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Probability and Geometry

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Summary

Fifty years back, Kiyosi Itô started the theory of *Stochastic Differential Equations* (SDE) in a fully geometric framework; in particular he constructed the heat process associated to an elliptic operator on a manifold by patching together Stochastic Differential Equations, the change

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of charts being mastered by Itô Calculus. We want to present here some aspects of the fastly growing exchange of concepts and methods existing betweeen Probability and Geometry in finite or infinite dimension. The topic is so wide that I am not able to fully cover it. I shall limitate myself to the subjects where I have get some personnal acquaintance in these last twenty five years. I gave to some parts of this paper an autobiographical flavour willing to emphasize here the decisive and unvaluable support that the japanese mathematical community bring to my program all around these years and singularly the Master for all of us: Professor Kiyosi Itô. To be called again to contribute to the Proceedings of a major Japanese Conference is for me a double privilege: it gives me the opportunity to recall the past and the possibility to act in the present.

What are the mathematical facts from which it could be possible to a priori advocate some relations between Probability and Geometry?

A major topic in Global Differential Geometry is the *passage from the infinitesimal to the global.* The elliptic operators constitute a basic tool for this passage: let us mention for instance the computation of the Atiyah-Singer Index by integrating a differential form built from curvature tensors or the cohomology vanishing theorems through positivity in the spirit of Bochner-Kodaira. The advantage of a probabilistic treatement of these elliptic operators is that the processes associated furnishes a *family of curves, geometrically highly significant.* In classical Differential Geometry some theorems are obtained by making *infinitesimal variations of the geodesic flow*: for instance a stricly positive lower bound of the sectional curvature leads to an a priori bound of the diameter of the manifold. It will be possible to proceed in the same way with the trajectories of the process by computing theirs infinitesimal variations by the Stochastic Calculus of Variations and these variations will be expressed in terms of curvatures.

In Probability the study of the joint distribution of a finite number of random variables involves *Geometric Integration* theory. Is it posssible to develop a similar programm directly on the probability space? If so we shall obtain conditional expectations defined for ALL VALUES of the conditioning, significant result from a purely probabilistic point of view.

Many others links between Probability and Geometry will appear below, in particular a full "Geometrization" of the concept of Stochastic Integral.

§1. Passage from the infinitesimal to the global through Com-

parison Lemmas for SDE

In 1972 I was looking to estimate the decreasing at the boundary of the complex Green function of a strictly pseudo-convex domain of C^n; finally I got that this decreasing goes as d^n, where d denotes the distance at the boundary; this result appeared soon as a key step for determining the behaviour of the zeroes of a function in the Nevanlinna class. My original proof, which still do not seem to have an alternative approach, was based on a comparison lemma for SDE [Ma74].

Given an abstract manifold M, an exhaustion function p and an elliptic operator Δ defined on M, the *heat conduction coefficient* and the *projected heat conduction coefficients* are defined respectively by

$$\begin{aligned} &a(m) := \frac{\Delta p}{\|\nabla p\|^2}(m); \\ &a^+(\xi) := \sup_{m\in p^{-1}(\xi)} (a(m)); \quad a^-(\xi) := \inf_{m\in p^{-1}(\xi)} (a(m)). \end{aligned} \tag{1.1}$$

Denote $x_\omega(t)$ the diffusion on M associated to Δ, denote $\xi^+_\omega(t^*)$ (resp. $\xi^-_\omega(t^*)$) the diffusion on R associated to the following ODE $y'' + a^+y'$, (resp. $y'' + a^-y'$). Then after a change of time $t \mapsto t^*$, we have

$$p(x_\omega(t^*)) \in [\xi^-_\omega(t); \xi^+_\omega(t)]; \tag{1.2}$$

the asymptotic behaviour of $x_\omega(t)$ on M is compared to two diffusions on R driven by two ODE. The heat conduction invariant is determined by infinitesimal computations when the behaviour of the Green function is a question of global character.

Comparison lemma quickly became an important tool in Riemmanian Geometry. J. J. Prat (75) showed that a Riemanian manifold with strictly negative sectional curvature has a space of bounded harmonic functions which is of infinite dimension. It was also established comparison between the heat kernel of a general riemannian manifold with the heat kernel of constant curvature spaces ([De-Ga-Ma]). These comparison are stated in terms of radial cordinates in the correspondind exponential charts. In the case of symmetric space of rank $r > 1$ the radial coordinate have to be understood as a point in a cone of R^r. Then it is possible to describe the full asymtotic behaviour of Brownian motions by splitting the probability space into a skew product ([Ma-Ma74]).

§2. Ground state of vector bundle

We give a riemanian manifold M, a vector bundle F above M, an euclidean metric on each fiber F_m and a connection preserving this metric. Then on the vector space of sections $\Gamma(F)$ we have two quadratic forms defined respectively by the L^2 and the H^1 metric. We are interested in $\lambda_0(F)$ which is the infimum of the second quadratic form under the constraint that the first take the value 1. For instance if M is compact and if F together with its connection is trivial then we have $\lambda_0(F) = 0$.

The special case where $F = M \times C$ with a non trivial connection corresponds to the ground state of a Schrödinger operator associated to a magnetic field H; this magnetic field H is the curvature of the given connection, which plays the rôle of the potential vector.

When the curvature is large, the holonomy of the connection is large, and therefore *the ground state* $\lambda_0(F)$ *must be large.* It is difficult to obtain a quantitative statement of this last sentence. Estimation of this ground state is a basic problem in Geometry: it is associated to the deformation of minimal surface or to vanishing theorem needed for realizing the Harish-Chandra discrete series of representations of a semi-simple Lie group. Also a famous conjecture of Selberg on arithmetic group of $\mathrm{Sl}(2, R)$ can be reformulated through inequalities of the form $\lambda_0(F) \geq 1/2$.

Let us state more precisely the problem in the special case of a magnetic field on R^d. The potential vector is 1-differential form $\omega = \sum A_k \, d\xi^k$; the Hamiltonian has the following expression:

$$\mathcal{H} = -\frac{1}{2} \sum_{k=1}^{d} \left(\frac{\partial}{\partial \xi^k} + \sqrt{-1} A_k \right)^2 .$$

Denoting by $x(t)$ the Brownian motion on R^d, then the corrrespond-ing semi-group has the following expression

$$\begin{aligned} &(\exp(-t\mathcal{H})f)(\xi_0) = E_{\xi_0}(\exp(\sqrt{-1}J_t)f(x(t))), \\ &\qquad \text{where } J_t := \int_0^t \langle \omega, o\, dx \rangle \end{aligned} \tag{2.1}$$

where the last integral is a Stratonovitch integral along the path of a differential form. Gaveau (77) give an exact expression for the ground state in the case of constant magnetic field.

The difficulty in (2.1) comes from the oscillating integral. I was lead in 1974 to the idea that this oscillating term could be estimated through some kind of stationary phase argument. I needed to compute for this reason the "gradient" ∇J_t and this was for me the first occasion to realize some kind of stochastic calculus of variations. The sharp following lower bound of this gradient was founded later by Prat (93):

$$\|\nabla J_t\|^2 \geq \int_0^t \left\| \int_0^s H \wedge o\, dx \right\|^2 ds - \frac{1}{t} \left\| \int_0^t ds \int_0^s H \wedge o\, dx \right\|^2 .$$

From this expression it can be shown that if $\|H\|(\xi) > c > 0$ then there exist $c' > 0$ such that

$$E\Big(\exp\Big(c' \frac{t}{\|\nabla J_t\|} \Big) \Big) < \infty. \tag{2.2}$$

This inequality is appealing for a stationary phase treatement: Taniguchi showed an infinite dimensional holomorphic Cauchy formula which implies an infinite dimensional stationary phase principle; in this context an hypothesis of type of (2.2) could imply an exponential decay of the stationary phase, UNDER the additional assumption of real analyticity for the potential vector.

Transverse analyticity concerns a similar problem; the time is now fixed at t_0 we look for the following exponential decay

$$\lim_{n \to \infty} \log \frac{|E(\exp(n\sqrt{-1} J_{t_0} \mid x(t) = 0)|}{|n|} < 0. \tag{2.3}$$

In 1985 at the autumn meeting of the Japanese Mathematical Society, I showed that (2.3) holds true for any C^3 differential form under the hypothesis that $\|H\|$ has pointwise a uniform lower bound; under the same hypothesis I obtain an estimate of the ground state [Ma85].

Important results related to this circle of idea can be found in [Ik-Ma], [Ma-Sh], [Ma-Ue], [Sh94], [Sh-Ta], [Ue].

