pi}{n} : n = 3, 4, \dots\} \cup [4, +\infty]$. As an analogue of the representation theory of compact groups as in H. Weyl, we see that the N-M bimodule ${}_N M_M$ essentially contains the same information as $N \subset M$. (We drop the symbol $L^2(\,\cdot\,)$ in the sequel.) That is, let g be the N-M bimodule ${}_N M_M$ as a generator and make the iterative tensor products over N and M $\cdots g \underset{M}{\otimes} \bar{g} \underset{N}{\otimes} g \underset{M}{\otimes} \bar{g} \cdots$. Then, we can decompose this into the direct sum of the irreducible components. Here is an example of the graph Γ coming from the decomposition of the tensor products by the generator ${}_N M_M$. Such a graph is called a *principal graph* of the inclusion $N \subset M$.

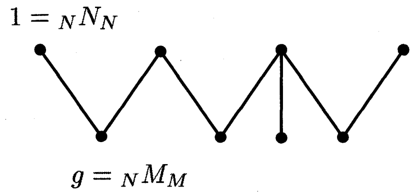

Fig. 1. An example of the principal graph.

In general, the graph norm $\|\Gamma\|$ of Γ coincides with $[M : N]^{1/2}$ if the number of irreducible objects is finite. (When the number of irreducibles is finite, then the inclusion $N \subset M$ is called of *finite depth.*) In particular, if $[M : N] < 4$, then we have $\|\Gamma\| < 2$. Hence the Coxeter graphs of type A, D and E are allowed to be such graphs coming from the decomposition of the tensor product by the generator ${}_N M_M$. In fact, the following finer result holds.

Theorem 1. *If $[M : N] < 4$, then only Coxeter graphs of A_n, D_{2n}, E_6 and E_8 appear as principal graphs. For E_6 and E_8, we have $(N \subset M) \not\cong (N^{opp} \subset M^{opp})$. In this sense, we have an algebraic chirality. In general case, if M is hyperfinite and the inclusion is of finite depth, then we can describe the complete position invariant.*

The position invariant in Theorem 1 is called a *paragroup.* In the case of infinite depth with amenablity, we have the spanning theorem due to S. Popa which guarantees that we have the complete position invariants.

Note that the Coxeter graphs of type D_{odd} and E_7 do not appear. It is not difficult to check the non-occurence of these subfactors. However, the existence of subfactors of D_{even}, E_6, and E_8 is checked with very complicated computations. It is an interesting question whether one can find a structure with irreducible objects labeled by E_7. Note that $E_7{}^{(1)}$, the affine E_7, comes from the isometry group of the binary icosahedron.

§3. Topological quantum field theory in three dimensions, braidings and quantum doubles

3.1. Topological quantum field theory in three dimensions

Roughly speaking, a topological quantum field theory (briefly, TQFT) is a functor from topological objects to algebraic objects, which was axiomatized by M. Atiyah. In particular, we are interested in 3-dimensional TQFT's which we abbreviate as TQFT_3, which have 3-dimensional manifolds as topological objects and Hilbert spaces as algebraic objects. Let us summarize what TQFT is in the case of a finite group G. Let α be an irreducible representation of G. We use a graphical way to describe an irreducible representation α and a set of homomorphisms $H_{\triangle} = \mathrm{Hom}(\alpha \otimes \beta, \gamma)$ as in the following figure.

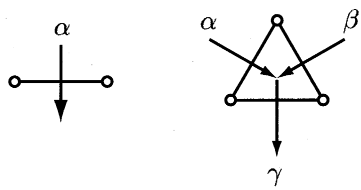

Fig. 2. Homomorphisms.

Naturally, $H_{\triangle}$ is a finite dimensional Hilbert space. Hence, $H_{\boxslash} = \bigoplus_\lambda \mathrm{Hom}(\alpha \otimes \beta, \lambda) \otimes \mathrm{Hom}(\lambda, \delta \otimes \bar{\gamma})$ is a finite dimensional Hilbert space as well. We have several orthonormal bases in $H_{\boxslash}$. In particular, the change of bases between two orthonormal bases given by the following figure gives rise to a vector $\zeta \in H_{\boxslash} \otimes \overline{H_{\boxbslash}} = H_{\partial\boxtimes}$, where $H_{\boxbslash}$ is the basis change of $H_{\boxslash}$ as in Figure 3.

From this observation, we see that ζ plays a role of composition of faces of tetrahedra in a triangulated 3-dimensional manifolds. Let S be a triangulated surface. Fix the label of the representations of G on the edges in S and assign a Hilbert space $H_{\triangle}$ on each triangle. Make tensor products of these Hilbert spaces for a given label of irreducible representations on edges and then take the direct sum over all the possible labels. Denote the resulting large Hilbert space by H_S, which has an exponential behavior for the disjoint union of two surfaces $H_{S_1 \sqcup S_2} = H_{S_1} \otimes H_{S_2}$. (This is known to give a baby model of a quantum field theory.) For

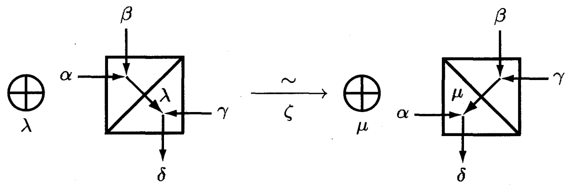

Fig. 3. Change of bases.

a triangulated 3-dimensional manifold M, we evaluate each tetrahedron by ζ and make a product of those with respect to the fixed labels of the edges and faces. Then, take a summation over all the possible label. This gives a topological invariant of M. Moreover, this construction gives a rational simplicial unitary TQFT_3. In the case of irreducible representations of $SU(2)$, similar constructions were done by Penrose and Ponzano-Regge in physics context and they showed the topological invariance although the summation is taken over the infinite labels. In the case of irreducible representations of the quantum group $SU(2)_N$ at a root of unity, this was first done by Turaev-Viro in a mathematically rigorous way. In a similar way, we can extend those construction to irreducible bimodules over II_1 factors coming from an inclusion of II_1 factors with finite Jones index and finite depth. Moreover, we see that all the rational simplicial unitary TQFT_3's appear in this way. One direction from bimodules to a TQFT_3 is similar to the case of a finite group as we have seen above. We explain how the other direction works in the next section.

3.2. From a TQFT_3 to bimodules

Assume that a rational simplicial unitary TQFT_3 is given, namely, labels of the vertices and edges, triangles for the finite dimensional Hilbert spaces and the evaluation for the tetrahedra ζ are given. Then, we can construct an increasing sequence of finite dimensional C^*-algebras in a specific way as follows. Let $\mathrm{Alg}_n(A)$ be a Hilbert space for the surface S in Figure 4, which gives rise to a finite dimensional C^*-algebra in an obvious way and is block-diagonalized into the direct sum in the right hand side of Figure 4.

