Advanced Studies in Pure Mathematics 30, 2001
Class Field Theory – Its Centenary and Prospect
pp. 313–323

Recent Progress on the Finiteness of Torsion Algebraic Cycles

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In this article we review recent results on the finiteness of torsion algebraic cycles on certain surfaces over number fields.

1. Let X be an algebraic variety over a field k. For an integer $i \geq 0$, let X^i be the set of schematic points of codimension i (equivalently, the set of integral closed subvarieties of codimension i of X). The Chow group of algebraic cycles of codimension d modulo rational equivalence is defined by

$$CH^d(X) := \mathrm{Coker}\left(\bigoplus_{x \in X^{d-1}} \kappa(x)^* \overset{\mathrm{div}}{\to} \bigoplus_{x \in X^d} \mathbb{Z}\right)$$

where $\kappa(x)$ denotes the residue field at x. Chow groups are natural generalization of Picard group, but little is known on their structure in general.

Bloch was the first to study the close relation between algebraic cycles and algebraic K-theory. Let $K_n(X)$ be the algebraic K-group defined by Quillen. Let $\mathcal{K}_n$ be the Zariski sheaf on X which is associated to the presheaf $U \mapsto K_n(\Gamma(U, \mathcal{O}_X))$. Define similarly $\mathcal{H}^n(\mathbb{Z}/l^m(r))$ by the sheafification of the étale cohomology functor $U \mapsto H^n_{\mathrm{et}}(U, \mu_{l^m}^{\otimes r})$ for a prime number $l \neq \mathrm{ch}(k)$. Then, if X is smooth we have the following isomorphisms called Bloch's formula:

$$CH^d(X) \simeq H^d_{\mathrm{Zar}}(X, \mathcal{K}_d),$$
$$CH^d(X)/l^m \simeq H^d_{\mathrm{Zar}}(X, \mathcal{H}^d(\mathbb{Z}/l^m(d))).$$

Received September 1, 1998.
Revised April 13, 1999.

On the other hand we have the Riemann-Roch theorem:

$$K_0(X)\otimes\mathbb{Q}\simeq\bigoplus_{d\geq 0}CH^d(X)\otimes\mathbb{Q}.$$

As to the structure of higher K-groups we have

Conjecture 1.1 (Bass). Let $\mathfrak{X}$ be a regular scheme of finite type over $\mathbb{Z}$. Then $K_i(\mathfrak{X})$ is a finitely generated abelian group for $i\geq 0$.

As a corollary of higher dimensional class field theory we have

Theorem 1.2 (Bloch[Bl3], Kato-Saito[K-S]). *For a scheme $\mathfrak{X}$ of finite type over $\mathbb{Z}$, the Chow group of zero-cycles $CH_0(\mathfrak{X})$ is a finitely generated abelian group.*

From now on, we mainly consider a projective smooth variety X over a number field k. Let $\mathfrak{X}$ be a proper smooth model of X over $\mathcal{O}_k[1/N]$ for some N. Then the natural maps

$$K_0(\mathfrak{X})\longrightarrow K_0(X),\ CH^d(\mathfrak{X})\longrightarrow CH^d(X)\ (d\geq 0)$$

are surjective, and we expect $CH^d(X)$ to be finitely generated.

Conjecture 1.3 (Tate[T], Beilinson[Be], Bloch[Bl4], Bloch-Kato [B-K]).
(i) For $d\geq 0$, $CH^d(X)$ is a finitely generated abelian group.
(ii) $\operatorname{rank}(CH^d(X)/CH^d(X)_{\mathrm{hom}})=\dim(H^{2d}_{\mathrm{et}}(\overline{X},\mathbb{Q}_p(d))^{\mathrm{Gal}(\overline{k}/k)})$
$=-\operatorname{ord}_{s=d+1}L(H^{2d}(X),s)$.
(iii) $\operatorname{rank}(CH^d(X)_{\mathrm{hom}})=\dim(H^1_f(k,H^{2d-1}_{\mathrm{et}}(\overline{X},\mathbb{Q}_p(d)))$
$=\operatorname{ord}_{s=d}L(H^{2d-1}(X),s)$.
Here, $CH^d(X)_{\mathrm{hom}}$ is the kernel of the cycle map $CH^d(X)\to H^{2d}_{\mathrm{et}}(\overline{X},\mathbb{Q}_p(d))$, and $H^1_f(k,-)\subset H^1(k,-)$ is a vector-space analog of the Selmer group defined by Bloch-Kato[B-K] (see §4).

