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The Intrinsic Hodge Theory of p-adic Hyperbolic Curves

Shinichi Mochizuki

Contents

§1. Uniformization Theory as a Hodge Theory at Arithmetic Primes

(A) Uniformization as a Catalogue of Rational Points

We begin our discussion by posing the following fundamental problem concerning algebraic varieties over the complex numbers (where, roughly speaking, an "algebraic variety over the complex numbers" is a geometric object defined by polynomial equations with coefficients which are complex numbers):

> **Problem**: Given an *algebraic variety* Z over $\mathbf{C}$, it is possible to give some sort of *natural explicit catalogue* of the rational points $Z(\mathbf{C})$ of Z?

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To gain a sense of what is meant by the expression "a natural explicit catalogue", it is useful to begin by thinking about some basic examples. Perhaps the simplest nontrivial examples of algebraic varieties are *plane curves*, i.e., subvarieties of $\mathbf{A}^2_{\mathbf{C}}$ (two-dimensional affine space over $\mathbf{C}$) defined by a single polynomial equation

$$f(X,Y) = 0$$

in two variables. In this case, the set of rational points $Z(\mathbf{C})$ of the corresponding variety Z is given by

$$Z(\mathbf{C}) = \{(x,y) \in \mathbf{C}^2 \mid f(x,y) = 0\}.$$

Moreover, we can classify plane curves by the degree of the defining equation $f(X,Y)$. We then see that the resulting sets $Z(\mathbf{C})$ may be explicitly described as follows:

(1) *The Linear Case* ($\deg(f) = 1$): Up to coordinate transformations, this is the case given by the equation $f(X,Y) = X$. In this case, we then obtain an explicit catalogue of the rational points by:

$$(0, ?) : \mathbf{C} \xrightarrow{\sim} Z(\mathbf{C})$$

(i.e., mapping $z \in \mathbf{C}$ to $(0,z) \in Z(\mathbf{C})$).

(2) *The Quadratic Case* ($\deg(f) = 2$): Up to coordinate transformations (and ruling out degenerate cases), we see that this is essentially the case where the equation $f(X,Y) = X \cdot Y - 1$. In this case, an explicit catalogue is given by the *exponential map*:

$$\exp : \mathbf{C} \longrightarrow Z(\mathbf{C}) = \mathbf{C}^{\times}$$

(In fact, the map may be defined intrinsically, without using coordinate transformations to render the defining equation in the "standard form" $X \cdot Y = 1$).

(3) *The Cubic Case* ($\deg(f) = 3$): Up to adding the point(s) at infinity, this is essentially the case where we are dealing with an *elliptic curve* E. In this case, as well, we have a natural exponential map:

$$\exp_E : T_E \longrightarrow E(\mathbf{C})$$

(where T_E is the one-dimensional complex vector space given by the tangent space to some fixed point – "the origin" – of E). This map allows us to think of E as being of the form "$\mathbf{C}/\Lambda$" (where $\Lambda \cong \mathbf{Z}^2$ is a lattice in $\mathbf{C}$).

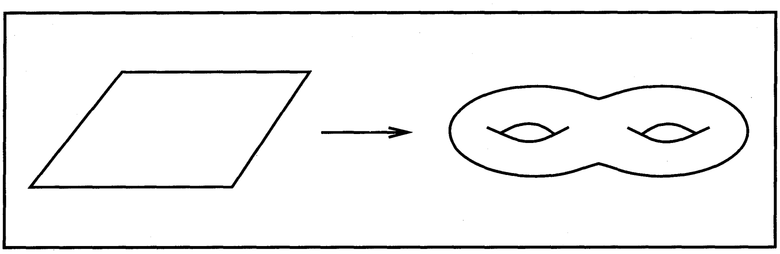
(4) <u>*Higher Degree*</u>: If $\deg(f) \geq 4$, or we wish to consider lower degree cases with lots of points removed, then we are led naturally to the following notion:

A *hyperbolic curve* Z is a smooth, proper connected algebraic curve of genus g with r points removed, where we assume that $2g - 2 + r > 0$.

According to the *uniformization theorem of Köbe*, hyperbolic curves may be uniformized by the upper half-plane $\mathfrak{H} \stackrel{\text{def}}{=} \{z \in \mathbf{C} \mid \mathrm{Im}(z) > 0\}$, i.e., we have a surjective (holomorphic) covering map:

$$\mathfrak{H} \longrightarrow Z(\mathbf{C})$$

which allows us to think of $Z(\mathbf{C})$ as being of the form $\mathfrak{H}/\Gamma$, where Γ is some discrete group acting on $\mathfrak{H}$.



(B) "Intrinsic" Hodge Theories

In the preceding section, we posed the problem of *explicitly cataloguing the rational points of a variety* (over $\mathbf{C}$). By looking at various examples, we saw that this problem may also be worded – in perhaps more familiar terms – as the problem of finding natural *uniformizations* of varieties. In the present section, we would like to further refine our understanding of the problem of finding natural uniformizations/explicit catalogues of rational points by rewording this problem in terms of the language of "Hodge theory".

First, let us discuss what we mean in general by the notion of a "*Hodge theory*". By a Hodge theory, we shall mean an *equivalence* of the following form:

$$\begin{pmatrix} \text{algebraic} \\ \text{geometry} \\ \text{(e.g., rational} \\ \text{points)} \end{pmatrix} \iff \begin{pmatrix} \underline{\text{Over } \mathbf{C}}\text{: topology +} \\ \text{differential geometry} \\ \\ \underline{\text{Over the } p\text{-adics}}\text{:} \\ \text{pro-}p \text{ étale topology +} \\ \text{Galois action} \end{pmatrix}$$

The most familiar example of such an equivalence is the *Hodge theory of cohomologies*. Over $\mathbf{C}$, this amounts to "classical Hodge theory", i.e., the well-known isomorphism between the de Rham cohomology of an algebraic variety (which is well-known to be an *algebro-geometric* invariant of the variety) and the singular cohomology of the variety (which is a *topological* invariant). More recently, the *p-adic Hodge theory of cohomologies* has been developed by Fontaine et al. (cf., e.g., [Falt2], [Falt3]). This theory asserts an equivalence between the (algebraic) de Rham cohomology of an algebraic variety over a finite extension K of $\mathbf{Q}_p$ and the p-adic étale cohomology of the variety, equipped with its natural Galois action (i.e., action of $\mathrm{Gal}(K)$).

This "Hodge theory of cohomologies" is the most basic example of a "Hodge theory" as defined above. In the present manuscript, however, we would like to consider a different kind of Hodge theory which we shall call an *intrinsic Hodge theory*, or IHT, for short. By an intrinsic Hodge theory, we mean a Hodge theory – i.e., an equivalence of the form discussed above – in which the invariant which appears on the "algebraic geometry" side is the "*variety itself*".

There are several different ways to interpret the phrase "the variety itself". In the present manuscript, we shall consider the following two interpretations:

(1) <u>The Physical Interpretation</u>: In this interpretation, one takes the phrase the "variety itself" to mean the "rational points of the variety".

(2) <u>The Modular Interpretation</u>: In this interpretation, one takes the phrase the "variety itself" to mean the "moduli of the variety".