§3. Transfer principle from ODE to SDE and Stochastic Calculus of Variations

A consequence of Itô theory of SDE is the possibility to use the probability space of the Brownian motion on R^d for realizing the diffusion process associated to any elliptic operator. The intrinsic construction of the *Itô map* is done in [Ma74'] as follows. Given a manifold M of dimension d and given q C^∞ vector fields $A_1, \ldots, A_q$ on M, we associate to any C^1 curve ϕ on R^n the non autonomous vector field defined on M by $Z_t^\phi := \sum \dot{\phi}^k A_k$. Then the *controll map* is the flow of C^∞ diffeomorphisms defined on $M \times R$ by

$$\frac{d}{dt} U^\phi_{t \leftarrow t'} = Z_t; \quad U^\phi_{t' \leftarrow t'}(m) = m.$$

Given an R^q-valued Broxnian motion $x(t)$ we denote by x_n a sequence of piecewise C^1-curves converging towards x and properly constructed. Then

Limit Theorem 3.1. *Almost surely $U^{x_n}_{t\leftarrow 0}$ converges locally uniformly in (m,t) towards $U^x_{t\leftarrow 0}$ which is a C^∞-flow of diffeomorphisms on $M \times R$; furthermore the trajectory $t \mapsto U^x_{t\leftarrow 0}(m_0)$ is the generic path of the diffusion associated to the operator $\frac{1}{2}\sum \mathcal{L}^2_{A_k}$.*

A big part of classical differential geometry is based on the machinery of ordinary differential equations. As SDE can be reduced *in a canonic way* to a limit of a sequence of ODE, a new geometry appeared: the *Stochastic Differential Geometry.* Furthermore the limit theorem provides a *transfer principle* giving AUTOMATIC PROOF of some statements of Stochastic Differential Geometry in terms of the correponding statements of Ordinary Differential Geometry.

The International Symposium on SDE held in june 1976 at Kyoto was for me an exceptional possibility to present this transfer principle with two of its consequences for SDE: the *Stochastic Flow of Diffeomorphism* and the *Stochastic Calculus of Variations* along a Stochastic Flow.

The limit theorem is a difficult and key result and it is fortunate that there exits now several independent proofs: [Ma76], [Ik-Wa], [Kun], [St-Ta].

The limit theorem gives as byproduct existence theorem of solutions of SDE. But it gives more, it embbed canonically the theory of SDE, as a limiting case, into the theory of ODE.

§4. Mean value formulae for harmonic differential forms

Given a Riemannian manifold M of dimension d, we consider $\pi_t(m_0, dm)$ the fundamental solution of the heat equation $\partial_t - \Delta_0$, where Δ_0 denotes the Laplace-Beltrami operator of M. Then for any bounded harmonic function h the following mean value formula holds true:

$$h(m_0) = \int h(m)\pi_t(m_0, dm) = E_{m_0}(h(p(t))), \quad \forall\, t > 0$$

where $p(*)$ denotes the Brownian motion on M: the diffusion process associated to Δ_0.

The De Rham Hodge operator of degree r is defined on exterior differential forms of degree r by the formula $\Delta_r := -\frac{1}{2}(\delta d + d\delta)$ where $\delta = d^*$ is the adjoint of the coboudary operator d. Given an harmonic differential form ω, is it possible to show that ω satisfies some kind mean value formula? Is it possible to construct a probabilistic representation of the heat semi-group $\exp(t\Delta_r)$?

The notion of mean values assumes implicitely that it is possible to add, in an intrinsic way, values of the differential form ω at two different points $m, m' \in M$, which is a strange assertion for a differential geometer.

The *procedure of scalarization* is a way to avoid this difficulty. A *frame* on M, let be r, is by definition an euclidean isomorphism of R^d onto $T_m(M)$; we denote $m = \pi(r)$. The collection $O(M)$ of all frames of M has a natural structure of C^∞ manifold; as the orthogonal group $SO(d)$ operates on $O(M)$ we get that $O(M)$ is a principal bundle over M. To a differential form ω on M we can associate its component in a frame; in this way is defined a function $f_\omega : O(M) \mapsto R^d$; then $f_\omega(rog) = gf_\omega(r), \forall g \in SO(d)$. Furthermore the correspondance $\omega \mapsto f_\omega$ identifies the differential forms on M to the functions on $O(M)$ satisfying the previous equivariance property.

Given a frame r_0, we denote $\gamma(\tau)$ the geodesic of M tangent to the first coordinate vector of r_0. Then the Levi-Civita parallel transport of r_0 along $\gamma(*)$ defines a curve $t \mapsto \tilde{\gamma}(\tau)$ on $O(M)$. The tangent vector to $\tilde{\gamma}(*)$ at $t = 0$ defines a tangent vector at r_0 which will be denoted $A_1(r_0)$. We obtain in this way d canonical tangent vector fields A_k on $O(M)$, which will be called the canonic *horizontal vector fields.* Finally the *horizontal laplacian* is defined by

$$\Delta_{O(M)} := \frac{1}{2}\sum_{k=1}^{d} \mathcal{L}^2_{A_k}$$

and the following Weitzenböck formula realizes the scalarization procedure at the level of differential operators

$$f_{\Delta_1(\omega)} = \Delta_{O(M)} f_\omega - \frac{1}{2}\,\mathrm{Ricc}(f_\omega),$$

where $\mathrm{Ricc}(r)$ is the $d \times d$ matrix obtained by expressing the Ricci tensor in the frame r.

The Stratanovitch SDE

$$dr_x = \sum_k A_k o\, dx^k$$

gives a global canonic parametrization by the Brownian motion x of R^d of the diffusion associated to $\Delta_{O(M)}$. The *Itô map* $\mathcal{I}$ is defined by

$$\mathcal{I}(x)(\tau) := \pi(r_x(\tau)); \tag{4.1}$$

if $r_0 \in \pi^{-1}(m_0)$ is fixed then $\mathcal{I}$ realizes a 1 to 1 parametrization of the Brownian motion of M starting from m_0 by the brownian motion in R^d.

The Itô parallel displacement, presented by Itô at ICM 1962, can be constructed by the transfer principle from the Levi-Civita parallel displacement along smooth curve. Within the formalism of the frame bundle it has the following expression

$$t^p_{\tau\leftarrow 0} = r_x(\tau) o [r_x(0)]^{-1} \quad \text{where } p(*) = (\mathcal{I}(x))(*). \tag{4.2}$$

The intrinsic character of (4.1) makes possible to get the following asymptotic expansion of the holonomy of the Stochastic parallel transport:

Theorem ([Ma75], [Be]). *Denote by* $\exp_{m_0}$ *the riemannian normal chart on* M *at* m_0*, then when* $\tau \to 0$ *we have*

$$t^{p_x}_{0\leftarrow\tau} o \big(\exp_{m_0}\big)_* = \text{Identity} + R_{m_0}\left(\int_0^\tau x \wedge dx\right) + o(t^{3/2-\epsilon}) \tag{4.3}$$

where R_{m_0} *denote the curvature tensor of* M *at* m_0.

Theorem ([Ma74']). *The Heat semi-group on* 1*-differential form has the following scalarised expression*

$$f_{\exp(\tau\Delta_1)(\omega)}(r_0) = E_{r_0}(\mathcal{R}_{0\leftarrow\tau}(f_\omega(r_x(\tau)))) \tag{4.4}$$

$$\textit{where } \frac{d}{d\tau}\mathcal{R}_{0\leftarrow\tau} = \frac{-1}{2}\mathcal{R}_{0\leftarrow\tau}\,\mathrm{Ricc}(r_x(\tau)),$$

and where $\mathcal{R}_{0\leftarrow 0} =$ *Identity.*

The Atiyah-Singer Theorem gives an expression of the index of an elliptic operator above M where the function $\lambda/\sinh\lambda$ mysteriously appears. Bismut [Bi84] starting from (4.3) and (4.4) gives a probalistic proof of Atiyah-Singer Theorem where this mysterious function arrive quite naturaly as the characteristic function of the random variable $\tau^{-1}\int_0^\tau x_1\,dx_2 - x_2\,dx_1$, expression which is the $(1,2)$ component of $\int_0^\tau x \wedge dx$.

§5. Path Space as a parallelized manifold; Tangent Process

Begining 1993, I received an invitation to teach at the Spring Quarter 1994 at Kyoto University. In the perpective of this course I start with Professor Cruzeiro a systematic study of the differential geometry on $P_{m_0}(M)$, the space of paths on a riemannian manifold, starting from

$m_0 \in M$. After numerous discussions with ours Kyoto colleagues and subsequent work at the Institute Mittag-Leffler, we produced [Cr-Ma96].

The elements of the tangent space $T_{p_0}(P_{m_0}(M))$ at the point p_0 are identified with the continuous maps $Z : [0,1] \mapsto T(M)$ such that $Z_\tau \in T_{p_0(\tau)}(M)$. The choice of some regularity for the map $(\tau, p) \mapsto Z_\tau(p)$ is a main issue which will be discussed below. We denote X' the Banach space $P_0(R^d)$ of continuous paths on R^d and by X the Wiener space that is the same space enriched with its structure of filtered probability space. We denote

$$\text{(5.1)} \qquad \begin{aligned} &\Omega_p : T_p^0(P_{m_0}(M) \longmapsto X' \\ &\text{defined by } \Omega_p(Z) = z, \;\; z(\tau) := t^p_{0\leftarrow\tau}(Z_\tau). \end{aligned}$$

Then Ω can be looked upon as an X'-valued 1-differential form defined on $P_{m_0}(M)$ which $\forall\, p$ realizes an isomorphism of the tangent space at p onto a fixed Banach space: we say that Ω defines the *canonic parallelism of the Path Space.*

We have now the concept *constant vector field*, which mean a vector field Z defined on $P_{m_0}(M)$ such that there exists a fixed $z \in X'$ verifying

$$\text{(5.2)} \qquad \begin{aligned} &\langle \Omega_p, Z(p) \rangle = z, \quad \forall\, p \in P_{m_0}(M); \\ &\text{or equivalently } Z_\tau(p) = t^p_{\tau\leftarrow 0}(z_\tau). \end{aligned}$$

A similar situation in a finite dimension setting is a Lie group parallelized throug its Maurer-Cartan differential form; then the constant vector fields correspond to left invariant vector fields; the bracket of left invariant vector fields defines the *structural constants* of the asssociated Lie algebra.