This C^*-algebra is embedded into a larger algebra $\mathrm{Alg}_{n+1}(A)$ and the embedding is compatible with the trace, using information of tetrahedra. Hence, we get a hyperfinite II_1 factor $\mathrm{Alg}(A) = \overline{\bigcup_n \mathrm{Alg}_n(A)}$, which is isomorphic to R. Note that $\mathrm{Alg}(A)$ is a factor after taking the

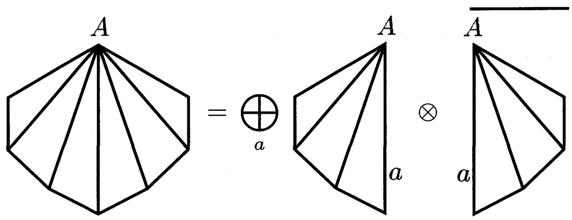

Fig. 4. A C^*-algebra $\mathrm{Alg}_n(A)$.

closure and in fact, $\mathrm{Alg}_n(A)$ is not a factor in general. This is an essential use of the argument from von Neumann algebras. Similarly, we can construct an $\mathrm{Alg}_n(A)$-$\mathrm{Alg}_n(B)$ bimodule $\mathrm{Mod}_n({}_A x_B)$ associated to a surface as in Figure 5, which is compatible with the embeddings as well.

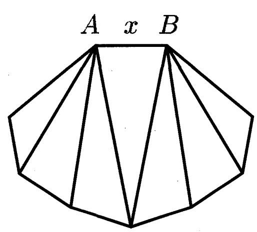

Fig. 5. A surface bimodule.

Then, we get an $\mathrm{Alg}(A)$-$\mathrm{Alg}(B)$ bimodule ${}_{\mathrm{Alg}(A)}\mathrm{Mod}({}_A x_B)_{\mathrm{Alg}(B)}$. Again, this bimodule is irreducible only after taking the closure. We see that all the information on a given TQFT_3 is contained in ${}_{\mathrm{Alg}(A)}\mathrm{Mod}({}_A x_B)_{\mathrm{Alg}(B)}$. Namely, in any TQFT_3, a Hilbert space corresponding to a triangle is exactly a space of homomorphisms and the evaluation ζ at each tetrahedron is a composition of homomorphisms.

3.3. Braidings in a TQFT_3

We will describe the "R-matrix" in a TQFT_3. As we see in the previous section we may assume that a TQFT_3 comes from bimodules over II_1 factors. A *braiding* of a system of bimodules is a distinguished choice of homomorphisms $\varepsilon = \bigoplus_{a,b} \varepsilon_{a,b} \in \bigoplus_{a,b} \mathrm{Hom}[a\otimes b, b\otimes a]$, which satisfies several axioms. (See [7] for the details of the axioms.) Graphically, a braiding is drawn as in Figure 6.

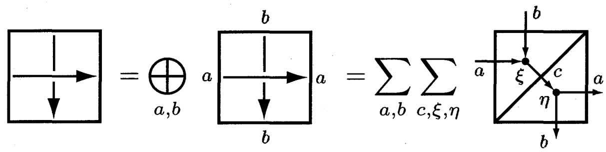

Fig. 6. A braiding.

It is natural to ask whether we can classify all braidings on a given fusion system, i.e., a system of bimodules. For the quantum group $SU(2)_N$ at a root of unity, this was already done by Kazhdan-Wenzl. In more general case, the solution was given by the author in the past Taniguchi Symposium on Operator Algebras. We will review the solution here. (Also see [7] for the details.) The main idea is in constructing a topological double. Namely, instead of using an usual wire, we use a tube. An element in $\bigoplus_{a,b,c} \mathrm{Hom}[a \otimes b, b \otimes c]$ is described as a tube as in the Figure 7. Hence, we get a tube as a homomorphism.

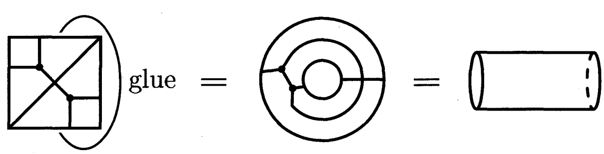

Fig. 7. Tube as a homomorphism.

The space of tubes makes a C^*-algebra in the following manner. For the multiplication, compose two tubes and take the inner product over $\bigoplus_{a,b,c} \mathrm{Hom}[a\otimes b, b\otimes c]$, then get another tube. For the $*$-structure, reverse the tube. (More precisely, we need a use of Frobenius reciprocity for the intertwiners in $\bigoplus_{a,b,c} \mathrm{Hom}[a \otimes b, b \otimes c]$ to reverse a tube. See [7] for the details of tube algebras.) Hence we get a finite dimensional C^*-algebra Tube called a *tube algebra*. Diagonalize Tube, then we get the isomorphism of C^*-algebras Tube $\cong \bigoplus_{c\in C} K_c \otimes \overline{K_c}$, where C is a set of minimal central projections in a tube algebra. We draw a tube which has a label c in C by a circle. (Note that such a tube looks like an annulus with the same labels of circles for both inside and outside. Shrink the band of the annulus, then we get a circle with label c. Also note that we had only edges for single labels.) Accordingly, we get new Hilbert spaces which are circles labeled with $c \in C$ and a vector ζ. Thus, we get a smooth theory (or *double*) from the initial theory (or *simple*). The difference between a simple theory and a double is graphically explained in Figure 8.

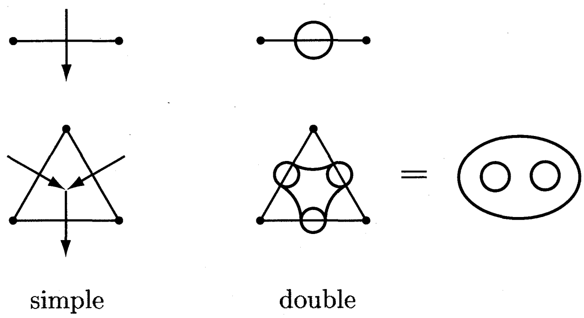

Fig. 8. Simple theory and double theory.

Naturally, the double has a braiding constructed from a fusion system as in the following figure.

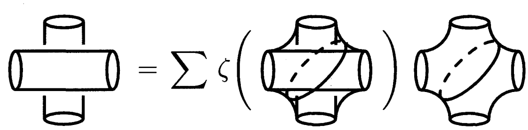

Fig. 9. Natural braiding.

In general, a simple theory cannot be embedded into a double. However, for a given braiding on a simple theory, we have a map φ_ε which is defined in Figure 10.

We get a theorem on this φ_ε.

Theorem 2. *For each α in a braided simple theory, $\varphi_\varepsilon(\alpha)$ is a minimal central projection in the center of the tube algebra. In consequence, $\varphi_\varepsilon(\alpha)$ is an irreducible object of the double.*

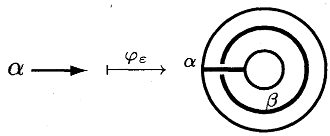

Fig. 10. A map φ_ε.

Note that φ_ε maps a triangle in a simple theory to trousers in a double as in Figure 8 and a crossing to a crossing of tubes.

Thus, we see that a braiding is identified with the embedding of simple objects into a double in this sense. Hence it allows us to classify possible braidings on all known systems. For example, a system of bimodules corresponding to the Coxeter graph E_8^+ has an abelian system of bimodules but no braiding on it.