Note that (it is a tautology of terminology that) a physical IHT essentially amounts to some sort of explicit description of the rational points of the variety in terms of topology and geometry/Galois theory. Thus, one may summarize the above discussion as follows:

$$\begin{aligned}\text{physical IHT} &= \text{uniformization of the variety,}\\ \text{modular IHT} &= \text{uniformization of the moduli space of}\\ &\qquad\quad \text{deformations of the variety.}\end{aligned}$$

Before concluding this section, we make some remarks on the relationship between the "IHT's" that we wish to discuss here and the more well-known "Hodge theories of cohomologies". First of all, although in general, IHT's are not the same as Hodge theories of cohomologies, typically in proving theorems which realize IHT's, the technical tools of the corresponding Hodge theory of cohomologies (e.g., in the p-adic case, the techniques of so-called "p-adic Hodge theory" as in [Falt2]) play an important role. Secondly, in the case of $\mathbf{G}_{\mathrm{m}} = V(X \cdot Y - 1)$ (i.e., Example (2) in §1, (A)), as well as in the case of elliptic curves (i.e., Example (3) in §1, (A)), it just so happens that the first cohomology module of the curve "is" the curve itself, i.e., in more sophisticated language, in these cases the curve in question is a 1-*motive.* Thus, in these cases, it turns out that the notions of IHT and Hodge theory of H^1 *coincide.* In particular, in these cases, the well-known Hodge theory of H^1 already provides a uniformization of the curve. Note that this differs quite substantially from the case of higher genus (Example (4) in §1, (A)).

(C) Completion at Arithmetic Primes

So far in our discussion, we have ignored the important issue of what sort of *ground field* should be considered in our discussion of uniformizations/explicit catalogues of rational points/IHT's. In the examples of §1, (A), we considered the situation over *the complex number field*, since this is the most elementary and well-known example of a ground field over which IHT's may be realized.

Of course, ideally, one would like to realize IHT's over any field, for instance, over a *number field* (i.e., a finite extension of $\mathbf{Q}$) – a case for which the problem of determining the set of rational points explicitly is of prime interest. Unfortunately, however, typically, in order to realize an IHT (or, indeed, any sort of "Hodge theory"), one must work over a base which is complete with respect to some sort of "arithmetic prime". The three main examples of this sort of base are the following:

(i) <u>the complex number field $\mathbf{C}$</u> (this also covers the case of $\mathbf{R}$ by working with objects over $\mathbf{C}$ equipped with an action of complex conjugation, i.e., of $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$)

(ii) <u>a p-adic field K</u> (i.e., a finite extension of $\mathbf{Q}_p$)

(iii) <u>power series over $\mathbf{Z}$</u> – typically arising as the completion of some sort of moduli space at a $\mathbf{Z}$-valued point corresponding to a *degenerate object* parametrized by the moduli space.

Indeed, all completions of number fields fall under cases (i) and (ii). Thus, if one is ultimately interested in rational points of number fields, IHT's over bases as in (i) and (ii) are of natural interest. Also, since the coefficients of the powers series appearing in (iii) are integers, case (iii) is also of substantial arithmetic interest.

In the following discussion, the following principle will serve as an important guide:

Guiding Principle: For every type of arithmetic prime (i.e., cases (i), (ii), and (iii) above), one expects that there should be a *canonical uniformization theory* at that prime.

In general, however, one does not expect that the canonical uniformization theories at different primes should be compatible with one another. We will return to this point in more detail later.

In terms of types of varieties, the main cases in which one has well-developed physical and modular IHT's are the following:

(1) abelian varieties.

(2) hyperbolic curves.

The physical and modular IHT's in these cases may be roughly summarized in the following charts:

(1) Abelian Varieties

C	p-adic	Degenerate Object
exponential map of abelian varieties/ Siegel upper half-plane uniformization	Tate's theorem/ Serre-Tate theory	Schottky uniformizations of Tate/Mumford/ Faltings/Chai

(2) Hyperbolic Curves

C	p-adic	Degenerate Object
Fuchsian uniformization/ uniformizations of Teichmüller and Bers	§2 (anabelian conjecture)/ §3 (p-adic Teichmüller theory)	formal algebraic Schottky uniformization of Mumford

Of these two examples, undoubtedly the example of abelian varieties is the more well-known. *Over* **C**, the exponential map of an abelian variety gives a uniformization of the abelian variety by a complex linear space. This generalizes Example (3) of §1, (A). Moreover, by using the periods that one obtains from this uniformization, one obtains a uniformization of the moduli space of abelian varieties by the Siegel upper half-plane. Thus, we see both the *physical* and *modular* aspects of the IHT of abelian varieties in evidence.

In the *p-adic case*, the IHT of an abelian variety essentially amounts to the p-adic Hodge theory of H^1 of the abelian variety. Although the p-adic Hodge theory of H^1 of an abelian variety has many different aspects, most of these may be traced to the fundamental paper of Tate

([Tate]) in the late 1960's. In this paper, the main theorem ("Tate's theorem" in the chart) states that homomorphisms between formal groups (e.g., the formal groups arising from abelian varieties) over p-adic fields are essentially equivalent to homomorphisms between the corresponding Tate modules. In some sense, this result is the analogue for abelian varieties of the main theorem (Theorem 1) of §2 below, and may be regarded as a sort of physical IHT for abelian varieties. On the other hand, the modular aspect of the IHT of abelian varieties may be seen most easily in Serre-Tate theory, which gives rise to p-adic parameters on the moduli space of abelian varieties that are very much analogous to the Siegel upper half-plane uniformization in the complex case.

Finally, in a neighborhood of a point in the moduli space corresponding to a *degenerating abelian variety*, one has the theory of Tate curves, generalized by Mumford and Faltings/Chai ([FC]). Moreover, it turns out that in the case of abelian varieties, the complex, p-adic, and degenerating object theories are all compatible with one another. For instance, if one specializes the uniformizing parameters that one obtains on the moduli space in a neighborhood of a point corresponding to a degenerating abelian variety to a p-adic (respectively, complex) base, one obtains parameters compatible with the Serre-Tate parameters (respectively, the Siegel upper half-plane uniformization).

Next, we consider the case of *hyperbolic curves. Over* $\mathbf{C}$, the physical aspect of the IHT of hyperbolic curves (cf. Example (4) of §1, (A)) essentially amounts to the Fuchsian uniformization. Then just as the exponential map uniformization of an abelian variety "induces" the Siegel upper half-plane uniformization of the moduli space of abelian varieties, the Fuchsian uniformization of a hyperbolic curve "induces" the Bers uniformization of the moduli space of hyperbolic curves (cf. §3, (A) below, as well as the Introduction of [Mzk4]).

In a neighborhood of a point in the moduli space corresponding to a totally degenerate (proper) hyperbolic curve, one has the theory of [Mumf]. Note, however, that this theory is *not* compatible with the theory of the Fuchsian uniformization in the following sense: If one specializes the ($\mathbf{Z}$-coefficient) power series in the base ring to some $\mathbf{C}$-valued point in a neighborhood of a point in the moduli space corresponding to a totally degenerate curve, the resulting uniformization over $\mathbf{C}$ that one obtains is the so-called *Schottky uniformization* of the curve. This uniformization is completely different from the Fuchsian uniformization.