We denote by $\mathcal{T}$ the vector space of smooth cylindrical functions on $P_{m_0}(M)$, that is such that there exits a finite set $\{\tau_i\}_{1\le i\le j}$ and a smooth function $F : M^j \mapsto R$ for which $f(p) = F(\ldots, p(\tau_i), \ldots)$; we define

$$(D_Z(f))(p) := \sum_{i=1}^{i=j} \langle d_i F, Z_{\tau_i}(p) \rangle.$$

Now for Z constant vector field the fact $f \in \mathcal{T}$ do not imply that $D_Z f \in \mathcal{I}$; this technical difficulty is eliminated by extending by closure the domain of D_Z as it will be seen in 5.8. It is then possible to define the bracket $Z^3 = [Z^1, Z^2]$ of two vector fields Z^1, Z^2 by the relation

$$D_{Z^3} f = (D_{Z^1} D_{Z^2} - D_{Z^2} D_{Z^1}) f.$$

The *structural equation of the Path Space* is defined by

$$[z^1, z^2]_p = \langle \Omega_p, [Z^1, Z^2] \rangle \tag{5.2}$$

where Z^s, $(s = 1, 2)$ are the constant vector fields associated to z^s. Then $\forall\, p_0 \in P_{m_0}(M)$ the structural equation defines a bilinear antisymmetric map $[*, *]_{p_0}$ of $X' \times X' \mapsto X'$.

Theorem ([Cruzeiro-Ma96]).

$$[z^1, z^2]_p(\tau) = Q_{z^1}(\tau) z^2_\tau - Q_{z^2}(\tau) z^1_\tau \tag{5.3}$$
$$\textit{where } Q_z(\tau) = \int_0^\tau R_\lambda(z_\lambda, o\, dx(\lambda))$$

where $p = \mathcal{I}(x)$ and where R_λ denotes the curvature tensor of M expressed in the frame $r_x(\lambda)$.

Remark. The curvature tensor depends upon four indices, the above integral saturates two indices, therefore $Q_{z^1}(\tau)$ is an antisymmetric matrix, which by operating on the vector z^2_τ gives a vector.

Corollary. *Denote $d_\tau z$ the differential in τ of z, then the Stratanovitch differential of the structural eqation is*:

$$d_\tau([z^1, z^2]) = R(z^1, z^2)(o\, dx) + Q_{z^1}(d_\tau z^2) - Q_{z^2}(d_\tau z^1). \tag{5.4}$$

Remark. For $z^1, z^2 \in H^1$ then $[z^1, z^2] \in H^1$ if and only if $R_*(z^1, z^2) = 0$.

We call a *tangent process on the Wiener space X* ([Cr-Ma96]) the data of a semimartingale ζ such that its Itô differential satisfies

$$d_\tau \zeta^j = a^j_i\, dx^i + c^j\, d\tau \quad \text{where } a^j_i + a^i_j = 0. \tag{5.5}$$

Example. For $z^1, z^2 \in H^1$ it results from (5.4) that $[z^1, z^2]$ is a tangent process on X.

Theorem ([Bi84'], [Dr92], [Fa-Ma93], [En-St95], [Cr-Ma96]). *The differerential operator D_ζ associated to a tangent process satisfies the following formula of integration by part*:

$$E(D_\zeta f) = E\left(f \int_0^1 c^j\, dx^j \right) \tag{5.6}$$

where the last integral is an Itô Stochastic Integral.

We call Z^* a *Tangent Process on the Path Space* if $\zeta^* := \langle \Omega, Z^* \rangle$ is a Tangent Process on the Wiener space.

Main Theorem. *Define the differential of the Itô map by*

$$\mathcal{I}'(x).\zeta := t^p_{0\leftarrow *}\left(\frac{d}{d\epsilon_{|\epsilon=0}}\mathcal{I}(x+\epsilon\zeta)\right) \quad \textit{where } p = \mathcal{I}(x);$$

then $\mathcal{I}'$ realizes an isomorphism of the space of tangent process on X onto the space of tangent process on $P_{m_0(M)}$; this isomorphism can be computed through the system of Stratonovitch SDE

$$d\zeta = d\zeta^* + \rho o\, dx; \quad d\rho = -R(o\, dx, \zeta^*) \tag{5.7}$$

all the initial conditions at $\tau = 0$ being equal to 0.

The combinaison of (5.7) and of (5.6) gives the

Corollary. *Tangent Processes on $P_{m_0}(M)$ satisfy formulae of integration by part; therefore the derivation operator along a Tangent Process defined on smooth cylindrical functions is closable.*

In order to explicit this formula we write the first equation of (5.7) in Itô form in the case where $\zeta^* = z \in H^1$, then

$$d\zeta = \left(\dot{z} + \frac{1}{2}\operatorname{Ricc}(z)\right)d\tau + \rho\, dx.$$

As the application of Itô preserves the probability measure the formula of integration by part can be pull back to X: introduce $\Phi := fo\mathcal{I}$; Φ is defined on X and

$$E(D_Z f) = E(D_\zeta \Phi) = E\left(\Phi \int_0^1 \left(\dot{z} + \frac{1}{2}\operatorname{Ricc}(z)\right)dx\right). \tag{5.8}$$

The origin of the introduction of tangent process has been the necessity to handle properly the structural equation. It is therefore interesting that this procedure stabilizes itself and that it not necessary to introduce new objects to compute braket of tangent processes:

Theorem 5.9 ([Cr-Ma96], [Dr99]). *Given two tangent process on $P_{m_0}(M)$, with differentiable coefficients, then their bracket is a tangent process.*

§6. Intertwinning formulae; Integration by part formulae

Intertwining at the level of PDE.

On the Riemannian manifold M we have the following classical intertwining formulae for Laplace-DeRham-Hodge operators:

$$d(d\delta + \delta d) = (d\delta + \delta d)d \quad \text{or} \quad d\Delta_r = \Delta_{r+1}d.$$

We write this intertwinning formula for $r = 0$ and we exponentiate we get

$$d\exp(\tau\Delta_0) = \exp(\tau\Delta_1)d. \tag{6.1}$$

Consider the final value problem for backward heat equation:

$$\Big(\frac{\partial}{\partial\tau} + \Delta_0\Big)\phi_\tau = 0, \quad \phi_1 = F \text{ given}$$

then this equation has for solution

$$\phi_\tau(m) = E_{p(\tau)=m}(F(p(1)) = (\exp(1-\tau)\Delta_0)F)(m).$$

In Mathematical Finance F will be the liquidative value at maturity $\tau = 1$ of a portfolio. Denote $\pi_\tau = d(\phi_\tau)$, then (6.1) and (4.4) give

$$\pi_\tau = E_{p(\tau)=m}(\tilde{t}^p_{\tau\leftarrow 1}(dF)_{p(1)})$$

where $\tilde{t}$ is the *damped parallel transport* [Fang-Ma93] defined by

$$\tilde{t}^p_{\tau\leftarrow\tau'} = Q(\tau,\tau')t^p_{\tau\leftarrow\tau'} \tag{6.2}$$
$$\text{where } d_\tau Q(\tau,\tau') = \frac{1}{2}(\text{Ricc})Q(\tau,\tau')\,d\tau, \;\; Q(\tau',\tau') = I.$$

The word "damped" has the following origin: if we suppose that Ricc $= 2I$ then $Q(\tau,1) = e^{\tau-1}$.

The function $\phi(\tau,m)$ is a parabolic function, therefore its variation in the space time is given by the following Itô stochastic integral

$$\begin{aligned} F(p(1) - \phi_\tau(p(\tau))) &= \int_\tau^1 \langle \pi_\lambda, dx(\lambda)\rangle \\ &= \int_\tau^1 E^{\mathcal{N}_\lambda}(\tilde{t}^p_{\lambda\leftarrow 1}(dF)_{p(1)})\,dx(\lambda), \end{aligned} \tag{6.3}$$

where $\mathcal{N}_t$ denotes the filtration on X ([Ai-Ma95]).

Given a smooth cylindrical function $f(p) := F(\ldots,p(\tau_i),\ldots)$ we defined its *damped gradient* $\tilde{D}_\tau f$, $\tau \in [0,1]$, by the formula

$$\tilde{D}_\tau f = \sum_{1\le i\le j} 1(\tau < \tau_i)\tilde{t}_{\tau\leftarrow\tau_i}(d_i F)$$

We get the *Clark-Bismut-Ocone formula*

$$f - E(f) = \int_0^1 E^{\mathcal{N}_\tau}(\tilde{D}_\lambda f)\, dx(\lambda) \tag{6.4}$$

firstly by specializing (6.3) to the case $\tau = 0$ which give the wanted formula for cylindrical functions of the form $f(p) = F(p(1))$ and finally by generalizing to arbitrarly cylindrical functions through Markov property applied at the sequence of times τ_i.