In general, for a given non degenerate braiding, the center of the tube algebra consists of the elements $\{p_{\alpha,\beta}\}$ graphically drawn in Figure 11.

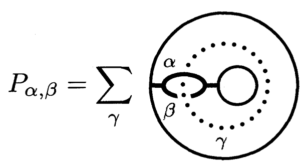

Fig. 11. An element in the center of Tube.

From this, we see that braidings are rigid and computable and get the solution of the problem stated above in a general scheme.

3.4. Algebraic doubles and topological doubles

Let $M_0 \subset M_1$ be an inclusion of factors of type II_1 with finite Jones index. Then, we can make a *Jones tower* of inclusions in a canonical way. Namely, $M_0 \subset M_1 \subset M_2 = \mathrm{End}({M_1}_{M_0}) \subset M_3 \subset \cdots \subset M_\infty = \overline{\bigcup_n M_n}$, where the completion is taken with respect to the trace tr. Hence we get the new inclusion $M_0 \vee ({M_0}' \cap M_\infty) \subset M_\infty$ out of $M_0 \subset M_1$. This inclusion is called an *asymptotic inclusion.* On the other hand, we have another way to construct a new inclusion using ultrafilter. Let $\omega \in \beta\mathbb{N}\backslash\mathbb{N}$ be a free ultrafilter on $\mathbb{N}$. For a fixed ω, we have a ultraproduct algebra

$M_\omega = \{(x_n)_n : \text{bounded sequence in } M\}/(x_n \underset{n\to\omega}{\longrightarrow} 0)$, which is a very big algebra but still a II_1 factor. Hence, we have the new inclusion $(M_0)_\omega \cap (M_1)_\omega \subset (M_1)_\omega$ again. This is called a *central sequence inclusion*. We reach the following theorem.

Theorem 3. *Constructions of an asymptotic inclusion, a central sequence algebra and a topological double are all equivalent as position invariants.*

We will make a brief comment on the relationship between topological invariants of 3-dimensional manifolds by Turaev-Viro and by Reshetikhin-Turaev. As shown in the proceedings of the Taniguchi Symposium on Operator Algebras, we can also make a Reshetikhin-Turaev type invariant from a system of bimodules if it admits (nondgenerate) braidings. For a given inclusion $M_0 \subset M_1$ with finite Jones index and finite depth, which may not have braidings at all, passing to the asymptotic inclusion $M_0 \vee (M_0{}' \cap M_\infty) \subset M_\infty$, we have the braidings on a system of M_∞-M_∞ bimodules. Hence we can make a Reshetikhin-Turaev invariant $\tau_{\mathcal{M}_\infty}$ in a similar manner to the original work of Reshetikhin-Turaev. On the other hand, from the asymptotic inclusion, we can construct a Turaev-Viro type invariant $\zeta_{\mathcal{M}_\infty}$. Recall we have a splitting type theorem by Turaev if the braiding is nondegenerate. Namely, $\zeta_{\mathcal{M}_\infty} = |\tau_{\mathcal{M}_\infty}|^2$ It is easy to see that $\zeta_{\mathcal{M}_\infty}$ is equal to the square of the modulus of the Turaev-Viro type invariant $\zeta_{\mathcal{M}}$ for the original inclusion. Hence, we have the equality $|\zeta_{\mathcal{M}}| = |\tau_{\mathcal{M}_\infty}|$. Actually, the following stronger result holds.

Theorem 4. *The Turaev-Viro type triangulation invariant of a* 3-*dimensional closed manifold M is equal to the Reshetikhin-Turaev type invariant of M computed with the double.*

§4. Maximal atlas and rigidity of topological quantum field theory

4.1. Maximal atlas

In TQFT_3, the basic data such as the labeled vertex A, edges, finite dimensional Hilbert spaces as triangles and the distinguished vector ζ for the evaluation of tetrahedra is given. In what follows, we will find the maximal atlas, namely, find all new vertex labels B and edges, triangles, and tetrahedra all labeled with new vertices, which are compatible with the given theory. More precisely, for a subdivided tetrahedron, the label of the center of a regular vertex can be changed between A and B. (See Figure 12.)

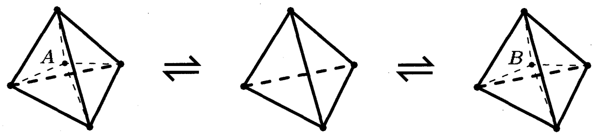

Fig. 12. Change of labels.

For example, when G is a finite group, each edge is labeled with $g \in G$ and a triangle is described as in Figure 13 and each tetrahedron takes its value at 1.

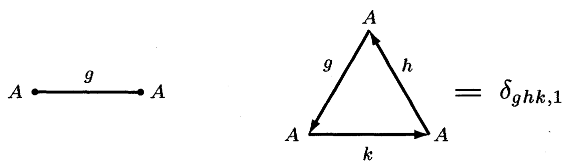

Fig. 13. Labels by G.

Hence, for a closed triangulated manifold M, the evaluation by ζ is exactly $|G|^{-1}\#\,\mathrm{Hom}(\pi_1(M), G)$. As a new label B, we put the initial data as follows. For edges with vertices A and B, assign the trivial representation of G, for each edge with vertices both B, assign an irreducible representation α of G, for each triangle labeled with an edge α with two B's, assign a representation space of α and for each triangle with vertices of B, assign $\mathrm{Hom}[\alpha \otimes \beta, \gamma]$.

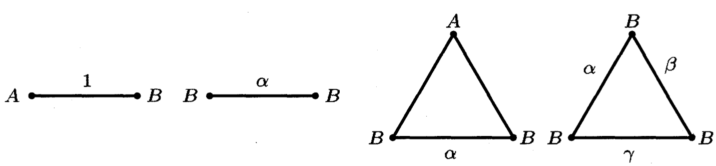

Fig. 14. Labels by $\mathrm{Rep}(G)$.

Let us compute the invariant of the lens space $L = L(2,1)$ in two different ways. Since $\pi_1(L) = \mathbb{Z}/2$, $\zeta(L) = \#\{\mathrm{Hom}(\pi_1(L), G)\} = \#\{g \in G : g^2 = 1\}$ when computed with vertices labeled only with A. On the other hand, $\zeta(L) = \sum \pm|\sigma|$, where summation is taken over $\sigma \in \mathrm{Irr}\, G$

satisfying $1 \in \sigma \otimes \sigma$ when computed with vertices labeled only with B. Hence, we get the equality $\#\{g \in G : g^2 = 1\} = \sum \pm|\sigma|$, which is called *Frobenius-Schur index.*

In the group G case, note that the vertices of the maximal atlas labeled with (H, μ) for a subgroup $H \subset G$ is up to inner conjugacy, i.e., $\mu \in H^2(H, \mathbb{T})$, which is called a *Schur multiplier* in theory of projective representations.