Finally, we come to the *p-adic case.* It seems that the IHT of p-adic hyperbolic curves has not been studied extensively until relatively recently ([Mzk1-5]). Thus, *It is the goal of this manuscript to report on recent developments concerning the intrinsic Hodge theory* (IHT) *of p-adic hyperbolic curves.*

The physical aspect, which concerns a theorem (Theorem 1) that gives a strong solution to *Grothendieck's anabelian conjecture*, will be the topic of §2. The modular aspect, which concerns a theory – which we call *p-adic Teichmüller theory* – which may be regarded as either the hyperbolic curve analogue of Serre-Tate theory or the p-adic analogue of the theory of the Fuchsian and Bers uniformizations, will be discussed in §3. We remark here that this p-adic Teichmüller theory is *not* compatible with the specialization of the theory of [Mumf] to the p-adic case. This may disappoint some readers, but is, in fact, natural in view of the fact that even over $\mathbf{C}$, the specialization of the theory of [Mumf] to the complex numbers is not compatible (as remarked above) with the theory of the Fuchsian uniformization. Moreover, it is in line with the general Guiding Principle discussed above that to each sort of arithmetic prime there should correspond a natural uniformization theory of hyperbolic curves. Thus, it seems to the author that it is meaningless to argue as to whether it is the specialization of the [Mumf] to the p-adic case or the theory of [Mzk1], [Mzk2], [Mzk3], [Mzk4] which is the "true" p-adic analogue of the Fuchsian uniformization. That is to say, it seems more natural to the author to regard the theory of [Mumf] as the "true" analogue of the Fuchsian uniformization at the "degenerating object prime", and the theory of [Mzk1], [Mzk2], [Mzk3], [Mzk4] as the "true" analogue of the Fuchsian uniformization at "the prime p".

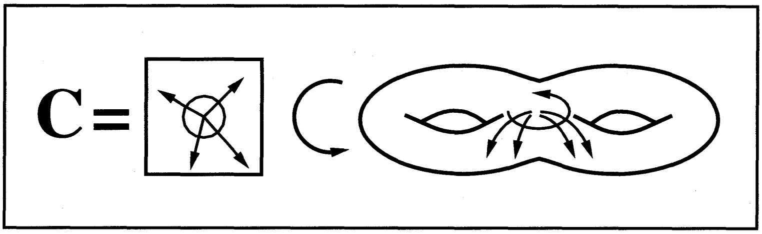
§2. The Physical Aspect: Embedding by Automorphic Forms

(A) The Complex Case

We begin our discussion by considering the complex case. The complex case is important to understand not only for reasons of philosophical analogy, but also because it provides the motivation for the proof of the main result in the p-adic case.

In the complex case, the physical IHT aspect of the Fuchsian uniformization may be summarized schematically as follows:

$$\begin{aligned}
&\underline{\text{algebraic}}\ \text{hyperbolic curve}\ X\\
&\quad\Longleftrightarrow SO(2)\backslash PSL_2(\mathbf{R})/\Gamma \quad \text{(physical/analytic obj.)}\\
&\quad\Longleftrightarrow \pi_1(\mathcal{X}) + \text{action of}\ \pi_1(\mathcal{X})\ \text{on}\ \mathfrak{H}\\
&\quad\Longleftrightarrow \underline{\text{topology}} + \underline{\text{arithmetic structure}}\ \text{(geometry)}
\end{aligned}$$



Here, the illustration is of an *algebraic* hyperbolic curve thought of as a topological surface equipped with an additional *arithmetic structure*, namely the geometry arising from the Poincaré metric on the upper half-plane $\mathfrak{H}$. This geometry is depicted as an "action of $\mathbf{C}$" on the topogical surface, given by the flows along geodesics defined by the geometry.

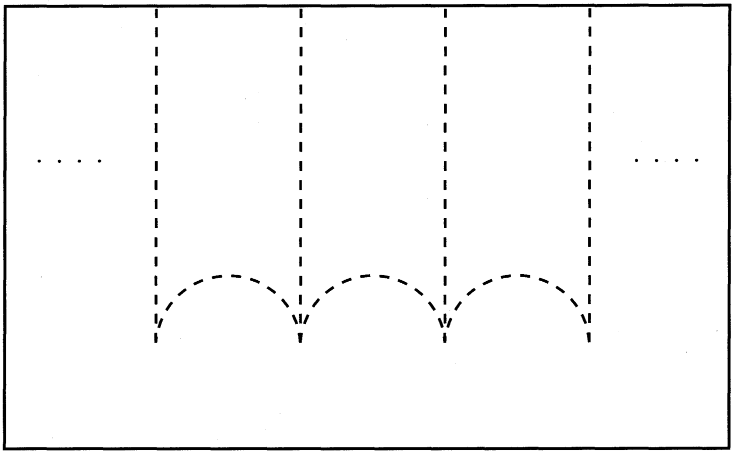
In order to generalize this picture to the p-adic case, it is necessary to recall the conceptual machinery that gives rise to the first "$\Longleftrightarrow$" in the chain of equivalences appearing above. If one starts from the right-hand side, i.e., the upper half-plane equipped with some action of $\pi_1(\mathcal{X})$, then the *algebraic* structure (i.e., the left-hand side) may be recovered by considering *automorphic forms* on $\mathfrak{H}$ which are invariant with the respect to the action of $\pi_1(\mathcal{X})$. These automorphic forms define a morphism from $\mathfrak{H}$ to projective space whose image is necessarily (by Chow's Theorem!) *algebraic* and, in fact, equal to the original algebraic curve X. That is to say, one has a commutative diagram:

$$\begin{array}{ccc} \text{Upper half-plane } \mathfrak{H} & \longrightarrow & \text{Projective Space} \\ \downarrow & & \| \\ \text{Algebraic Curve } X & \hookrightarrow & \text{Projective Space} \end{array}$$

Put another way, the main point is the following: Although ultimately $\pi_1(\mathcal{X})$-invariant differential forms on $\mathfrak{H}$ define algebraic forms on X, such forms may be defined *directly* from the data of the action $\{\pi_1(\mathcal{X}) \curvearrowright \mathfrak{H}\}$ and, moreover, by the above diagram, allow one to recover the *algebraic structure* of X from the analytic data $\{\pi_1(\mathcal{X}) \curvearrowright \mathfrak{H}\}$. That is, to put the matter more succinctly, the key idea is the following:

Key Idea: *Consider analytic reprepresentations of algebraic differential forms.*

It turns out that the proof of the main p-adic result (Theorem 1 below) of this section consists precisely of implementing this key idea in the p-adic case by using the technical machinery of p-adic Hodge theory.



The Case of $SL_2(\mathbf{Z})$

(B) The Arithmetic Fundamental Group

In this section, we prepare for the statement of Theorem 1 in the following section by introducing various notations and terminology.