Given a constant vector field Z on $P_{m_0}(M)$ which has for image in the parallelism $z \in H^1$ we denote

$$\tilde{D}_Z f := \int_0^1 \tilde{D}_\tau f \dot{z}(\tau)\, d\tau;$$

using the energy identity for Itô stochastic integral we get

$$E(\tilde{D}_Z f) = E\bigg(\Big(\int_0^1 \dot{z}\, dx\Big)\Big(\int_0^1 E^{\mathcal{N}_\tau}(\tilde{D}_\tau f)\, dx(\tau)\Big)\bigg);$$

using Clark-Bismut-Ocone formula we get finally the following *formula of integration by part*

$$= E\Big(f \int \dot{z}\, dx\Big). \tag{6.5}$$

It can be remarked the proof of (6.5) given here depends only upon Itô formula and re-presentation of the heat semi-group on 1-differential forms. As it is possible by elementary computations to deduce (5.8) from (6.5), we could have used this approach to give an elementary proof of (5.9); nevertheless the use of Geometry on $P_{m_0}(M)$ is essential for the following intertwinning.

Intertwining by transference at the level of Path space.

Denote $\pi_\tau(m_0, dm)$ the fundamental solution of the heat with pole at m_0. We call *transference* a formula which replace a derivation relatively to the starting point m_0 by a derivation relatively to the ending point m:

Transference Theorem ([Cruzeiro-Ma98]). *Given a vector $a \in T_{m_0}(M)$, consider its damped parallel transport along generic trajectory of the Brownian motion of M starting from m_0, then for every $F \in C_b^1(M)$ we have*

$$\frac{d}{d\epsilon}_{|\epsilon=0} \int_M \pi_\tau(m_0 + \epsilon a, dm) F(m) = E_{m_0}(\langle \tilde{t}^p_{\tau\leftarrow 0}(a), (dF)_{p(\tau)}\rangle). \tag{6.6}$$

The proof is based on the machinery of tangent process. It is possible to deduce from this transference theorem an intertwinning result for first order PDE.

Corollary. *Define a vector field Y on M by the formula*

$$Y_m := E^{p(\tau)=m}_{p(0)=m_0}(\tilde{t}^p_{\tau\leftarrow 0}(a))$$

then

$$(\partial_a(\exp(\tau\Delta_0)F))(m_0) = (\exp(\tau\Delta_0)(\partial_Y F))(m_0). \tag{6.7}$$

Remark. This formula differs from (6.1) in the sense that the tangent vector a is arbitrarly choosen when from the other hand the point m_0 where the intertwinnig is computed is fixed.

Theorem (Bismut-Harnack identity) ([Bi84'], [En-St], [El-Li], [El-Le-Li], [Fa-Ma]).

$$\begin{aligned}\frac{d}{d\epsilon}_{|\epsilon=0}\int_M \pi_\lambda(m_0+\epsilon a, dm)F(m) \\ = E_{m_0}\left(F(p(\lambda))\frac{1}{\lambda}\int_0^\lambda \langle \tilde{t}_{\tau\leftarrow 0}(a), dx(\tau)\rangle\right)\end{aligned} \tag{6.8}$$

where the stochastic integral is an Itô stochastic integral.

We can deduce a proof by applying to (6.6) Girsanov type theorem.

§7. Construction of measures in infinite dimension through infinitesimal geometry

Our point of view is to define a probability measure μ as the invariant mesure of a second order elliptic operator. This point of view illustrate the paradigm of *going from the infinitesimal to the global* as it produces a global object, the measure μ, from infinitesimal constructions. Our methodology will be presented in the framework of finite dimensional manifold but with *dimension free constants.*

We consider an elliptic operator $\tilde{\Delta}'$ on a manifold M such that $\Delta'(1) = 0$. Then the second order terms define a riemannian metric and there exits a unique vector field Z on M such that

$$\Delta' f = \frac{1}{2}\Delta_0 f - \langle Z, df\rangle.$$

The scalarization procedure associates to Z an R^d-valued function f_Z defined on $O(M)$. We denote $\partial_*^h f_Z$ the $(d \times d)$ matrix which has for k^{th} column $\partial_{A_k} f_Z$ where $\ldots, A_k, \ldots$ are the canonic horizontal vector fields. We define a $(d \times d)$ matrix valued function Ricc$'$ on $O(M)$ by

$$\mathrm{Ricc}' = \mathrm{Ricc} + 2\partial_* f_Z.$$

Lemma 7.1. *Replacing* Ricc *by* Ricc$'$ *then the heat kernel* π'_τ *associated to* $\exp(\tau\Delta')$ *satisfies* (6.6).

The proof follows the same line as in Section 6 and the Stochastic Calculus of Variations along the diffusion generated by Δ' brings into the computation the matrix $\partial_*^h f_Z$.

Theorem 7.2 ([Cr-Ma98]). *Assume that* $\exists\, \delta > 0$ *such that*

$$\mathrm{Ricc}' + (\mathrm{Ricc}')^* \geq 2\delta \times \mathrm{Identity}$$

then Δ' *has a unique invariant probability measure* μ.

The proof is based on the extension of the intertwining formula (6.7) to derivative of second order, extension which is obtained by differentiating the identity (6.6) and computing its r.h.s. by the Stochastic Calculus of Variations. The derivation of the semi group $\frac{d}{d\tau}\exp(\tau\Delta') = \Delta' \exp(\tau\Delta')$ gives then

$$\frac{d}{d\tau}\int \pi'_\tau(m_0, dm) f(m) = \Delta'_{m_0} \int_M \pi_\tau(m_0, dm) f(m).$$

We transfert the first (by (6.7)) and second order derivatives appearing in Δ'_{m_0} from the starting point to the end point: we get a second order elliptic operator $\mathcal{Q}_\tau$, called the *desintegrated adjoint operator* such that

$$\frac{d}{d\tau}\int_M \pi'_\tau(m_0, dm) f(m) = \int_M \pi'_\tau(m_0, dm)(\mathcal{Q}_\tau f)(m) \tag{7.3}$$

where the norm of $\mathcal{Q}_\tau$ relatively to the underlying riemannian metric of M satisfies

$$\|\mathcal{Q}_\tau\| \leq c \exp(-\delta\tau). \tag{7.4}$$

We call the *desintegrated adjoint process* the process $\hat{p}(\tau)$ associted to $\exp(\tau\mathcal{Q}_\tau)$ and starting at $\tau = 0$ from m_0. Then the law at time τ of the desintegrated adjoint diffusion is equal to $\pi'_\tau(m_0, *)$; furthermore (7.4) implies that

$$\hat{p}(\infty) := \lim_{\tau \to \infty} \hat{p}(\tau)$$

exists.

We construct μ as the law of $\hat{p}(\infty)$.

Corollary. *Assume furthermore that* Ricc$'$ *is symmetric, then the measure* μ *is reversible and the following formula of integration by part holds true:*

$$\int_M \partial_V \Phi \, d\mu = \int_M \Phi(2(Z|V) - \mathrm{trace}(\partial_* f_V)) \, d\mu. \tag{7.5}$$

§8. Vector fields on Probability spaces, their divergence and their flow

An *admissible vector field* on a probability space is a derivation operator defined on an algebra of cylindrical functions and satisfying a formula of integration by part part relatively to the probability measure μ:

$$E(D_Z \phi) = E(\phi \delta_\mu(Z));$$

the function $\delta_\mu(Z)$ is called the *divergence of* Z *relatively to* μ.

On the Wiener space X of the Brownian motion on R^d, classically is considered the Cameron-Martin space

$$H^1 = \left\{ h \in X \; ; \; \int \|\dot{h}\|^2_{R^d} \, d\tau := \|h\|^2_{H^1} < \infty \right\}.$$

Given $h \in H^1$, we have the following Cameron-Martin integration by part formula:

$$E(D_h \phi) = E\Big(\phi \int_0^1 \dot{h} \, dx\Big)$$

where the last integral is an Itô Wiener Stochastic Integral.

The existence of a formula of integration by part implies that the derivation operator D_h is closable in L^r, $r \in]1, \infty[$. Then the intersection $\forall\, h \in H^1$ of the domains these closures defines a Banach space, the *Gross-Stroock Sobolev space* $D_1^r(X)$. In the same way the Gross-Stroock Sobolev space $D_s^r(X)$ of s-times differentiable functions can be defined. Given an abstract Hilbert space $\mathcal{H}$, the Gross-Stroock Sobolev space $D_1^r(X; \mathcal{H})$ of $\mathcal{H}$-valued functions is similarly defined. Then

Theorem (Watanabe 84). *For all* $r > 1$, $\exists\, c_r < \infty$ *such that*

$$\|\delta_\mu(Z)\|_{L^r} \leq c_r \|Z\|_{D_1^r(X; H^1)} \tag{8.1}$$

and the finiteness of the r.h.s. implies the existence of the divergence appearing in the l.h.s.

We have then the following infinite dimensional analog of the finite dimensional fact that smooth vector field generates a flow:

Theorem (Cruzeiro 83). *Given $Z \in D_1^r(X; H^1)$, $\forall\, r < \infty$ such that $\exists\, c > 0$ for which*

$$
\begin{aligned}
&E(\exp c|\delta(Z)|) < \infty, \quad E(\exp(c\|Z\|_{H^1}) < \infty, \\
&E(\exp(c\|DZ\|_{H^1 \otimes H^1}) < \infty,
\end{aligned}
\tag{8.2}
$$

then, $\forall\, \tau \in R$, there exists a map $U_\tau : X \mapsto X$, defined μ-a.e., preserving the class of the Wiener measure, satisfying the group property $U_\tau o U_\sigma = U_{\tau+\sigma}$ such that denoting $\mu_\tau := (U_\tau)_* \mu$ the image of the Wiener measure, we have

$$
\begin{aligned}
&\frac{d\mu_\tau}{d\mu}(x_0) = \exp\Big(\int_0^\tau (\delta Z)(U_{-\lambda}(x_0))\, d\lambda\Big); \\
&\frac{d}{d\tau} \int f\, d\mu_\tau = \int f \delta(Z)\, d\mu_\tau.
\end{aligned}
\tag{8.3}
$$

In the case of a constant vector field we get back a Theorem of Cameron-Martin. Several papers amplify this theorem see [Bo-Mw].