In the case of Turaev-Viro type TQFT_3 coming from the quantum group $SU(2)_N$ at a root of unity, we have a Coxeter graph A_{N-1} with vertices labeled with irreducible representations, where N is the Coxeter number. The maximal atlas for this system is given by all the Coxeter graphs with Coxeter number N. For instance, in the case of the quantum group $SU(2)_{29}$, we have the Coxeter graph of A_{28} and the Coxeter number is 29. Thus there is no vertex other than the initial system of A_{29}. In the case of quantum group $SU(2)_{30}$, we have the Coxeter graph of A_{30} and its Coxeter number is 30. Hence, the other vertices in the maximal atlas come from the vertices of D_{16} and E_8.

More examples are given in terms of bimodules over type II_1 factors. The maximal atlas for bimodules is formulated as before. Namely, for given family of bimodules $\{{}_A x_A\}$, which is closed under the operations of tensor product, direct sum and conjugation, find all family of bimodules $\{{}_A y_B\}$ with $\{{}_A y_1 \underset{B}{\otimes} \overline{y}_{2A}\}$ is in the span of $\{{}_A x_A\}$. For example from a subfactor $N \subset M$, we have families of bimodules $\{{}_N x_N\}$ and $\{{}_M y_M\}$ which are on the maximal atlas because we have $\{{}_N z_M\}$ bimodules to make a closed system of bimodules. The maximal atlas for subfactors is described in terms of intermediate subfactors $N \subset P \subset M$.

Theorem 5. *For a given inclusion $N \subset M$ of type II_1 factors with finite index and finite depth, the edges of the maximal atlas which connect vertices correspond to intermediate subfactors*

$$pM_0 \subset Q_0 \subset Q_1 \subset pM_np$$

for the Jones tower $M_0 = N \subset M_1 = M \subset M_2 \subset M_3 \subset \cdots$ and $p \in \mathrm{Proj}(M_0{}' \cap M_n)$.

For the inclusion generated by the Jones projections at $N = 30$, $M_0 = \langle e_i \rangle_{i \geq 1} \subset M_1 = \langle e_i \rangle_{i \geq 0}$, it contains $E_8{}^{\pm}$ on the maximal atlas.

We have some rigidity results on TQFT's.

Theorem 6. *For a rational TQFT, the maximal atlas is finite.*

Theorem 7. *Rational TQFT's are rigid. Namely, for any finite fusion rule algebra, there are finitely many possibilities for the $6j$-symbols, up to gauge. Moreover, the $6j$-symbols ζ can be chosen to be algebraic numbers.*

For the proofs on the rigidity results, in both theorems use a perturbation and compactness argument for the unitary orbits of $6j$-symbols.

As a corollary of Theorem 7, we have

Corollary 8. *The hyperfinite II_1 factor R has countably many finite depth subfactors $M \subset R$, up to conjugacy.*

We shall list several problems for our future directions. The main objective is to find structure behind rigidity as for semisimple Lie groups.

1. Find a complete set of numerical invariants.
2. Find all relations between invariants.
3. Find discrete structure, i.e., each invariant takes its value in $\{0, 1, 2, \dots\}$.
4. Classify structures...

4.2. The topological interpretation of the maximal atlas

Assume we have a smooth theory of TQFT_3. Namely, we have a finite dimensional Hilbert space H_S for a surface S, a vector $\zeta_M \in H_{\partial M}$ for a 3-dimensional manifold M with boundary, a complex number ζ_M for a closed 3-dimensional manifold M. However, we want numerical invariants even for embedded surfaces and 3-dimensional manifold with boundary. For this purpose, we need a distinguished surface, called *rind.*

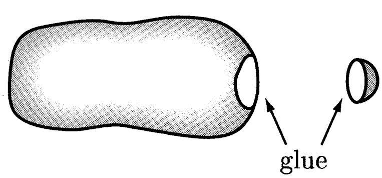

Fig. 15. 3-manifold with rind.

In Figure 15, the right small part gives a Hilbert space with rind instead of the Hilbert space labeled with a circle.

To evaluate a 3-dimensional manifold with rind, we construct a cone over the rind. Namely, for a given vertex label A in the maximal atlas, build the cone over the rind. (See Figure 16.)

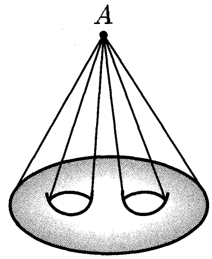

Fig. 16. A cone built over rind.

Hence we get a closed manifold with an additional vertex A and evaluate it by the given smooth theory with the maximal atlas and get a complex number. Conversely, for a given TQFT_3 with rind, use information from the rind to build one or more labels for vertices in the maximal atlas. Henceforth, a vertex label in the maximal atlas corresponds to an irreducible extension of TQFT_3 up to a TQFT_3 with rind. A vertex label A in the maximal atlas can make a cone over the empty rind $S^1 \times S^1$ but it does not have enough information to rebuild a vertex label in the maximal atlas. Modular invariants are described from such a cone over $S^1 \times S^1$. (See 6.1.)

§5. Modular invariants

5.1. Modular invariants and quantum symmetries

Classification of modular invariants is one of the most important problem in conformal field theory and in fact, the classification program is complete by Capelli-Itzykson-Zuber in the case of $SU(2)$ and by Gannon in the case of $SU(3)$. (See [1] and [3] for the details.) In the following, first we consider the case of $SU(2)$ modular invariants.

Let n be a positive integer and $N = 2n$. We denote the standard basis of $\mathbb{C}^{n-1}$ by $\{\chi_1, \dots, \chi_{n-1}\}$. Define a unitary representation ρ of $SL(2,\mathbb{Z})$

$$\rho : SL(2,\mathbb{Z}) \longrightarrow \mathcal{U}(n-1,\mathbb{C})$$

by specifying the actions of the generators

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

as follows

$$\rho(S)(\chi_k) = \frac{-2i}{\sqrt{N}} \sum_{l=1}^{n-1} \sin\left(\frac{\pi k l}{n}\right) \chi_l,$$

$$\rho(T)(\chi_k) = \exp\left(\pi i \left(\frac{k^2}{N} + \frac{1}{4}\right)\right) \chi_k.$$

Problem. Find all $(n-1) \times (n-1)$ matrices $M = (m_{k,l})$ which satisfy the following conditions;

1. $\sum_{k,l=0}^{n-1} m_{kl} \chi_k \overline{\chi}_l$ is an $SL(2, \mathbb{Z})$-invariant sesquilinear form on the dual space of $\mathbb{C}^{n-1}$.
2. $m_{kl} \geq 0$ for all k, l.
3. $m_{11} = 1$.

The above problem is equivalent to the following problem, where the two $(n-1) \times (n-1)$ matrices $\rho(S)$ and $\rho(T)$ are defined as above.

Problem. Find all intertwiners $M = (m_{k,l})_{k,l=1,\dots,n-1}$ such that

$$\rho(g)M = M\rho(g), \quad g \in SL(2, \mathbb{Z})$$

with $m_{k,l} \in \mathbb{N} = \{0, 1, 2, \dots\}$ and $m_{1,1} = 1$.