Remark 1.4. If $d=1$, then $CH^1(X)=\operatorname{Pic}(X)$ and (i) follows from the Néron-Severi theorem and the Mordell-Weil theorem.

If X is an elliptic curve and $d=1$, (iii) is a part of the Birch-Swinnerton-Dyer conjecture.

2. On the finiteness of torsion part of Chow group of codimension two, there has been considerable progress for varieties over various fields not only number fields. In particular, for varieties with $H^2_{\mathrm{Zar}}(X,\mathcal{O}_X)=0$, we have many general results (see [CT2] and [CT3] for much more). The first example was the following:

Theorem 2.1 (Bloch[Bl2]). *Let X be a rational surface over a number field which is a conic bundle over $\mathbb{P}^1$. Then,* $\mathrm{Ker}(CH^2(X) \overset{\deg}{\to} \mathbb{Z})$ *is finite.*

This was generalized:

Theorem 2.2 ([Colliot-Thélène[CT1]]). *The same holds for every rational surface.*

While Bloch used the theory of quadratic forms, Colliot-Thélène proved rather simply using the following significant theorem of Merkurjev and Suslin called Hilbert 90 for K_2.

Theorem 2.3 ([Merkurjev-Suslin[M-S]). *For a field k and a prime number $l \neq \mathrm{ch}(k)$, we have an isomorphism $K_2(k)/l^n \simeq H^2(k, \mu_{l^n}^{\otimes 2})$.*

With this theorem and the Bloch-Ogus theory[B-O], Bloch gave the following exact sequence[Bl1] (cf. [CT2]):

$$\begin{aligned}(2.1)\qquad 0 \longrightarrow H^1_{\mathrm{Zar}}(X, \mathcal{K}_2) \otimes \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow\ & NH^3_{\mathrm{et}}(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \\ &\longrightarrow CH^2(X)\{p\} \longrightarrow 0\end{aligned}$$

where

$$\begin{aligned}&NH^3_{\mathrm{et}}(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \\ &\qquad := \mathrm{Ker}(H^3_{\mathrm{et}}(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \to H^3_{\mathrm{et}}(k(X), \mathbb{Q}_p/\mathbb{Z}_p(2))).\end{aligned}$$

Therefore, to show the finiteness of torsion of $CH^2(X)$, we are to show that the first group $H^1_{\mathrm{Zar}}(X, \mathcal{K}_2)$ is sufficiently large. The most general result known so far is the following:

Theorem 2.4 (Colliot-Thélène-Raskind[CT-R], Salberger[Sa]). *Let X be a projective smooth variety over a number field. If $H^2_{\mathrm{Zar}}(X, \mathcal{O}_X) = 0$, then $CH^2(X)_{\mathrm{tor}}$ is finite.*

3. One of the crucial difficulties in the situation $H^2_{\mathrm{Zar}}(X, \mathcal{O}_X) \neq 0$ is that we need essentially new elements in $H^1_{\mathrm{Zar}}(X, \mathcal{K}_2)$ called indecomposable in the sense that they are not contained in the image of the product map

$$\mathrm{Pic}(X) \otimes k^* = H^1_{\mathrm{Zar}}(X, \mathcal{K}_1) \otimes H^0_{\mathrm{Zar}}(X, \mathcal{K}_1) \longrightarrow H^1_{\mathrm{Zar}}(X, \mathcal{K}_2)$$

(even after any finite extension of the base field).