Let K be a field of characteristic zero. Let us denote by $\overline{K}$ an algebraic closure of K. Let $\Gamma_K \stackrel{\text{def}}{=} \text{Gal}(\overline{K}/K)$. Suppose that X_K is a *variety over* K. Then we will denote by

$$\pi_1(X_K)$$

the *algebraic fundamental group of* X_K (cf. [SGA1]). This group is a compact, profinite topological group, well-defined up to inner automorphism (since we did not specify a "base-point"), and which has the following property: The category of finite sets with a continuous $\pi_1(X_K)$-action is naturally equivalent to the category of finite étale coverings of X_K. Moreover, if K is, for instance, an algebraically closed subfield of $\mathbf{C}$, then $\pi_1(X_K)$ may be identified with the *profinite completion* (= inverse limit of all finite quotients) *of the usual topological fundamental group* $\pi_1^{\text{top}}(X_{\mathbf{C}})$ (where $X_{\mathbf{C}} \stackrel{\text{def}}{=} X_K \otimes_K \mathbf{C}$).

Now let X_K be a *hyperbolic curve over* K; write $X_{\overline{K}} \stackrel{\text{def}}{=} X \times_K \overline{K}$. Then one has an exact sequence

$$1 \longrightarrow \pi_1(X_{\overline{K}}) \longrightarrow \pi_1(X_K) \longrightarrow \Gamma_K \longrightarrow 1$$

of *algebraic fundamental groups.* We shall refer to $\pi_1(X_{\overline{K}})$ as the *geometric fundamental group* of X_K. Note that, by the above discussion of the case where $\overline{K} \subseteq \mathbf{C}$, it follows that the structure of $\pi_1(X_{\overline{K}})$ is determined entirely by (g, r) (i.e., the "type" of the hyperbolic curve X_K). In particular, $\pi_1(X_{\overline{K}})$ *does not depend on the moduli of* X_K. Of course, this results from the fact that K is of *characteristic zero.* In positive characteristic, on the other hand, preliminary evidence ([Tama2]) suggests that the fundamental group of a hyperbolic curve over an algebraically closed field (far from being independent of the moduli of the curve!) may in fact *completely determine* the moduli of the curve.

We shall refer to $\pi_1(X_K)$ (equipped with its augmentation to Γ_K) as the *arithmetic fundamental group* of X_K. Although it is made up of two "parts" – i.e., $\pi_1(X_{\overline{K}})$ and Γ_K – which do not depend on the moduli of X_K, it is not unreasonable to expect that the extension class defined by the above exact sequence, i.e., the structure of $\pi_1(X_K)$ as a group equipped with augmentation to Γ_K, may in fact depend quite strongly on the moduli of X_K. Indeed, according to the *anabelian philosophy* of Grothendieck (cf. [LS]), for "sufficiently arithmetic" K, one expects that *the structure of the arithmetic fundamental group* $\pi_1(X_K)$ *should be enough to determine the moduli of* X_K. Although many important versions of Grothendieck's anabelian conjectures remain unsolved (most notably the so-called *Section Conjecture* (cf., e.g., [LS], p. 289, 2)), in the remainder of this section, we shall discuss various versions that have been resolved in the affirmative. For instance, such a version of these conjectures which will be discussed in (B) below (Theorem 1) states roughly that (*nonconstant*) *morphisms from a smooth* K*-variety to* X_K *should be in bijective correspondence with* (*open*) *homomorphisms* (*over* Γ_K) *between the corresponding arithmetic fundamental groups.* Thus, there is an obvious analogy between this (form of Grothendieck's) conjecture and the Tate conjecture on abelian varieties, which states roughly that morphisms between abelian varieties are equivalent to morphisms between their arithmetic fundamental groups.

Note that this anabelian philosophy is a *special case of the notion of "intrinsic Hodge theory"* discussed above: indeed, on the algebraic geometry side, one has "the curve itself", whereas on the topology plus arithmetic side, one has the arithmetic fundamental group, i.e., the purely (étale) topological $\pi_1(X_{\overline{K}})$, equipped with the structure of extension given by the above exact sequence.

In fact, it is interesting to note – especially relative to the discussion at the beginning of §1, (C) – that Grothendieck's anabelian philosophy arose as an *approach to diophantine geometry.* It is primarily for this reason that it was originally thought that the most natural sort of "arithmetic" base field K over which one should expect Grothendieck's anabelian conjectures to hold was a *number field.* Another reason for the idea that the base field in these conjectures should be a number field was the analogy with Tate's conjecture on homomorphisms between abelian varieties (cf., e.g., [Falt1]). Indeed, in discussions of Grothendieck's anabelian philosophy, it was common to refer to statements such as that of Theorem 1 below as the "anabelian Tate conjecture", or the "Tate conjecture for hyperbolic curves". In fact, however, there is an important difference between Theorem 1 and the "Tate conjecture" of, say, [Falt1]: Namely, whereas Theorem 1 below holds over *local fields* (i.e., finite extensions of $\mathbf{Q}_p$), *the Tate conjecture for abelian varieties is false over local fields.* Moreover, until the proof of Theorem 1, it was generally thought that, just like its abelian cousin, the "anabelian Tate conjecture" was essentially global in nature. That is to say, it appears that the point of view of the author, i.e., that Theorem 1 should be regarded as a p-adic version of the "physical aspect" of the Fuchsian uniformization of a hyperbolic curve, does not exist in the literature (prior to the work of the author).

Finally, we remark, relative to the issue of locating an analogue of Theorem 1 in the theory of abelian varieties, that it seems that it is more natural to think of "Tate's theorem" (cf. the discussion in §1, (C)) as the proper analogue of Theorem 1 for abelian varieties. Indeed, not only does Tate's theorem hold over local fields, but it plays an important technical role in the proof of Theorem 1 below (cf. [Mzk5]).

(C) The Main Theorem

Building on earlier work of H. Nakamura and A. Tamagawa (see, especially, [Tama1]), the author applied the p-adic Hodge theory of [Falt2] and [BK] to prove the following result (cf. Theorem A of [Mzk5]):

Theorem 1. *Let p be a prime number. Let K be a subfield of a finitely generated field extension of $\mathbf{Q}_p$. Let X_K be a hyperbolic curve over K. Then for any smooth variety S_K over K, the natural map*

$$X_K(S_K)^{\mathrm{dom}} \longrightarrow \mathrm{Hom}^{\mathrm{open}}_{\Gamma_K}(\pi_1(S_K), \pi_1(X_K))$$

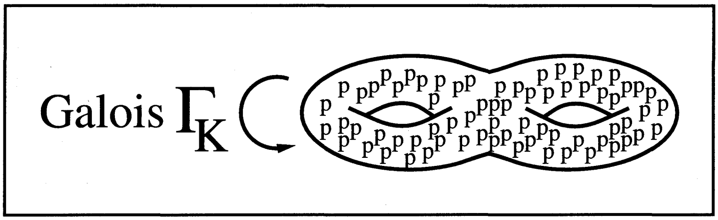
is bijective. Here, the superscripted "dom" denotes dominant ($\Leftrightarrow$ nonconstant) K-morphisms, while $\mathrm{Hom}^{\mathrm{open}}_{\Gamma_K}$ denotes open, continuous homomorphisms compatible with the augmentations to Γ_K, and considered up to composition with an inner automorphism arising from $\pi_1(X_{\overline{K}})$.