The above self-intertwiners M are called the $SU(2)$ *modular invariants.* The next answer to this problem is due to Capelli-Itzykson-Zuber [1].

Theorem 9. *Any sesquilinear form satisfying the above conditions is given by one of the following table. These sesquilinear form are called modular invariant partition functions in conformal field theory.*

type	modular invariant partition function
A_n	$\sum_{k=1}^{n} \lvert\chi_k\rvert^2$
D_{2n+1}	$\sum_{k=1}^{4n-1} \lvert\chi_k\rvert^2 + \sum_{k=2, k \in 2\mathbb{Z}}^{2n-2} (\chi_k \overline{\chi}_{4n-k} + \chi_{4n-k} \overline{\chi}_k) + \lvert\chi_{2n}\rvert^2$
D_{2n+2}	$\sum_{k=1}^{2n-1} \lvert\chi_k + \chi_{4n+2-k}\rvert^2 + 2\lvert\chi_{2n+1}\rvert^2$
E_6	$\lvert\chi_1 + \chi_7\rvert^2 + \lvert\chi_4 + \chi_8\rvert^2 + \lvert\chi_5 + \chi_{11}\rvert^2$
E_7	$\lvert\chi_1 + \chi_{17}\rvert^2 + \lvert\chi_5 + \chi_{13}\rvert^2 + \lvert\chi_7 + \chi_{11}\rvert^2 + \lvert\chi_9\rvert^2$ $+(\chi_3 + \chi_{15})\overline{\chi}_9 + \chi_9(\overline{\chi}_3 + \overline{\chi}_{15})$
E_8	$\lvert\chi_1 + \chi_{11} + \chi_{19} + \chi_{29}\rvert^2 + \lvert\chi_7 + \chi_{13} + \chi_{17} + \chi_{23}\rvert^2$

The corresponding matrix M which is a self-intertwiner of the modular representation of $SU(2)$ by Hurwitz-Verlinde is also called the $SU(2)$ *modular invariant matrix.* Some of them are given in Figures 17, 18 and 19, where a dot means 0 in the matrix entries. Note that each diagonal term is equal to the number of exponents of the corresponding Coxeter graph Γ.

$$A_8 \begin{bmatrix} 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1 \end{bmatrix} \quad D_6 \begin{bmatrix} 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1 \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&1&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&2&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&1&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1 \end{bmatrix} \quad D_7 \begin{bmatrix} 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot \\ \cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1 \end{bmatrix}$$

Fig. 17. Modular invariant matrices of type A_8, D_6, D_7.

$$E_6 \begin{bmatrix} 1&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&1 \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&1 \end{bmatrix} \quad E_7 \begin{bmatrix} 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1 \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ \cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot \\ 1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1 \end{bmatrix}$$

Fig. 18. Modular invariant matrices of type E_6 and E_7.

However, an interpretation of off-diagonal terms has been unknown until very recently. Therefore it is natural to consider the following problem.

Problem. Relate the modular invariants labeled with Γ, which is one of the A-D-E Coxeter graphs, to the graph Γ.

At first sight the Coxeter graphs themselves do not have enough symmetries. However it turns out that they have many *quantum symmetries* which are enough to interpret the off-diagonal terms of modular invariant matrices. Here the quantum symmetry in our sense is

$$
E_8 \quad \begin{bmatrix}
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1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{bmatrix}
$$

Fig. 19. Modular invariant matrix of type E_8.

defined by the following procedure. First we replace the edges of the graph Γ by a set of linear combinations of paths $\mathbb{C}^{\mathrm{Path}\,\Gamma}$ on Γ which has a natural Hilbert space structure and we denote this Hilbert space by $\mathrm{HPath}\,\Gamma$. Then the *quantum symmetry* of the graph Γ is a linear map Φ which preserves the length of the paths in $\mathrm{HPath}\,\Gamma$ and respects contracted concatenation, i.e., $\Phi(\xi \circ_n \eta) = \Phi(\xi) \circ_n \Phi(\eta)$, where $\xi \circ_n \eta$ represents the contracted concatenation. (For the details see [8].) A gauge transformation of a quantum symmetry is given by a change of bases in $\mathrm{End}(\mathrm{HPath}\,\Gamma)$.

The set of quantum symmetries modulo gauge transformation has a natural product defined in the following way.

$$\Phi \circ \Psi : \mathrm{HPath}\,\Gamma \longrightarrow \mathrm{Mat}_m(\mathrm{HPath}\,\Gamma) \longrightarrow \mathrm{Mat}_m(\mathrm{Mat}_n(\mathrm{HPath}\,\Gamma))$$

Moreover with this product it becomes a fusion rule algebra by introducing the natural notions of irreducibility and direct sum. (See [8].) Here we get a surprising and unexpected result, that is, if a graph Γ is one of the *A*-*D*-*E* Coxeter graphs there are only finitely many irreducible quantum symmetries. The following theorem describes the relationship between the modular invariant matrices for Γ and the quantum symmetries on Γ.

Theorem 10. *Let $M = (m_{kl})$ be the modular invariant matrix for an A-D-E graph Γ. Then the number of irreducible quantum symmetries is equal to the sum of squares of entries of the modular matrix M, i.e., $\sum m_{kl}^2$. The fusion rule algebra of the quantum symmetries is isomorphic to the direct sum of finite dimensional matrix algebras of the form $\bigoplus_{k,l} \mathrm{Mat}_{m_{kl}}(\mathbb{C})$. Hence the entries of a modular invariant matrix M represent the number of characters of the fusion rule algebra.*

If the graph Γ is one of the affine A-D-E Coxeter graphs, the quantum symmetries are labeled by double cosets $G \setminus SU(2)/G$ of $SU(2)$ by a finite subgroup G of $SU(2)$ corresponding to the graph Γ. For the quantum $SU(2)$ case, i.e., the case when the graph Γ is one of the A-D-E Coxeter graphs, the quantum symmetries and their fusion rules are given as in Figures 20, 21, 22 and 23, respectively.

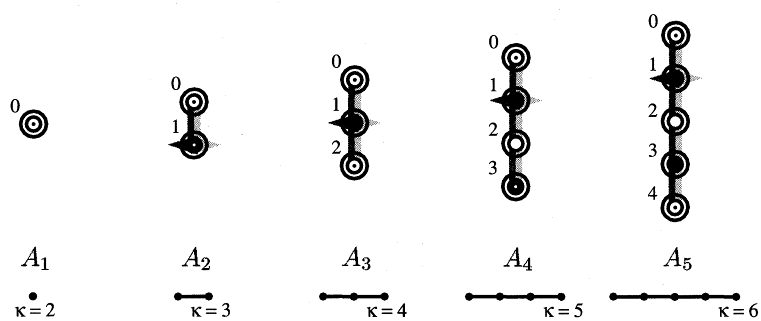


Fig. 20. Quantum symmetries for Coxeter graphs A_n.