We have the following extension to the Riemannian Path Space.

Theorem 8.4 (Driver 92, Hsu 95). *Fix $z \in H^1(R^d)$ then define a constant vector field Z on $P_{m_0}(M)$ by $Z(p) = \Omega_p^{-1}(z)$ then the flow associated to Z exists.*

This theorem cannot be reduced by the Itô map $\mathcal{I}$ to Cruzeiro Theorem: in fact the inverse image of a constant vector field is a tangent process and no more an H^1 vector field. The original proof of Driver is a "tour de force": at the same time he constructed the flow and produced a new approach to the Bismut formula of integration by part (5.6). From the Bismut book (1984) to the Driver paper (1992) the study of the Riemannian Path pace stayed essentially quiet. After the Driver breakthrough, the subject became very active.

§9. Geometrization of the Anticipative Stochastic Calculus by Divergences

On a finite dimensional Riemannian manifold the *energy identity for*

1-*differential form* ω has the following expression

$$(9.1)\qquad \|d\omega\|^2_{L^2_\mu} + \|\delta_\mu(\omega)\|^2_{L^2_\mu} = \int_M \|\nabla\omega\|^2\, d\mu + \int_M (\mathrm{Ricc}''(\omega)|\omega)\, d\mu$$

where $d\mu = \exp(-\phi)\, dm$, dm being the riemannian volume measure, where δ_μ is the *divergence* that is the ajoint in L^2_μ of d which means:

$$\int_M (\omega|df)\, d\mu = \int_M f\delta_\mu(\omega)\, d\mu,$$

where ∇ denotes the Levi-Civita connection on M and where $\mathrm{Ricc}'' := \mathrm{Ricc} + \nabla^2\phi$.

Take for instance $M = R^d$, $\phi = \frac{1}{2}\|\xi\|^2$, then μ is the canonic gaussian measure on R^d and $\mathrm{Ricci}'' = \mathrm{Identity}$.

Theorem 9.2 (Shigekawa 86). *On the Wiener space X the identity* (9.1) *holds true with* $\mathrm{Ricc}'' =$ *Identity.*

A far reaching discovery of Gaveau-Trauber is that, on the Wiener space, $\delta_\mu(\omega)$ is equal to the Skorokhod anticipative integral; their proof is based on an L^2 chaos expansion and therefore is of **global** nature. Zakai-Nualart-Pardoux ([Nu-Za] and [Nu-Pa]) define Anticipative Integral as the divergence and produce a constructive scheme of approximation by a sequence of finite sums producing in this way a **local** construction of the Anticipative Integral. As a differential form ω is an H^1 valued functional we introduce $a := d_\tau\omega$ and then a takes its value in $L^2([0,1])$; we assume that $a \in D^2_1(X; L^2([0,1]))$. Then Zakai-Nualart-Pardoux prove the following energy identity

$$(9.3)\qquad E\left(\left(\int_0^1 a_\tau\, dx(\tau)\right)^2\right) = E\left(\int_0^1 |a_\tau|^2\, d\tau + \int_0^1\int_0^1 D_\tau a_\lambda D_\lambda a_\tau\, d\lambda\, d\tau\right).$$

For a is adapted, $D_\tau a_\lambda = 0$ for $\tau > \lambda$; the second term of the r.h.s. vanishes and the Itô energy identity appears.

A very short proof of (9.3) can be obtained by a direct application of (9.1), and (9.2).

Using Schwarz inequality we deduce from (9.3) the inequality

$$(9.4)\quad E\left(\left(\int_0^1 a_\tau\, dx(\tau)\right)^2\right) \le E\left(\int_0^1 |a_\tau|^2\, d\tau + \int_0^1\int_0^1 |D_\tau a_\lambda|^2\, d\tau\, d\lambda\right).$$

On a riemanian manifold for a constant vector field, the formula (6.5) proves that the Itô type Stochastic Integral is the transposed of the **damped** gradient $\tilde{D}$. It could be natural to define the anticipative integral of a non adapted process as the operator $(\tilde{D})^*$. We shall not develop this point of view and will look for majorations of $\delta := D^*$. As the passage from the gradient to the damped gradient is an operator which is uniformly bounded with its inverse, a majoration for D^* will imply majoration of the stochastic integral.

Manageable generalization to $P_{m_0}(M)$ of (9.1) seems presently not available. The Levi-Civita connection appearing in (9.1) can be explicitely computed on $P_{m_0}(M)$ but leads quickly to exploding formulas ([Cr-Ma96]). These two unfortunate facts make difficult the approach to the following inequality (9.6).

We define a bounded inversible map $H^1 \mapsto H^1$ let $z \mapsto \hat{z}$ through the relation

$$\dot{\hat{z}} = \dot{z} + \frac{1}{2}\operatorname{Ricc}(z)$$

For constant vector fields z, the divergence is given by the following Itô Stochastic integral:

$$\delta(z) = \int_0^1 \dot{\hat{z}}\, dx$$

A connection ∇_u on $P_{m_0}(M)$ will be given by its action on constant vector fields $\nabla_u z$, defined by the following formula

$$d_\tau \widehat{\nabla_u z} := \Gamma_u(\tau)\dot{\hat{z}}(\tau) \qquad \text{where } \Gamma_u(\tau) := \int_0^\tau R_\lambda(u(\lambda), o\, dx(\lambda)).$$

A *new riemannian metric* on $P_{m_0}(M)$ is introduced

$$\|z\|^2 = \int_0^1 |\dot{\hat{z}}|^2 d\tau,$$

and for this metric we have

Theorem (Cruzeiro-Fang 97).

$$\|\delta(z)\|^2 + \|\tilde{d}z\|^2 = \|\nabla z\|^2 + \|z\|^2 \tag{9.6}$$

where $\tilde{d}$ is the antisymmetrized covariant derivative.

This formula is a perfect analog of (9.1). It implies L^2 inequality analog to (9.4). Later Fang 98 has obtained the analogous of (9.4) for L^r-norms.

§10. Geometric measure theory. Principle of descent

I was invited to a Taniguchi Conference at Katata, on lake Biwa, in 1982. After the push that the Stochastic Calculus of Variations received from its presentation at the International Symposium on SDE, RIMS 1976, some participants of this 1982 Conference kindly discussed some analytical aspects of this subject. I present an *implicit function theorem* valid for $D^\infty(X) := \bigcap_{r,s<\infty} D^r_s(X)$ functions.

In the finite dimensional case, we have the classical Sobolev embedding $D^r_s(R^d) \subset C(R^d)$ the space of continuous funtions. In the case of the Wiener space $D^\infty(X)$ is not contained in $C(X)$ and this was the difficulty to overcome to build a local inversion needed to construct the implicit function. This isolated result develop into new subject: the *quasi-sure analysis.*

The dual space of $D^\infty(X)$ can be written as

$$D^{-\infty}(X) = \bigcup_{r>1,\, s<\infty} D^r_{-2s}(X),$$

$D^r_{-2s}(X)$ being characterized as the image by $(-\Delta_X + 1)^s$ of the space $L^r(X)$, where Δ_X denotes the Ornstein-Uhlenbeck elliptic operator on X.

Theorem 10.1 (Sugita 88). *Given $\Psi \in D^{-\infty}(X)$ we say that $\Psi > 0$ if $\Psi(f) \geq 0$ for all $f \geq 0$, $f \in D^\infty(X)$. Then given $\Psi > 0$ there exists on X a positive Radon measure ν_Ψ of finite total mass such that*

$$\Psi(u) = \int_X u(x)\nu_\Psi(dx)$$

for all u smooth cylindrical functions.

We says that a borelian $A \subset X$ is *slim* if $\forall\, \Psi > 0$, $\Psi \in D^{-\infty}(X)$, $\nu_\Psi(A) = 0$.

Redefinition Theorem 10.2 (in the version of K. Itô 90). *Given $f \in D^\infty(X)$ we can construct a borelian function f^*, defined everywhere and almost everywhere equal to f, and a decreasing sequence of open subsets $O_n \subset O_{n-1} \subset \cdots X$ such that $\bigcap O_n$ is slim and such that the restriction of f^* to O_n^c the complement of O_n is continuous. Two such functions f^* are equal ouside a slim set; identifying two functions which differ on a slim set we call f^* the redefinition of f.*

Given a map $g \in D^\infty(X; R^d)$, we defined its *covariance matrix* $\mathcal{C}(g)$ as the $(d \times d)$ matrix $(\nabla g^k | \nabla g^l)_{H^1}$. We say [Ma76] that

$$\text{(10.3)} \qquad g \text{ is } \textit{non degenerated} \text{ if } \det(\mathcal{C}^{-1}) \in L^r, \quad \forall r < \infty.$$

In finite dimension the hypothesis $\mathcal{C}(x_0)$ invertible implies the existence of an implicit in a neighborhood of x_0; therefore (10.3) appears as global quantitative version of this finite dimensional hypothesis (see Lescot [Le] for a qualitative version).