Note that this result constitutes an analogue of the "physical aspect" of the Fuchsian uniformization, i.e., it exhibits the *scheme* X_K (in the sense of the functor defined by considering (nonconstant) K-

morphisms from arbitrary smooth S_K to X_K) as equivalent to the "physical/analytic object"

$$\mathrm{Hom}^{\mathrm{open}}_{\Gamma_K}(-, \pi_1(X_K))$$

defined by the topological $\pi_1(X_{\overline{K}})$ together with some additional canonical arithmetic structure (i.e., $\pi_1(X_K)$). This sort of equivalence is depicted in the following illustration, which is meant to remind the reader of the first illustration in §2, (A):



In fact, various slightly stronger versions of Theorem 1 hold. For instance, instead of the whole geometric fundamental group $\pi_1(X_{\overline{K}})$, it suffices to consider its maximal pro-p quotient $\pi_1(X_{\overline{K}})^{(p)}$. Indeed, this is natural relative to the general philosophy discussed in §1, (B) – i.e., one typically expects that the right-hand side of the equivalence of a p-adic Hodge theory should only involve the *pro-p* étale topology. Another strengthening allows one to prove the following result (cf. Theorem B of [Mzk5]), which generalizes a result of Pop ([Pop]):

Theorem 2. *Let p be a prime number. Let K be a subfield of a finitely generated field extension of $\mathbf{Q}_p$. Let L and M be function fields of arbitrary dimension over K. Then the natural map*

$$\mathrm{Hom}_K(\mathrm{Spec}(L), \mathrm{Spec}(M)) \longrightarrow \mathrm{Hom}^{\mathrm{open}}_{\Gamma_K}(\Gamma_L, \Gamma_M)$$

is bijective. Here, $\mathrm{Hom}^{\mathrm{open}}_{\Gamma_K}(\Gamma_L, \Gamma_M)$ *is the set of open, continuous group homomorphisms $\Gamma_L \to \Gamma_M$ over Γ_K, considered up to composition with an inner homomorphism arising from* $\mathrm{Ker}(\Gamma_M \to \Gamma_K)$.

As discussed in §2, (A), the proof of Theorem 1 consists of implementing the ideas discussed in §2, (A), in the p-adic case by using p-adic Hodge theory.

More precisely, suppose that one is given two *hyperbolic curves* X_K, Y_K over K. For simplicity, let us assume that both X_K and Y_K are both *proper* and *non-hyperelliptic*, and that K is a *finite* extension of $\mathbf{Q}_p$. Suppose, moreover, that we are given an isomorphism

$$\alpha : \pi_1(X_K) \cong \pi_1(Y_K)$$

of the respective arithmetic fundamental groups which is compatible with the projections to Γ_K. Then Theorem 1 states that α necessarily *arises geometrically*, i.e., from a K-isomorphism $X_K \cong Y_K$. In the following, we would like to give a rough sketch of the ideas used to prove this result.

First, observe that α induces an isomorphism

$$\pi_1^{(p)}(X_{\overline{K}})^{\mathrm{ab}} \cong \pi_1^{(p)}(Y_{\overline{K}})^{\mathrm{ab}}$$

between the abelianizations of the maximal pro-p quotients of the respective geometric fundamental groups. Then it follows from p-adic Hodge theory that if one tensors this isomorphism with $\mathbf{C}_p$ (i.e, the p-adic completion of $\overline{K}$), and then takes Γ_K-invariants, one obtains (naturally) on both sides the respective spaces of global differentials, $D_X \stackrel{\mathrm{def}}{=} H^0(X_K, \omega_{X_K})$ and $D_Y \stackrel{\mathrm{def}}{=} H^0(Y_K, \omega_{Y_K})$. Thus, one obtains an isomorphism

$$D_X \cong D_Y$$

induced by α. Let $P_X \stackrel{\mathrm{def}}{=} \mathbf{P}(D_X)$, $P_Y \stackrel{\mathrm{def}}{=} \mathbf{P}(D_Y)$ be the corresponding projective spaces. Thus, one obtains an isomorphism $P_X \cong P_Y$. On the other hand, since we assumed that X_K and Y_K are *non-hyperelliptic*, it follows from elementary algebraic geometry that we have canonical embeddings $X_K \subseteq P_X$, $Y_K \subseteq P_Y$. In other words, we have a diagram:

$$\begin{array}{ccc} P_X & \cong & P_Y \\ \cup & & \cup \\ X_K & \xrightarrow{?} & Y_K \end{array}$$

Thus, the problem of constructing an isomorphism $X_K \cong Y_K$ as desired is reduced to showing that the isomorphism $P_X \cong P_Y$ that we have already constructed maps X_K into Y_K. This is proven precisely by considering certain *p-adic analytic representations* of the differentials of D_X and D_Y as differentials on a certain p-adic space (= the spectrum of a certain large p-adic field) in a fashion reminiscent of the way in which analytic representations (i.e., automorphic forms) of differential forms appeared in the above discussion of the complex case. We refer to [Mzk5], [NTM], for more details.

§3. The Modular Aspect: Canonical Frobenius Actions

(A) The Complex Case

In this section, we discuss the ideas in the complex case that form the philosophical background underlying the theory of [Mzk1], [Mzk2],

[Mzk3], [Mzk4]. Just as in §2, the key phrase was "embedding by automorphic forms", in the present discussion, the key phrase is *"canonical Frobenius actions"*.

First, let us observe that the Fuchsian uniformization of the Riemann surface $\mathcal{X}$ associated to a hyperbolic algebraic curve X gives rise to an action of $\pi_1(\mathcal{X})$ on $\mathfrak{H}$, hence defines a *canonical representation*

$$\rho_{\mathcal{X}} : \pi_1(\mathcal{X}) \longrightarrow PSL_2(\mathbf{R}) \stackrel{\text{def}}{=} SL_2(\mathbf{R})/\{\pm 1\} = \mathrm{Aut}_{\mathrm{Holomorphic}}(\mathfrak{H}).$$

Note that $\rho_{\mathcal{X}}$ may also be regarded as a representation into $PGL_2(\mathbf{C}) = GL_2(\mathbf{C})/\mathbf{C}^{\times}$, hence as defining an action of $\pi_1(\mathcal{X})$ on $\mathbf{P}^1_{\mathbf{C}}$. Taking the quotient of $\mathfrak{H} \times \mathbf{P}^1_{\mathbf{C}}$ by the action of $\pi_1(\mathcal{X})$ on both factors then gives rise to a projective bundle with connection on $\mathcal{X}$. It is immediate that this projective bundle and connection may be algebraized, so we thus obtain a projective bundle and connection $(P \to X, \nabla_P)$ on X. This pair (P, ∇_P) has certain properties which make it an *indigenous bundle* (terminology due to Gunning).

In general, the notion of an "indigenous bundle on $\mathcal{X}$" may be thought of as the datum of a *projective structure* on $\mathcal{X}$, i.e., a subsheaf of the sheaf of holomorphic functions on $\mathcal{X}$ such that locally any two sections of this subsheaf are related by a linear fractional transformation (with constant coefficients). Thus, the Fuchsian uniformization defines a special *canonical indigenous bundle*, or *canonical projective structure*, on X.