A quantum symmetry $\Phi = (\Phi_{kl}) : \mathrm{HPath}\,\Gamma \to \mathrm{Mat}_n(\mathrm{HPath}\,\Gamma)$ is determined by its action on edges of Γ. We define a map W from a set of cells of Γ to $\mathbb{C}$ by the first equality in Figure 24. This map W is called a *connection* which is an analogue of Boltzmann weights in the RSOS models in the statistical physics. Then the (k,l)-th entry of $\Phi(\alpha)$ is written as in the second equality in Figure 24 in terms of a connection W, where α is an element in $\mathrm{HPath}\,\Gamma$. There are two generator symmetries $\Phi^+ = (\Phi^+_{kl})$ and $\Phi^- = (\Phi^-_{kl})$ of Γ. We denote the corresponding connections by W^+ and W^- respectively. One of the two generator symmetries, say, W^+ is defined by the following equality as in Figure 25, where N is the Coxeter number of Γ, $\varepsilon = i\exp(\pi i/2N)$ and k and l are taken from edges of Γ. The other generator W^- is a complex conjugate of W^+. Diagramatically these two connections W^+ and W^- are expressed as in Figure 26.

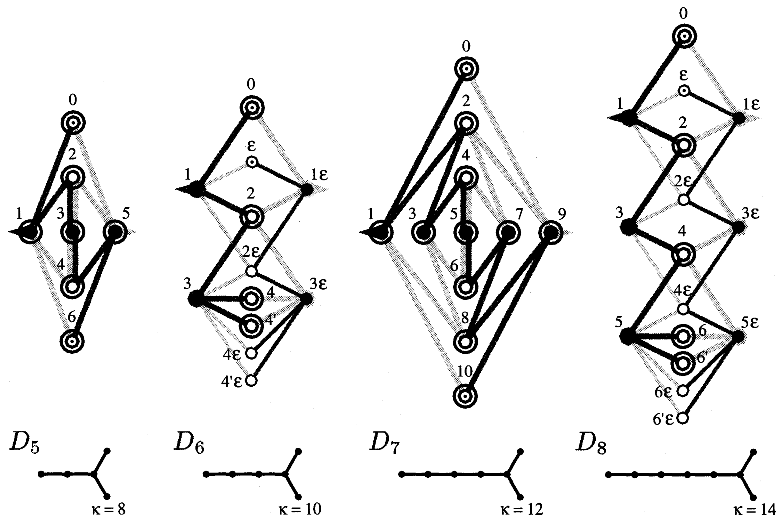


Fig. 21. Quantum symmetries for Coxeter graphs D_n.

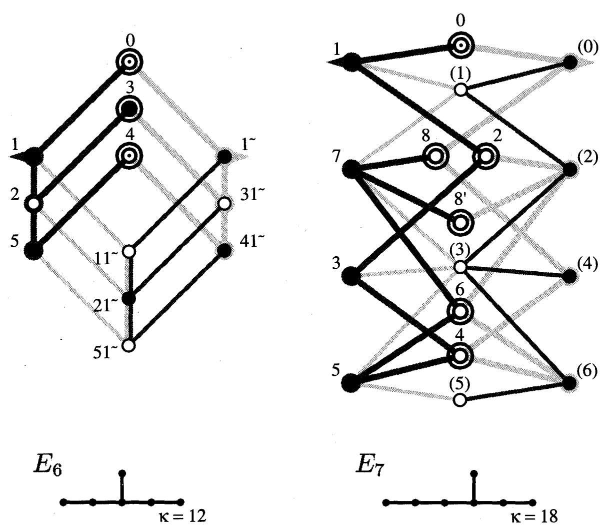


Fig. 22. Quantum symmetries for Coxeter graphs E_6 and E_7.

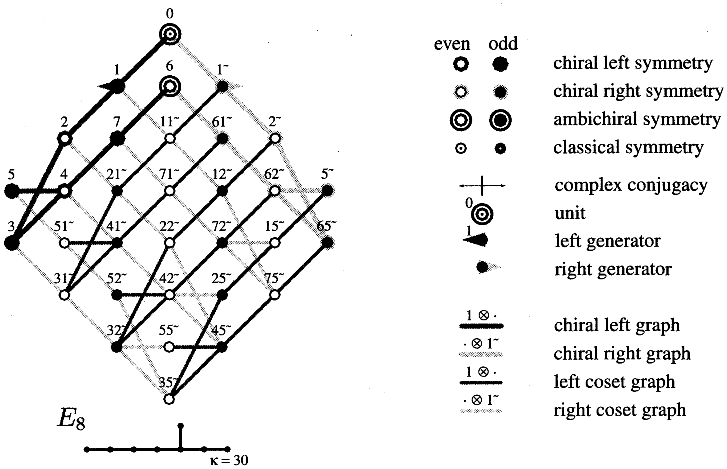


Fig. 23. Quantum symmetries for Coxeter graphs E_8.

These quantum symmetries of an *A-D-E* Coxeter graph Γ can be naturally considered as representations of a *quantum subgroup* of the quantum group $SU(2)_N$ at $N = \kappa$, the Coxeter number of Γ. A subset of quantum symmetries of Γ generated by Φ^+ (resp. Φ^-) is called the *chiral left* (resp. *right*) *part* in $SU(2)_N$. The intersection of the chiral left and right parts is called the *ambichiral part.* In general chiral left (or right) part is not braided though $SU(2)_N$ itself is braided. But there is a *relative braiding* between chiral left part and chiral right part. Hence there is a braiding on the ambichiral part.

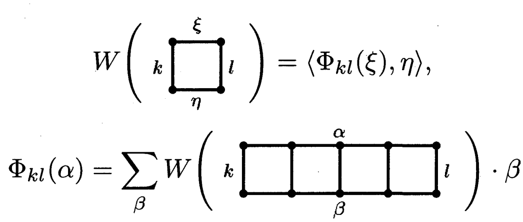


Fig. 24. Quantum symmetry Φ of a graph Γ and a connection W.

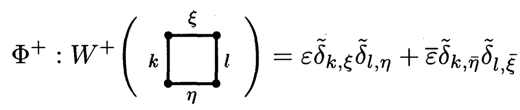


Fig. 25. A generator symmetry Φ^+ and W^+ of a graph Γ.

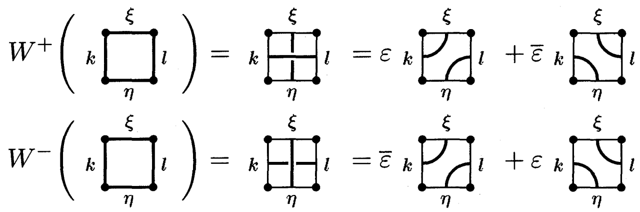


Fig. 26. Two generator symmetries W^+ and W^-.

5.2. Quantum kleinian invariants

In the following we review the classical $SU(2)$ invariant theory of F. Klein. Then as an analogy of classical invariants of $SU(2)$, we describe quantum (Kleinian) invariants of $SU(2)_N$. First we introduce diagrams as shown in Figure 27. Here each oriented line represents a generator of (irreducible) representations of quantum $SU(2)_N$ such as the fundamental representation, and we denote it by σ. Their tensor product $\sigma \otimes \sigma$ is drawn as two parallel lines. And n parallel lines with a rectangle in Figure 27 represents the quantum symmetrizer, i.e., the Jones-Wenzl projector of $SU(2)_N$. (See [4, Chapter 3].)