In abstract theory of conditioning, the conditional expectation or the conditional law can be defined only **almost surely**. The following coarea theorem provides conditional laws defined for all resonnable conditioning.

Coarea Theorem (Airault-Ma 88). *Under the hypothesis* (10.3), *the law of the random variable g has a C^∞ density $k(\xi)$ relatively to the Lebesgue measure $d\xi$ of R^d. Denoting Q the interior of the support of k, there exists a continuous map $Q \mapsto D^{-\infty}$, which associates to $\xi \in Q$ a probability measure ν_ξ such that for all $f \in D^\infty(X)$*

$$\text{(10.4)} \qquad E(f) = \int_Q k(\xi)\, d\xi \left[\int_{(g^*)^{-1}(\xi)} f^*(x) \nu_\xi(dx) \right].$$

Fixing $\xi \in Q$ it is possible to write $(g^*)^{-1}(\xi) = \bigcup V_n$ where V_n is an increasing sequence of manifolds; it is possible to define an gaussian area measure λ on V_n such that

$$\text{(10.5)} \qquad \nu_\xi(dx) = \frac{1}{k(\xi)} \frac{1}{\sqrt{\det(\mathcal{C})}} \lambda(dx).$$

10.6. Principle of Descent.

A property true outside a slim set is said to be true *quasi-surely.* For instance given a sequence of functions f_n converging in $D^\infty(X)$ then theirs redefinitions f_n^* converge pointwise quasi-surely.

The principle of descent states that a property true quasi-surely remains true almost surely under the conditioning by $g^*(x) = \xi_0$, $\xi_0 \in Q$. Its proof results from the fact that $\nu_{\xi_0} \in D^{-\infty}$ and therefore do not charge slim sets.

Kusuoka has developed a *quasi-sure theory of differential forms* on the Wiener space.

§11. From Path Group to Loop Group

The reading of Pressley-Segal book "Loop Groups" engaged me in 1986 to try to work on probabilistic questions linked to this framework. The first obvious question was to find *quasi-invariant measures.* The French-Japanese joint Seminar run in Paris in the spring 1987 in the context of the centenary of the birth of Paul Lévy, push us to this work which finally appeared in [Ma-Ma90].

We denote by G a compact simply connected Lie group; we choose on its Lie algebra $\mathcal{G}$ an euclidean metric which is invariant under the adjoint action; for simplicity in the following we shall restrict ourselfs to matrix group, as for instance $SU(2)$; picking an orthonormal basis e_k of $\mathcal{G}$, we consider the matrix Stratonovitch SDE:

$$dg_x = g_x \sum_k e_k o\, dx^k \tag{11.1}$$

where $x \in X$ the Wiener space of the Brownian motion on R^d. The infinitesimal generator associated to this process is $\Delta^r := \frac{1}{2}\sum_k (\partial^r)_k^2$ where $\partial_k^r \phi(g) = \frac{d}{d\epsilon_0}\phi(g\exp(\epsilon e_k))$. We can exchange in all the previous constructions right action by left action, consider left derivatives ∂_k^l and the left operator $\Delta^l := \frac{1}{2}\sum_k (\partial_k^l)^2$. The fact that the adjoint action of G acts on $\mathcal{G}$ by orthogonal transformations implies that $\Delta^r = \Delta^l$. The metric on $\mathcal{G}$ induces a riemannian metric on G; denoting Δ_0 its Laplace-Beltrami operator, we have also $\Delta_0 = \Delta^r$.

We denote $P_e(G)$ the space of paths on G starting from the unit element e. This is a special case of a Riemannian Path Space to which we could apply the results of Section 5. But the Stochastic Calculus of Variations depends upon the type of variations we use along a path of the diffusion. We shall use variations fitted to the SDE (11.1): writting

$$\frac{d}{d\epsilon}_{|\epsilon=0} g_{x+\epsilon h}(\tau) = u(\tau) g_x(\tau)$$

and differentiating (11.1) we get

$$\dot{u} g_x = g_x \sum e_k \dot{h}^k \quad \text{or} \quad \dot{u}(\tau) = \mathrm{Ad}(g_x^{-1}(\tau))(\dot{h}(\tau)). \tag{11.2}$$

As Ad is an orthogonal transformation the map $h \mapsto k$ is a unitary automorphism of H^1: the situation is much more simpler than in (5.7) where we need to introduce tangent processes in order to find an isomorphism. Given $u \in P_0(\mathcal{G})$ and given a cylindrical function $f(x) =$

$F(\dots, g_x(\tau_i), \dots)$ we define the derivative

$$(D_u f)(x) := \frac{d}{d\epsilon}_{|\epsilon=0} F(\dots, \exp(\epsilon u(\tau_i)) g_x(\tau_i), \dots).$$

Then the relation (11.2) implies the following formula of integration by part:

$$E(D_u f) = E(q(x) f(g_x)) \tag{11.3}$$
$$\text{where } q(x) := \int_0^1 \langle \mathrm{Ad}(g_x(\tau)) u_x(\tau), dx(\tau) \rangle$$

We denote by $L_e(G)$ the *loop group* constituted by the paths satisfiying $p(0) = p(1) = e$. We take on $L_e(G)$ the measure of the Brownian bridge $\mu_{L_e(G)}$. We want to get on $L_e(g)$ a formula of integration by part on $L_e(G)$ for admissible variations constituted by $u \in H^1$ such that $u(0) = u(1) = 0$. The two approachs that we consider below are valid in the more general context of $L_{m_0}(M)$, the Loop Space of a Riemannian manifold M.

The first approach [Ma-Ma90] is based on *quasi-sure analysis.* We denote by $\Phi(x) := g_x(1)$; then $\Phi \in D^\infty(X; G)$. Fixing x the map $h \mapsto u$ defined in (11.2) is unitary: the matrix $\mathcal{C}$ = Identity; therefore Φ is a non degenerated map.

The function q defined in (11.3) satisfies $q \in D^\infty(X)$. We denote by q^* its redefinition, by the descent principle q^* is defined $\mu_{L_e(G)}$ almost everywhere. As $D_u \Phi = 0$, we have $\forall\, \theta \in C^\infty(G)$

$$E((\theta o \Phi) D_u f) = E(D_u(\theta(\Phi) f)) = E((\theta o \Phi) q f);$$

we apply the coarea formula along Φ to the first and the third term of the above identity; we obtain an identity valid for all the conditionning $(\Phi^*)^{-1}(g_0)$. Taking $g_0 = e$ we obtain the localization of (11.3) to $L_e(G)$.

The second approach is based on the *Doob h-theory.* We denote $\pi_\tau(g)\, dg$ the law of $g_x(\tau)$. Then the measure $\mu_{L_e(G)}$ is parametrized by the following Stratonovitch SDE, driven by a new Brownian motion y:

$$dl_y(\tau) = l_y(\tau) \Big(\sum_k e_k o\, dy^k + \nabla^r \log(\pi_{1-\tau}(l_y(\tau)))\, d\tau \Big).$$

The Stochastic Calculus of Variations associated to this new parametrization is computed by considering $y \mapsto y + \epsilon h$. This computation involve the second derivative:

$$\nabla^l \nabla^r \log \pi_\lambda, \text{ where } \nabla^l \text{ denotes the left gradient.}$$

For bounding these derivatives the following thorem is usefull

Theorem (Stroock-Ma 96). *Given a Riemannian manifold M, fix $m_0 \in M$ and consider $\pi_\lambda(m_0, m)\, dm$ the fundamental solution of the heat. The Levi-Civita covariant derivative ∇ defines a canonic Hessian ∇^2. Then*

$$\lim_{\lambda \to 0} \lambda \nabla^2 \log \pi_\lambda = -\frac{1}{2} \nabla^2 d^2$$

where d denotes the Riemannian distance to m_0, this limit being uniform $m \in K$ where K is a compact which do not intersect the cutt locus of m_0.

Some others theorems linked to differential analysis on Loop group

Ergodicity Theorem (Gross 93). *Assume G simply connected. A function $f \in D_1^r(L_e(G))$ satisfying $D_u f = 0$, $\forall u \in H^1$, such that $u(0) = u(1) = 0$ is constant.*

Research of Logarihmic Sobolev inequalities on Loop spaces was started by Gross and after has been very much discussed; this important topic is at the frontier of our subject, therefore we quote only few recent results [Ai-El], [Fa99'], [Ma-Se].

§12. Heat equation on some infinite dimensional groups

Invited to the Hashibara Forum, a satellite Conference of ICM Kyoto 1990, the Group Γ of diffeomeorphism of the circle, in reason of its relation to Mathematical Physics, seems to me a possible subject for this pluridisciplinar Conference under the general thema of "Special Functions". We constructed [Ma-Ma91] a quasi-invariant measure, which was used afterwards to construct some unitary representation of Γ.