In fact, it is not difficult to see that the notion of an indigenous bundle is entirely *algebraic*. Thus, one has a natural moduli stack

$$\mathcal{S}_{g,r} \longrightarrow \mathcal{M}_{g,r}$$

of hyperbolic curves of type (g, r) equipped with an indigenous bundle, which forms a torsor (under the affine group given by the sheaf of differentials $\Omega_{\mathcal{M}_{g,r}}$ on $\mathcal{M}_{g,r}$) – called the *Schwarz torsor* – over the *moduli stack* $\mathcal{M}_{g,r}$ of hyperbolic curves of type (g, r). Moreover, $\mathcal{S}_{g,r}$ is not only algebraic, it is defined over $\mathbf{Z}[\frac{1}{2}]$. Thus, if, for instance, X is a hyperbolic curve over $\mathbf{C}$, the space of indigenous bundles (or equivalently, of projective structures) on X is a complex affine linear space of dimension $3g - 3 + r$. In particular, (in general) X admits many more indigenous bundles than the canonical one arising from the Fuchsian uniformization.

The canonical indigenous bundle defines a *canonical real analytic section*

$$s : \mathcal{M}_{g,r}(\mathbf{C}) \longrightarrow \mathcal{S}_{g,r}(\mathbf{C})$$

of the Schwarz torsor at the infinite prime. Moreover, not only does s "contain" all the information that one needs to define the Fuchsian uniformization of an individual hyperbolic curve (indeed, this much is obvious from the definition of s!), it also essentially "is" (interpreted properly) the *Bers uniformization* of the universal covering space (i.e., "Teichmüller space") of $\mathcal{M}_{g,r}(\mathbf{C})$. More precisely,

(1) $\overline{\partial}\, s$ is equal to the *Weil-Petersson metric* (a natural real analytic Kähler metric) on $\mathcal{M}_{g,r}(\mathbf{C})$.

(2) In general, real analytic Kähler metrics may be integrated locally to form *canonical* (*holomorphic*) *coordinates* on the given complex manifold. If one applies this general theory to the Weil-Petersson metric, one obtains the *Bers coordinates* (i.e., the coordinates arising from the Bers uniformization). That is to say, (cf. (1) above), the Bers uniformization may thought of as being precisely the "$\overline{z}$-part" or "*anti-holomorphic part*" of the canonical real analytic section s.

(cf. the discussions in the Introductions of [Mzk1], [Mzk4] for more details). In short, the study of this canonical section s may be regarded as the realization of the Fuchsian uniformization as a *modular IHT*.

Alternatively, from the point of view of classical Teichmüller theory, one may regard the uniformization theory of the moduli of hyperbolic curves as the theory of (so-called) *quasi-fuchsian deformations of the representation* $\rho_{\mathcal{X}}$. Briefly summarized, this point of view runs as follows: Inside $\mathcal{S}_{g,r}(\mathbf{C})$, then is an *open subset* consisting of projective structures defined by certain *quasi-fuchsian groups*. We denote this open subset by

$$\mathrm{Rep}^{\mathrm{QF}}(\pi_1(\mathcal{X}), PGL_2(\mathbf{C})) \subseteq \mathcal{S}_{g,r}(\mathbf{C}).$$

That is to say, this subset parametrizes representations $\pi_1(\mathcal{X}) \rightarrow PGL_2(\mathbf{C})$ that arise from Bers' *simultaneous uniformizations* of pairs of hyperbolic Riemann surfaces of type (g, r).

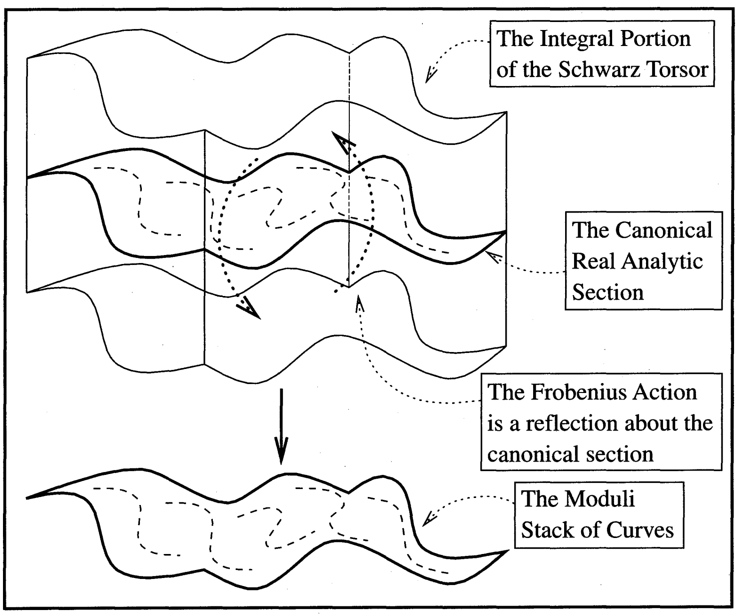
On each of the fibers of $\mathcal{S}_{g,r}(\mathbf{C}) \rightarrow \mathcal{M}_{g,r}(\mathbf{C})$ (which are complex affine spaces of dimension $3g - 3 + r$), this open subset is a *bounded contractible subset* of the complex affine space which forms the fiber. From the point of view of Arakelov theory, it is natural to regard such a bounded contractible subset as an *integral stucture* (at the infinite prime) on the complex affine space. Thus, we shall also write

$$\mathcal{S}_{g,r}^{\mathrm{int}_\infty} \stackrel{\mathrm{def}}{=} \mathrm{Rep}^{\mathrm{QF}}(\pi_1(\mathcal{X}), PGL_2(\mathbf{C}))$$

for $\mathrm{Rep}^{\mathrm{QF}}(\pi_1(\mathcal{X}), PGL_2(\mathbf{C}))$.

Next, let us observe that there is a natural action of *complex conjugation* – which we would like to think of as an action of *the Frobenius* Fr_∞ *at the infinite prime* – on $\mathrm{Rep}^{\mathrm{QF}}(\pi_1(\mathcal{X}), PGL_2(\mathbf{C}))$ induced by the action of complex conjugation on the various components of the (equivalence classes of) matrices which form $PGL_2(\mathbf{C})$. Relative to this action of Fr_∞, it is not difficult to see that the image of the canonical real analytic section $s : \mathcal{M}_{g,r}(\mathbf{C}) \to \mathcal{S}_{g,r}(\mathbf{C})$ considered above, i.e., the subset of the set of quasi-fuchsian groups consisting of the *Fuchsian groups*, is precisely equal to the set of Fr_∞-invariants of $\mathcal{S}_{g,r}^{\mathrm{int}_\infty} = \mathrm{Rep}^{\mathrm{QF}}(\pi_1(\mathcal{X}), PGL_2(\mathbf{C}))$. That is to say, we have a natural commutative diagram:

$$\begin{array}{ccc}
`\mathcal{S}_{g,r}^{\mathrm{int}_\infty}\text{'} \curvearrowleft \mathrm{Fr}_\infty & & \\
\| & & \\
\mathrm{Rep}^{\mathrm{QF}}(\pi_1(\mathcal{X}), PGL_2(\mathbf{C})) & \overset{\text{open}}{\hookrightarrow} & \mathcal{S}_{g,r}(\mathbf{C}) \\
\cup & & \downarrow \\
\mathrm{Fr}_\infty\text{-invariants} = \mathrm{Image}(s) & \xrightarrow{\sim} & \mathcal{M}_{g,r}(\mathbf{C})
\end{array}$$