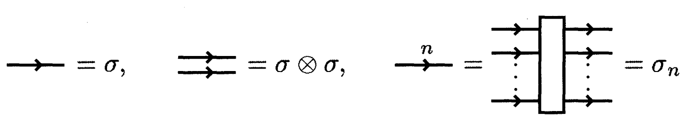


Fig. 27. A generator of Irr $SU(2)_N$ and a quantum symmetrizer.

For the classical $SU(2)$, we have invariant theory of Icos$\tilde{}$, the binary icosahedral group, which is a finite subgroup of $SU(2)$ and the corresponding Coxeter graph is $E_8^{(1)}$ as in Figure 29. In this case we have three invariant polynomials of two variables u and v of degree 12, 20 and 30. We denote them by X_{12}, Y_{20} and Z_{30}, respectively. The roots of these three polynomials correspond to vertices, each barycenter of faces and edges of icosahedron, respectively. Moreover, these three polynomials satisfy the equation $X^5 + Y^3 + Z^2 = 0$. See Figure 29 for other classical cases.

$X_{12}(u,v) = \overset{12}{\longrightarrow} X$ Roots : vertices of icosahedron

$Y_{20}(u,v) = \overset{20}{\longrightarrow} Y$ Roots : barycenters of faces of icosahedron

$Z_{30}(u,v) = \overset{30}{\longrightarrow} Z$ Roots : barycenters of edges of icosahedron

Fig. 28. The classical $SU(2)$ invariant theory of F. Klein for icosahedral group.

For the quantum $SU(2)_{30}$, we have a quantum subgroup which corresponds to the Coxeter graph E_8. We can regard it as a quantum icosahedral group and we denote it by QIcos. Similarly to the classical case, the chiral left part QIcos$^+$ of QIcos has three invariants of degree 10, 18 and 28. We denote them by X^+, Y^+ and Z^+, respectively. (See Figure 31.) We remark that the degrees of these invariants correspond to the the column number of non-zero entries in the first row of the modular invariant matrix of type E_8. (See Figure 19.)

In the classical invariant theory for Icos$\tilde{}$, the set of all invariants are given by $\mathbb{C}[X,Y,Z]/(X^5 + Y^3 + Z^2 = 0)$. Note that the polynomial Y of degree 20 is obtained by taking the Hessian of X and similarly Z is obtained by the Jacobian of X and Y. (See Figures 28 and 30.)

In the quantum case, we have the chiral left part as well as the chiral right part. So all invariants consist of ones in the chiral left part, ones in the chiral right part and mixed ones of them. Now we describe another relationship between the modular invariants and the quantum symmetries on Γ. The (k,l)-th entry m_{kl} of the modular invariant matrix represents the number of invariants of degree (k,l), i.e., the number of mixed invariants appears in the product of chiral left representation of degree k and chiral right representation of degree l. (See Figure 32.)

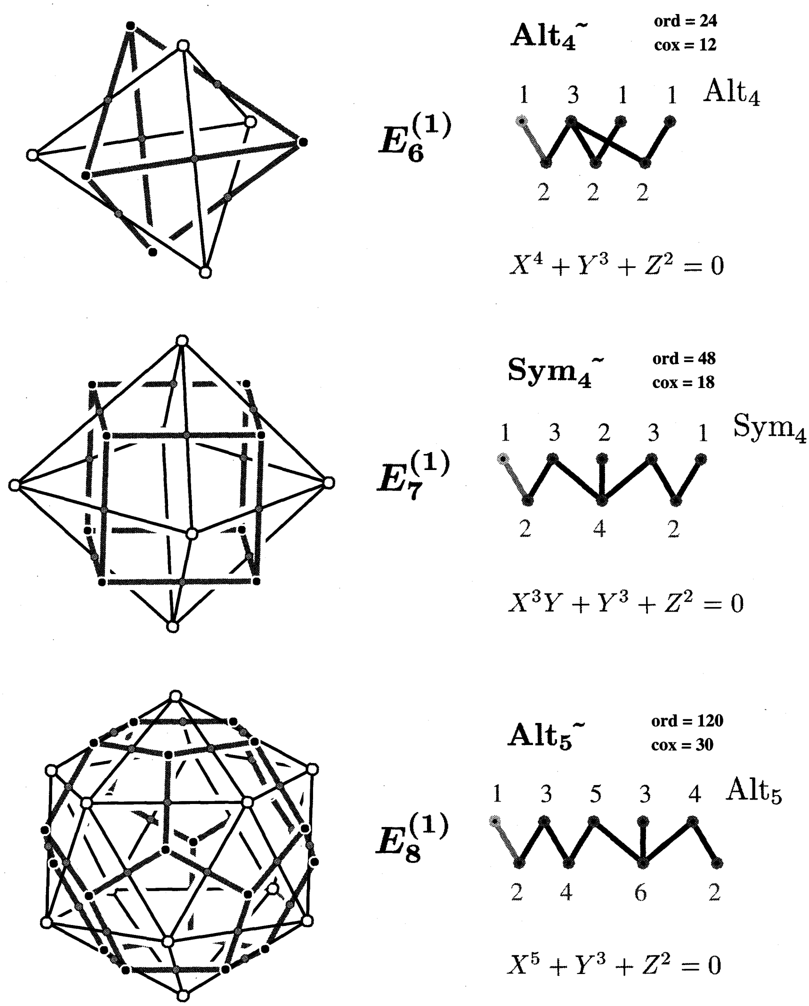
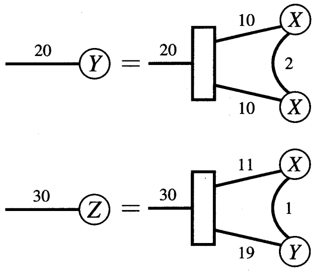


Fig. 29. Platonic solids and finite subgroups of $SU(2)$.

§6. Quantum subgroups of a braided quantum group

6.1. A general method of constructing maximal atlas of a given braided system

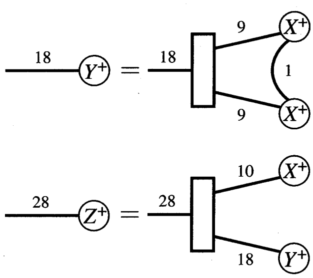
Suppose a commutative fusion rule algebra of type A-A (such as the one from $SU(2)_N$) and a nondegenerate braiding on it are given. We



$Y = \mathrm{Hess}(X)$

$Z = \mathrm{Jac}(X, Y)$

Fig. 30. The classical $SU(2)$ invariant theory for Icos˜.



$Y^+ = \mathrm{Jac}(X^+, X^+)$

$Z^+ = X^+ \cdot Y^+$

Fig. 31. The quantum Kleinian invariants of QIcos$^+$.