Mathematical Physic is mainly interested in a central extension $\tilde{\Gamma}$ of Γ. Identifying the Lie algebra of Γ with the C^∞ functions on the circle T, the central extension is then defined by the cocycle

$$\gamma(u, v) := \int_T \dot{u}(v - \ddot{v})\, d\theta.$$

Denoting by $H(v)$ the Hilbert transform on the crcle of v, and by $\hat{v}(n)$ its n^{th} Fourier coefficient, we have the identity

$$\gamma(H(v), v) = \sum_n (|n| + |n|^3) |\hat{v}(n)|^2.$$

The r.h.s. correspond to the Sobolev norm $H_{3/2}$.

Theorem 12.1 [Ma99]. *The Sobolev norm 3/2 defines on Γ a "Riemannian metric" for which the law π_T of the associated Brownian is carried by a group of homemorphisms.*

The question of quasi-invariance of this measure is open.

Given G as in the preceeding section the *Free Loop Group* is defined as

$$L(G) := \bigcup_{g\in G} gL_e(G).$$

We can identify $L(G)$ with the space of maps $T \mapsto G$; its Lie algebra is identified $L(\mathcal{G})$ the space of free loops on $\mathcal{G}$. The rotation group operates on these two spaces. The $L(\mathcal{G})$-valued Brownian motion is defined by

$$\psi_t(\theta) := \sum_{n=1}^{+\infty} \frac{1}{n}(B_n(t)\cos n\theta + B'_n(t)\sin n\theta),$$

where B_n, B'_n are independent Brownian motions.

The brownian motion on $L(G)$ is defined by solving the following family of G-valued SDE depending upon the parameter θ

$$d_t g_{x,\theta}(t) = g_{(x,\theta)}(t) o\, d_t\psi_t(\theta).$$

It is possible to construct a continuous version relatively to (t,θ) of these equations: denoting $l_t = g_{x,*}(t)$ this version we get an $L(G)$-valued process with continuous trajectories.

There exists on $L(G)$ a left invariant "elliptic operator" $\Delta_{L(G)}$ which is the infinitesimal generator of the process l_t.

For each $t > 0$, consider the Brownian bridge measure $\mu^t_{L_e(G)}$ constructed in Section 11 for on $L_e(G)$. Define $\mu^t := \int g\mu^t_{L_e(G)}\, dg$ where dg is the Haar measure of G.

Theorem 12.2 (Airault-Ma 92). *Given f a cylindrical functions on $L(G)$, then we have*

$$\frac{d}{dt}\int f\, d\mu^t = \int \left(\Delta + \frac{1}{t}a\right) f\, d\mu^t$$

where

$$a = b - E(b), \quad b = \left\| \int_0^{2\pi} odl(\theta) l^{-1}(\theta) \right\|^2_{\mathcal{G}}.$$

Another point of view is to denote π_t the law at time t of the Brownian motion on $L(G)$, starting for $t = 0$ from the constant loop equal to e. A pending problem for about five years was the question of quasi-invariance of π^t:

Theorem 12.3 (Driver 97'). *Under the left action of $L^1(\mathcal{G})$, the measure π^t is quasi-invariant.*

The difficulty to prove such kind of theorem was the fact that the adjoint action $\mathrm{Ad}(l_x(t))$ is an unbouded operator on $L^1(\mathcal{G})$.

Driver takes the alternative approach to replace left invariant connection on $L(G)$ by the Levi-Civita connection for the underlying Hilbert-Riemann metric of $L(G)$; then it become possible to transfer to $L(G)$ the Bismut-Harnack estimate proved for a compact manifold M. The same strategy as before is used that is study of the geometry of the path space sitting above: $P(L(G))$. As in finite dimension this study depends upon a computation of the Ricci tensor of $L(G)$. Developing this point of view, Fang obtained the logarihmic Sobolev inequality on $P(L(G))$.

For all present and prospective developpements the proof of the existence of Ricci tensor (i.e., the summability of the trace defining Ricci) and its effective computation are questions of paramount importance: see [Sh-Ta], [Fa99] and also all the litterature in Mathematical Physic concerning this point.

Heat operator acts as an homotopy operator realizing a continuous deformation between the considered family of measures μ^t towards the initial measure, the Dirac mass at the starting point. In the case of gaussian measure of variance t on a vector space, for instance in the case of $L(\mathcal{G})$ considered in the previous paragraph, this homotopy can be reealized by the linear operator $\psi \mapsto e^{\lambda}\psi$ which is generated by the *dilatation vector field*: $\psi \mapsto -\psi$. An interesting fact is that the compression vector field exists also in a finite dimensional non linear setting (Airault-Ma96) and also on $L(G)$ (Mancino).

Conclusion. In these exchange of ideas between Geometry and Probability, what is the benefit for each discipline considered from its own sake?

In the case of finite dimensional riemannian geometry, stochastic comparison equation of Section 1 started to raise natural questions which have been subsequently considerably developed nowadays mainly by purely analytic tools do not involving probability. The advantage of probabilistic approach is to be often more conceptual. Probabilistic methods seems to be more difficult to replace in the case of study of boundaries at infinity, in particular for symmetric space of rank > 1.

The mean values formulas for harmonic forms leaded to a fully intrinsic proof of Atiyah-Singer Theorem. After the publication of this proof, purely analytic approach appears using on the frame bundle probabilistic ideas in a disguised analytic form. Here again probability methods have a conceptual impact upon some analytical subsequent developments.

Estimate of ground state of vector bundles stays a very intriguing problem which is not covered untill now by analytic methods. Nevertheless the interesting results avalaible by probabilistic methods do not answer simple geometrical questions as the stability of minimal surfaces.

Some cohomology vanishing theorems obtained by probabilistic approach have not get untill now an alternative proof by classical analytical methods see [El-Li-Ro] as a recent exemple.

The geometry of groups of infinite dimension appearing in Mathematical Physics seems to rely heavily on probabilistic arguments.

From the point of view of probability it can be sayed that the geometric point of view has fully revolutionized the field: Anticipative calculus, Quasi-sure Analysis, Stochastic Calculus of Variations, Stochastic Flows of Diffeomorphisms, Harnack inequalities, representation of martingales through stochastic integral, conditionning, our vision of all these purely probabilistic objects have been fully changed by the geometric point of view.

References

[Ai-El] S. Aida and D. Elworthy, Logarithmic Sobolev inequality on Path Spaces, C. R. Acad. Sci. Paris, **321** (1995), 97–102.

[Ai-Ma88] H. Airault et P. Malliavin, Intération géométrique sur l'espace de Wiener, Bul. Sci. Math., **112** (1988), 3–52.

[Ai-Ma92] H. Airault and P. Malliavin, Integration on Loop Group II: Heat Equation for the Wiener measure, J. Funct. Anal., **104** (1992), 71–109.

[Ai-Ma95] H. Airault and P. Malliavin, Semi-martingales with values in an euclidean vector bundle and Ocone's formula on a Riemannian manifold, Proc. Symp. Pure and Appl. Math., **57**, (M. Cranston and M. Pinsky ed.), Cornell (1993), Amer. Math. Soc. (1995), 173–193.

[Ai-Ma96] H. Airault and P. Malliavin, Integration by part formula and dilatation vector fields on elliptic probability spaces, Prob. Th. Rel. Fields, **106** (1996), 447–494.

[An-Dr] L. Anderson and B. Driver, Finite dimensional Approximations to Wiener Measure and Path space integral formulas on Manifolds, J. Funct. Anal., **165** (1999), 431–499.

[Be] G. Ben Arous, Flots et séries de Taylor stochastiques, Prob. Th. Rel. Fields, **81** (1989), 73–99.

[Bi84] J. M. Bismut, The Atiyah-Singer theorems, a probabilistic approach, J. Funct. Analysis, **57** (1984), 56–99.

[Bi84'] J. M. Bismut, "Large deviations and the Malliavin Calculus", Birkhäuser Progress in Math., **45**, 1984, pp. 216.

[Bo-Mw] V. Bogachev and E. Mayer-Wolf, Absolutely Continuous Flows Generated by Sobolev Class Vector Fields in finite or Infinite Dimensions, J. Funct. Analysis, **167** (1999), 1–68.

[Cl] J. M. C. Clark, The representation of functionals of the Brownian motion by stochastic integrals, Ann. Math. Statistics, **41** (1970), 1282–1295.

[Cr] A. B. Cruzeiro, Equations diffentielles sur l'espace de Wiener, J. Funct. Anal., **54** (1983), 206–227.

[Cr-Fa] A. B. Cruzeiro and S. Fang, L^2-estimate for Riemannian Anticipative Stochastic integrals, J. Funct. Anal., **143** (1997), 400–414.

[Cr-Ma96] A. B. Cruzeiro and P. Malliavin, Renormalized Differential Geometry on path spaces, I – Structural equation, curvature, J. Funct. Anal., **139** (1996), 119–181.

[Cr-Ma98] A. B. Cruzeiro and P. Malliavin, Non pertubative construction of invariant mesaures through confinement by curvature, J. Math. pures et appl., **77** (1998), 527–537.

[De-Ga-Ma] A. Debiard, B. Gaveau et E. Mazet, Théorèms de comparaison en géométrie, J. Res. Inst Kyoto Univ., **12** (1976), 391–429.

[Dr92] B. Driver, A Cameron-Martin type quasi-invariance theorem for Brownian motion on a Riemannian manifold, J. Funct. Anal., **110** (1992), 272–376.

[Dr97] B. Driver, Integration by part for heat kernel measure revisited, J. Math. pures et appl., **76** (1997), 703–737.