Moreover, relative to this point of view, the Bers uniformization is the uniformization of $\mathcal{M}_{g,r}(\mathbf{C})$ by a fiber of $\mathcal{S}_{g,r}^{\mathrm{int}_\infty}$ given by the formula:

$$\text{Bers uniformization} = \mathrm{pr}_{\mathcal{M}_{g,r}} \circ (\mathrm{Fr}_\infty |_{\mathrm{Fiber}})$$

i.e., the composite morphism

$$\text{Fiber (of } \mathcal{S}_{g,r}^{\text{int}_\infty} \to \mathcal{M}_{g,r}(\mathbf{C})) \hookrightarrow \mathcal{S}_{g,r}^{\text{int}_\infty} \xrightarrow{\text{Fr}_\infty} \mathcal{S}_{g,r}^{\text{int}_\infty} \longrightarrow (\mathcal{M}_{g,r})_{\mathbf{C}}$$

In other words, from this point of view:

Key Point: *"The Bers uniformization is a Frobenius action!"*

Formulated from this point of view, the ideas of classical Teichmüller theory carry over fairly transparently to the p-adic case. This will be the theme of our discussion of *p-adic Teichmüller theory* in the section's to follow.

(B) Teichmüller Theory in Characteristic p

Let p be an *odd* prime. Then the p-adic theory of Shimura curves (cf., e.g., [Ihara]) suggests that a natural condition to expect of canonical indigenous bundles in characteristic p is that they should have *square nilpotent p-curvature.* The "p-curvature" is a natural invariant of bundles with connection in characteristic p, which, philosophically, may be thought of as a measure of the extent to which the connection ∇ is compatible with (i.e., "commutes with") Frobenius, i.e.:

$$\underline{p\text{-curvature}} = \text{"[Frobenius, } \nabla]\text{"}.$$

In the case, of $\mathbf{P}^1$-bundles, the p-curvature may be thought of as a 2×2-matrix of differentials (conjugated by Frobenius) whose trace is zero. Thus, to say that the p-curvature is "square nilpotent" means simply that the square (in the sense of ordinary matrix multiplication) of this 2×2-matrix is zero.

Let $\mathcal{N}_{g,r} \subseteq (\mathcal{S}_{g,r})_{\mathbf{F}_p}$ denote the closed algebraic substack of indigenous bundles with square nilpotent p-curvature. Then one has the following key result ([Mzk1, Chapter II, Theorem 2.3]):

Theorem 3. *The natural map $\mathcal{N}_{g,r} \to (\mathcal{M}_{g,r})_{\mathbf{F}_p}$ is a finite, flat, local complete intersection morphism of degree p^{3g-3+r}. Thus, up to "isogeny" (i.e., up to the fact that this degree is not equal to one), $\mathcal{N}_{g,r}$ defines a canonical section of the Schwarz torsor $(\mathcal{S}_{g,r})_{\mathbf{F}_p} \to (\mathcal{M}_{g,r})_{\mathbf{F}_p}$ in characteristic p, i.e., it gives rise to a diagram*

$$\begin{array}{ccc} (\mathcal{S}_{g,r})_{\mathbf{F}_p} & = & (\mathcal{S}_{g,r})_{\mathbf{F}_p} \\ \cup & & \downarrow \\ \mathcal{N}_{g,r} & \longrightarrow & (\mathcal{M}_{g,r})_{\mathbf{F}_p} \end{array}$$

reminiscent of the diagram appearing in §3, (A).

It is this stack $\mathcal{N}_{g,r}$ of *nilcurves* – i.e., hyperbolic curves in characteristic p equipped with an indigenous bundle with square nilpotent p-curvature – which is the central object of study in the *p-adic Teichmüller theory* of [Mzk1], [Mzk2], [Mzk3], [Mzk4].

Many facts are now known about the *finer structure of* $\mathcal{N}_{g,r}$. One interesting consequence of this structure theory of $\mathcal{N}_{g,r}$ is that *it gives rise to a new proof of the connectedness of* $(\mathcal{M}_{g,r})_{\mathbf{F}_p}$ (for p large relative to g). This fact is interesting – relative to the claim that this theory is a p-adic version of Teichmüller theory – in that one of the first applications of classical complex Teichmüller theory is to prove the connectedness of $\mathcal{M}_{g,r}$. Also, it is interesting to note that F. Oort has succeeded in giving a proof of the connnectedness of the moduli stack of principally polarized abelian varieties by applying the structure theory of certain natural substacks of this moduli stack in characteristic p.

(C) p-adic Teichmüller Theory

So far, we have been discussing the characteristic p theory. Ultimately, however, we would like to know if the various characteristic p objects discussed in §3, (B), *lift canonically* to objects which are flat over $\mathbf{Z}_p$. Unfortunately, it seems that it is unlikely that $\mathcal{N}_{g,r}$ itself lifts canonically to some sort of natural $\mathbf{Z}_p$-flat object. If, however, we consider the open substack – called the *ordinary locus* – $(\mathcal{N}^{\mathrm{ord}}_{g,r})_{\mathbf{F}_p} \subseteq \mathcal{N}_{g,r}$ which is the étale locus of the morphism $\mathcal{N}_{g,r} \to (\mathcal{M}_{g,r})_{\mathbf{F}_p}$, then (since the étale site is invariant under nilpotent thickenings) we get a canonical lifting, i.e., an étale morphism

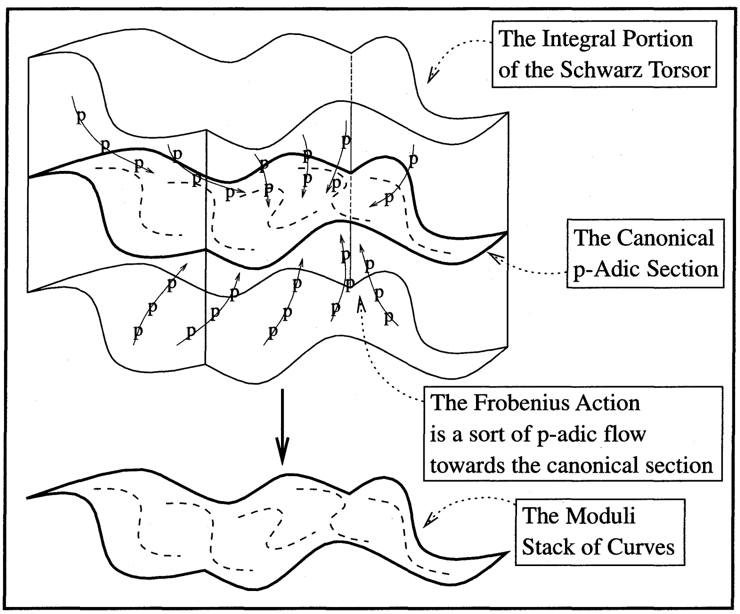
$$\mathcal{N}^{\mathrm{ord}}_{g,r} \longrightarrow (\mathcal{M}_{g,r})_{\mathbf{Z}_p}$$

of p-adic formal stacks. Over $\mathcal{N}^{\mathrm{ord}}_{g,r}$, one has the sought-after *canonical p-adic splitting of the Schwarz torsor*, i.e., the p-adic analogue of the canonical real analytic splitting $s : \mathcal{M}_{g,r}(\mathbf{C}) \to \mathcal{S}_{g,r}(\mathbf{C})$ discussed in §3, (A) (cf. Theorem 0.1 of the Introduction of [Mzk1]; [Mzk4, Chapter X, §3]):