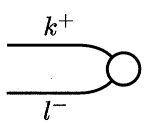


Fig. 32. A mixed invariants of degree (k, l).

draw diagrams for a fusion and a braiding as in Figure 33. Define S- and T-matrices by the equalities in Figure 34. These matrices S and T become the images of canonical generators

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

in a unitary representation of the modular group $SL(2, \mathbb{Z})$, respectively. Additionally to the A-A fusion rule algebra, we suppose a new system of a fusion rule algebra of type A-B is given. We draw thin wires for A-A objects and thick wires for A-B objects as in Figure 35. Now the problem is how to find all irreducible objects of type B-B, which are naturally considered as irreducible representations of a quantum subgroup of a given quantum group of type A-A labeled by the symbol B. We draw very thick wires for B-B objects as in Figure 35.

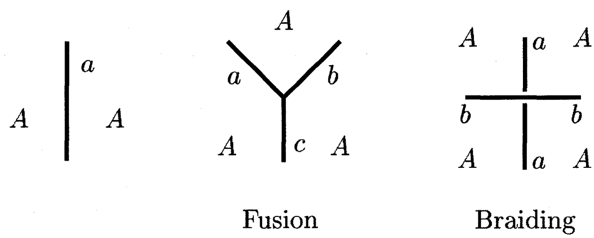


Fig. 33. Fusion and braiding.

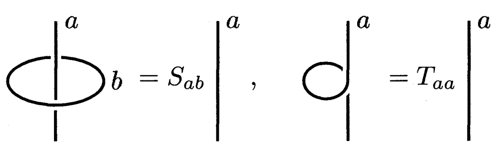


Fig. 34. S- and T-matrices.

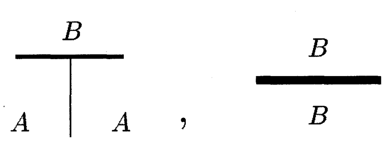


Fig. 35. Thick wires as A-B objects and very thick wires as B-B objects.

A very general method to find all irreducible objects of a fusion rule algebra of type B-B is as follows. First we define an algebra $\hat{A}$ called a *double triangle algebra* with the horizontal product as in Figure 36. (See [8] for the definition and properties of double triangle algebras.) One sees that to find all irreducibles of type B-B is equivalent to diagonalize the algebra $\hat{\mathcal{A}}$ as in Figure 37. We have two embeddings Ψ^+ and Ψ^- into a larger system using braidings as in Figure 38. Note that these embeddings give the characters of $\hat{\mathcal{A}}$. By these two embeddings, we obtain the chiral left part and the chiral right part. In particular, we get chiral generators p_a^+ and p_a^- as in Figure 39. These elements p_a^+ and p_a^- in $\hat{\mathcal{A}}$ become minimal central projections in $\hat{\mathcal{A}}$ in some cases. (See [8] for more details.) In this setting the entries m_{kl} of the modular invariant matrix M are described by the equality in Figure 40. Then, we compute all irreducible representations of $\hat{\mathcal{A}}$ from the above data such as chiral left and right parts, ambichiral part, modular invariant matrices, characters of the fusion rule algebra, and so on. (See [8] for more details.)

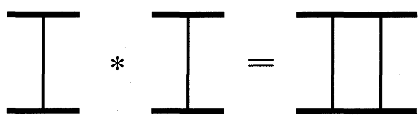


Fig. 36. The (horizontal) product of the double triangle algebra $\hat{\mathcal{A}}$.

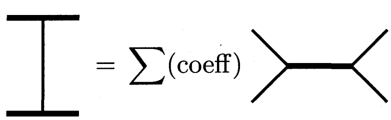


Fig. 37. Diagonalization of the double triangle algebra $\hat{\mathcal{A}}$.

The entries of the modular invariant matrix $M = (m_{a,b})$ have another description by the inner product from TQFT_3 with the singularities. (See Figure 41.) Take the modular invariant vector $\xi \in H_{S^1 \times S^1}$ and make the suspension with the vertex labeled by B. Then, TQFT_3 with the new label B gives the vector $\hat{\xi}$, which is canonically associated with ξ. For the vector coupled with ξ, take a double basis of $H_{S^1 \times S^1}$ as in Figure 41. The inner product of these two vectors gives $m_{a,b}$, an entry of the modular invariant matrix M.

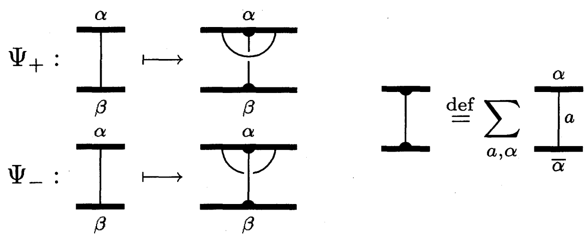


Fig. 38. Two embeddings Ψ^+ and Ψ^-.

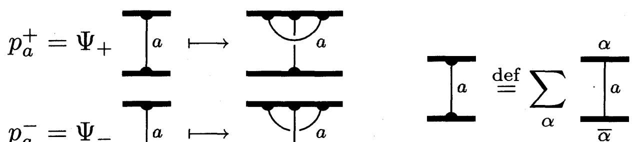


Fig. 39. Chiral generators p_a^+ and p_a^-.

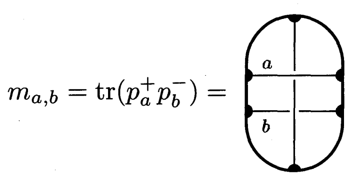


Fig. 40. A description of the modular invariant matrix.

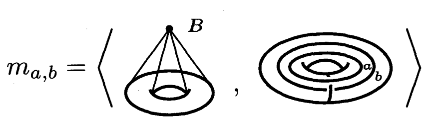


Fig. 41. Another description of the modular invariant matrix by TQFT_3.

6.2. Quantum subgroups of a nondegenerate braided quantum group

For quantum subgroups of a nondegenerate braided quantum group, we have the following structural results.

Theorem 11. *Subgroups of the nondegenerate braided quantum group $SU(2)_N$ have the following structures.*

1. *A subgroup is a product of the chiral left part and the chiral right part fibred over the ambichiral part;*
   $Subgroup = Subgroup^+ \times_{Ambichiral} Subgroup^-$,
2. *A subgroup is a product of the groups modulo Kleinian invariants;*
   $Subgroup = (Group \times Group)/Kleinian\ invariants,$
3. *The chiral left part of a subgroup is the group modulo Kleinian invariants of degree $(k, 0)$;*
   $Subgroup^+ = Group/Kleinian\ invariants\ of\ degree\ (k, 0)$.

Moreover we have the following structural results concerning the quantum doubles of subgroups of the nondegenerate braided quantum group $SU(2)_N$.

Theorem 12. *The quantum doubles of subgroups of the nondegenerate braided quantum group $SU(2)_N$ have the following structures.*

1. *The quantum double of a subgroup is same as the quantum double of the original quantum group and they are isomorphic to the product of the group and its complex conjugate;*
   $Double(Subgroup) = Double(Group) = Group \times \overline{Group},$
2. *The quantum double of the chiral left part of a subgroup is isomorphic to the product of the original group and a complex conjugate of the ambichiral part;*
   $Double(Subgroup^+) = Group \times \overline{Ambichiral}.$

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