[Dr97'] B. Driver, Integration by prt and quasi-invariance for heat kernel measures on loop groups, J. Funct. Anal., **149** (1997), 470–547.

[Dr99] B. Driver, The Lie bracket of adapted vector fields on Wiener spaces, Appl. Math. Optim., **39,** No.2 (1999), 179–210.

[Dr-Lo] B. Driver and T. Lohrentz, Logarihmmic Sobolev inequalities for pinned loop groups, J. Funct. Anal., **140** (1996), 381–448.

[El-Le-Li] D. Elworthy, Y. Le Jan and X. M. Li, Integration by part for degenerated diffusions on path space and diffeomorphism group, C. R. Acad. Sci. Paris, **325** (1996), 921–926.

[El-Li] D. Elworthy and X. M. Li, Formula for derived heat semi-group, J. Funct. Anal, **125** (1994), 252–286.

[El-Li-Ro] D. Elworthy, X. M. Li and S. Rosenberg, Bounded and L^2 harmonic forms on universal cover, Geom. Funct. Anal., **8** (1998), 283–305.

[En-St] O. Enchev and D. Stroock, Towards a Riemannian Geometry on the path space over a riemannian manifold, J. Funct. Anal., **134** (1996), 392–416.

[Fa98] S. Fang, Stochastic Anticipative Calculus over the Path space over a compact riemannian manifold, J. Math. pures et appl., **77** (1998), 249–282.

[Fa99] S. Fang, Ricci tensors on some infinite dimensional Lie Algebras, J. Funct. Anal., **161** (1999), 132–161.

[Fa99'] S. Fang, Integration by part and logarithmic Sobolev inequality on the path space over loop groups, The Annals of Probability, **27** (1999), 664–683.

[Fa-Ma] S. Fang and P. Malliavin, Stochastic Analysis on the Path space of a riemannian manifold I. Markovian Stochastic Calculus, J. Funct. Anal., **118** (1993), 249–274.

[Ga] B. Gaveau, Principe de moindre action, propagation de la chaleur et estimées sous elliptiques sur certains groupes nilpotents, Acta Mathematica, **139** (1977), 95–153.

[Ga-Tr] B. Gaveau et P. H. Trauber, L'intégrale stochastique comme opérateur de divergence dans l'espace fonctionnel, J. Funct. Anal., **46** (1982), 230–238.

[Gr] L. Gross, Uniqueness of ground state for Schrödinger operators over Looop Groups, J. Funct. Anal., **112** (1993), 373–441.

[Hs] E. P. Hsu, Quasi invariance of the Wiener measure on the path space over a compact Riemann manifold, J. Funct. Anal., **134** (1995), 417–450.

[Ik-Ma] N. Ikeda and H. Matsumoto, Brownian motion on the hyperbolic plane and Selberg trace formula, J. Funct. Anal., **163** (1998), 63–110.

[Ik-Wa] N. Ikeda and S. Watanabe, "Stochastic differential equations and diffusion processes", North Holland, 1989, pp. 554.

[It50] K. Itô, Stochastic differential equation on a manifold, Nagoya Math. J., **1** (1950), 35–47.

[It62] K. Itô, The Brownian motion and tensor fields on Riemannian manifolds, Proc. ICM-62 Stockholm, 536–539.

[It90] K. Itô, "Positive generalized functionals on R^{∞}", Proc. Coll. White Noise analysis, Bielefeld 1989, World Scientific 1990, pp. 166–179.

[Kun] H. Kunita, "Stochastic flows and stochastic differential equations", Cambridge Univ. Press, 1988.

[Kus] Kusuoka, Analysis on Wiener Space II. Differential forms, J. Funct. Anal., **103** (1992), 229–274.

[Le] P. Lescot, Un théorème de désintégration en Analyse quasi-sure, Sém. Proba. XXVII, Springer Lecture notes, **1557** (1993), 256–275.

[Ma74] P. Malliavin, Asymptotic of the Green function of a riemannian manifold and Its stochastic integral, Proc. Nat. Acad. Sci., **71** (1974), 381–383.

[Ma74'] P. Malliavin, Formules de la moyenne, calcul des pertubations et théorémes d'annulation pour les formes harmoniques, J. Funct. Anal., **17** (1974), 274–291.

[Ma75] P. Malliavin, Paramétrix trajectorielle, C. R. Acad. Sciences Paris, **281** (1975), 241–245.

[Ma76] P. Malliavin, Stochastic Calculus of Variations and hypoelliptic operators, in "Proc. Int. Symp. on SDE", (K. Itô, ed.), Kyoto, 1976, pp. 195–263.

[Ma77] P. Malliavin, Un principe de transfert et son application au calcul stochastique, C. R. Acad. Sci. Paris, **284** (1977), 187–189.

[Ma78] P. Malliavin, "Géométrie Différentielle stochastique", Sém. Univ. Montréal **64**, 1978, p. 132.

[Ma82] P. Malliavin, Implicit functions in finite corank on the Wiener Space, Proc. the Taniguchi Symp. Stoch. Anal., Katata/Kyoto K. Itô ed., 369–386.

[Ma85] P. Malliavin, Intégrales Stochastiques oscillantes, C. R. Acad. Sciences Paris, **301** (1985), 141–143; 767–770 * **302** (1986), 73–78; 359–362; 481–486.

[Ma97] P. Malliavin, "Stochastic Analysis", Grundlehren der mathematischen Wissenschaften, Springer, 1997, pp. 343.

[Ma99] P. Malliavin, The canonic diffusion above the diffeomorphism group of the circle, C. R. Acad. Sci. Paris, **329** (1999), 325–330.

[Ma-Ma74] M. P. Malliavin et P. Malliavin, Factorisation et lois limites de la diffusion horizontale au-dessus d'un espace riemannien symétrique, Springer Lecture Notes in Math., **404** (1974), 155–210.

[Ma-Ma90] M. P. Malliavin and P. Malliavin, Integration on loop groups I: quasi invariant measures, J. Funct. Anal., **93** (1990), 207–237.

[Ma-Ma91] M. P. Malliavin and P. Malliavin, An infinitesimally quasi-invariant measure on the group of diffeomorphisms of the circle, in "Special functions", (M. Kashiwara and T. Miwa, eds.), ICM 90 Satellite Conference, Springer, Tokyo, 1991, pp. 234–244.

[Ma-Se] Z. Ma and Sen, Logarithmic Sobolev inequalities on Loop spaces, J. Funct. Anal. (1999).

[Ma-Sh] S. Manabe and I. Shigekawa, A Comparison Theorem for Eigenvalue of the covariant Laplacian, J. Funct. Anal., **94** (1990), 349–357.

[Ma-St] P. Malliavin and D. Stroock, Short time behaviour of the heat kernel and of its logarihmic derivatives, J. Differential Geom-

etry, **44** (1996), 550–570.
[Man] M. E. Mancino, Dilatation vector field on the Loop group, J. Funct. Anal., **166** (1999), 130–147.
[Ma-Ue] H. Matsumoto and N. Ueki, Spectral analysis of Schrödinger operators with magnetic field, J. Funct. Anal., **140** (1996), 218–255.
[Nu-Pa] D. Nualart and E. Pardoux, Stochastic Calculus with anticipating integrand, Prob. Th. Rel. Fields, **78** (1988), 535–581.
[Nu-Za] D. Nualart and M. Zakai, Generalized stochastic integrals and the Malliavin calculus, Prob. Th. Rel. Fields, **73** (1986), 255–280.
[Pr75] J. J. Prat, Convergence angulaire de la diffusion sur une variété à courbure négative, Comp. Rend. Acad. Sci. Paris, **286** (1975), 687–690.
[Pr93] J. J. Prat, Résolution de l'équation de Schrödinger par intégration stochastique du champ magnétique, Prob. Th. Rel. Fields, **95** (1993), 217–237.
[Sh86] I. Sigekawa, The de Rham-Hodge-Kodaira decomposition on an Abstract Wiener space, J. Math. Kyoto Univ., **26** (1986), 191–202.
[Sh94] I. Shigekawa, Eigenvalue problem for Schrödinger operator with magnetic field, J. Funct. Anal., **120** (1994), 344–379.
[Sh-Ta] I. Shigekawa and S. Taniguchi, A Kähler metric on based Loop Group and a covariant differentiation, in "Its Stochstic calculus and Probability Theory", (Ikeda, Watanabe, Fukushima, Kunita, eds.), Springer, Tokyo, 1996, pp. 327–347.
[St-Ta] D. W. Stroock and S. Taniguchi, Diffusions as integral curves of vector fields. Stratonovitch diffusion without Integration, in "Vol. in honour of E. Dynkin", (Freidlin, ed.), 1994, Birkhäuser Prog. in Prob., **34**, pp. 333–369.
[Su] H. Sugita, Positive generalized functions and potential theory over an abstract Wiener space, Osaka Math. J., **25** (1988), 665–696.
[Tan] S. Taniguchi, On the exponential decay of oscilatory Integrals on an abstract Wiener space, J. Funct. Anal., **154** (1998), 424–443.
[Ue] N. Ueki, Lower Bounds for the spectra of Schrödinger operators with magnetic fields, J. Funct. Anal., **120** (1994), 344–379.
[Wa] S. Watanabe, Lectures on stochastic differential equations and Malliavin Calculus, in "Lecture Tata Institute for fundamental research **73**", 1984, p. 111.

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