Theorem 4. *There is a canonical section $\mathcal{N}^{\mathrm{ord}}_{g,r} \to \mathcal{S}_{g,r}$ of the Schwarz torsor over $\mathcal{N}^{\mathrm{ord}}_{g,r}$ which is the unique section having the following property: $\exists$ a lifting of Frobenius $\Phi_{\mathcal{N}} : \mathcal{N}^{\mathrm{ord}}_{g,r} \to \mathcal{N}^{\mathrm{ord}}_{g,r}$ such that the indigenous bundle on the tautological hyperbolic curve over $\mathcal{N}^{\mathrm{ord}}_{g,r}$ defined by the section $\mathcal{N}^{\mathrm{ord}}_{g,r} \to \mathcal{S}_{g,r}$ is* invariant *with respect to the Frobenius* *action defined by $\Phi_{\mathcal{N}}$. Moreover, this canonical section and canonical Frobenius lifting give rise to a diagram*

$$\begin{array}{ccc} (\mathcal{S}_{g,r})_{\mathbf{Z}_p} & = & (\mathcal{S}_{g,r})_{\mathbf{Z}_p} \\ \cup & & \downarrow \\ \Phi_{\mathcal{N}} \curvearrowright (\mathcal{N}^{\mathrm{ord}}_{g,r})_{\mathbf{Z}_p} & \stackrel{p\text{-adic et}}{\longrightarrow} & (\mathcal{M}_{g,r})_{\mathbf{Z}_p} \\ \cdot\|\cdot & & \\ \mathrm{Rep}^{\mathrm{Crys}}(\pi_1(X_{\mathbf{Q}_p}), PGL_2(\mathbf{Z}_p)) & & \end{array}$$

reminiscent of the diagram appearing in §3, (A). *Here, "$\cdot\|\cdot$" means "roughly may be identified with", and "*$\mathrm{Rep}^{\mathrm{Crys}}(\pi_1(X_{\mathbf{Q}_p}), PGL_2(\mathbf{Z}_p))$*" is a certain natural space of crystalline respresentations of the arithmetic fundamental group of the tautological hyperbolic curve into $PGL_2(\mathbf{Z}_p)$. Finally, the resulting action of $\Phi_{\mathcal{N}}$ on "*$\mathrm{Rep}^{\mathrm{Crys}}(\pi_1(X_{\mathbf{Q}_p}), PGL_2(\mathbf{Z}_p))$*" is the natural Frobenius action on this space of crystalline representations* (*cf.* [Mzk4, Chapter X, especially §3], *for more details*).



Next, we observe that the *Frobenius lifting* $\Phi_{\mathcal{N}} : \mathcal{N}^{\mathrm{ord}}_{g,r} \to \mathcal{N}^{\mathrm{ord}}_{g,r}$ (i.e., morphism whose reduction modulo p is the Frobenius morphism) has the special property that $\frac{1}{p} \cdot \mathrm{d}\Phi_{\mathcal{N}}$ induces an isomorphism

$$\Phi^*_{\mathcal{N}}\Omega_{\mathcal{N}^{\mathrm{ord}}_{g,r}} \cong \Omega_{\mathcal{N}^{\mathrm{ord}}_{g,r}}$$

Such a Frobenius lifting is called *ordinary*. It turns out that *any* ordinary Frobenius lifting (i.e., not just $\Phi_{\mathcal{N}}$) defines (by integration) a set of *canonical multiplicative coordinates* in a formal neighborhood of any point α valued in an algebraically closed field k of characteristic p, as well as a *canonical lifting* of α to a point valued in $W(k)$ (Witt vectors with coefficients in k).

Moreover, there is a certain analogy between this general theory of ordinary Frobenius liftings and the theory of *real analytic Kähler metrics* (which also define canonical coordinates by integration). Relative to this analogy, the canonical Frobenius lifting $\Phi_{\mathcal{N}}$ on $\mathcal{N}_{g,r}^{\mathrm{ord}}$ may be regarded as corresponding to the *Weil-Petersson metric* on complex Teichmüller space (a metric whose canonical coordinates are the coordinates arising from the Bers uniformization of Teichmüller space – cf. the discussion of §3, (A)), i.e.,

$$\Phi_{\mathcal{N}} \longleftrightarrow \text{Weil-Petersson metric,}$$
$$`\int \Phi_{\mathcal{N}}' \longleftrightarrow \text{Bers coordinates.}$$

Thus, $\Phi_{\mathcal{N}}$ is, in a very real sense, a p-adic analogue of the Bers uniformization in the complex case. Moreover, there is, in fact, a canonical ordinary Frobenius lifting on the "ordinary locus" of the tautological curve over $\mathcal{N}_{g,r}^{\mathrm{ord}}$ whose relative canonical coordinate is analogous to the canonical coordinates arising from the Köbe uniformization of a hyperbolic curve (i.e., from the canonical real analytic Kähler metric obtained by descending the Poincaré metric on $\mathfrak{H}$ via the Köbe uniformization $\mathfrak{H} \to \mathcal{X}$).

Next, we observe that Serre-Tate theory for ordinary (principally polarized) abelian varieties *may also be formulated as arising from a certain canonical ordinary Frobenius lifting*. Thus, the Serre-Tate parameters (respectively, Serre-Tate canonical lifting) may be identified with the canonical multiplicative parameters (respectively, canonical lifting to the Witt vectors) of this ordinary Frobenius lifting. That is to say, *in a very concrete and rigorous sense, Theorem 4 may be regarded as the analogue of Serre-Tate theory for hyperbolic curves.* Nevertheless, we remark that it is *not* the case that the condition that a nilcurve be ordinary (i.e., define a point of $(\mathcal{N}_{g,r}^{\mathrm{ord}})_{\mathbf{F}_p} \subseteq \mathcal{N}_{g,r}$) either implies or is implied by the condition that its Jacobian be ordinary.

Although this fact may disappoint some readers, it is in fact very natural when viewed relative to the general analogy between ordinary Frobenius liftings and real analytic Kähler metrics discussed above. Indeed, relative to this analogy, we see that it corresponds to the fact that, when one equips $\mathcal{M}_g$ with the Weil-Petersson metric and $\mathcal{A}_g$ (the moduli stack of principally polarized abelian varieties) with its natural metric arising from the Siegel upper half-plane uniformization, *the Torelli map* $\mathcal{M}_g \to \mathcal{A}_g$ *is not isometric.*

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Research Institute for Mathematical Sciences
Kyoto University
Sakyo-ku
Kyoto, 606-8502
Japan
`motizuki@kurims.kyoto-u.ac.jp`