Importance of the group $H^1_{\mathrm{Zar}}(X,\mathcal{K}_2)$ can also be seen from the following localization sequence in K-theory. Let $\mathfrak{X}$ be a proper smooth model of X over $\mathcal{O}_k[1/N]$ and X_v be its closed fiber over a prime v. Then we have an exact sequence:

(3.1)

$$\longrightarrow H^1_{\mathrm{Zar}}(X,\mathcal{K}_2) \xrightarrow{\partial} \bigoplus_{v\nmid N} \mathrm{Pic}(X_v) \longrightarrow CH^2(\mathfrak{X}) \longrightarrow CH^2(X) \longrightarrow 0.$$

It is known by [CT-R] that the p-primary torsion subgroup $CH^2(\mathfrak{X})\{p\}$ is a cofinitely generated $\mathbb{Z}_p$-module (i.e. a direct sum of finite copies of $\mathbb{Q}_p/\mathbb{Z}_p$ and a finite p-group). Therefore, if $\mathrm{Ker}(CH^2(\mathfrak{X}) \to CH^2(X)) = \mathrm{Coker}(\partial)$ is torsion then $CH^2(X)\{p\}$ is also cofinitely generated as a $\mathbb{Z}_p$-module and hence the n-torsion of $CH^2(X)$ is finite for any n.

For the self-product $X = E\times E$ of a modular elliptic curve over $\mathbb{Q}$, we have the following elements in $H^1_{\mathrm{Zar}}(X,\mathcal{K}_2)$ constructed by Flach[Fl2] and Mildenhall[M] using the theory of modular curves and modular units. For a prime p where E has good reduction there is an element of $H^1_{\mathrm{Zar}}(X,\mathcal{K}_2)$ whose image by the boundary map ∂ of (3.1) is trivial at $l \neq p$, and a non-zero constant multiple of the class of the graph of the Frobenius endomorphism of E (mod p) at $l = p$. Mildenhall used them to show the torsionness of $\mathrm{Ker}(CH^2(\mathfrak{X}) \to CH^2(X))$, and Flach used them to detect the Selmer group associated to the Galois representation $\mathrm{Sym}^2(T_p(E))$.

Based on their results Langer and Saito proved

Theorem 3.1 ([Langer-Saito[L-S]). *Let E be a semi-stable elliptic curve over $\mathbb{Q}$ with conductor N. Then $CH^2(E\times E)\{p\}$ is finite for $p \nmid 6N$, and trivial for almost all primes.*

When E has complex multiplication we have

Theorem 3.2 ([Langer[L1], Langer-Raskind[L-R], [O1]). *Let E be an elliptic curve over $\mathbb{Q}$ with complex multiplication by the ring of integers in an imaginary quadratic field K. Let N be its conductor and $p \nmid 6N$ be a prime number. Then, under some assumption which is satisfied if N is a power of a prime, $CH^2(E\times E)\{p\}$ and $CH^2(E_K \times E_K)\{p\}$ are finite, where $E_K := E \otimes_{\mathbb{Q}} K$. Moreover, the same holds for the associated Kummer surfaces* $\mathrm{Km}(E\times E)$ *and* $\mathrm{Km}(E_K \times E_K)$.

Remark 3.3. For an abelian surface A over a field with characteristic $\neq 2$, the Kummer surface $\mathrm{Km}(A)$ associated to A is a $K3$-surface obtained by blowing up sixteen singularities of the quotient $A/\{\pm 1\}$ corresponding to the points of order 2 on A. Also in the semi-stable case, we can show in the same manner the finiteness of $CH^2(\mathrm{Km}(E\times E))\{p\}$ for E and p as in Theorem 3.1.

Finally we introduce

Theorem 3.4 ([O1]). *Let F be the Fermat quartic surface over $\mathbb{Q}$ defined by*

$$x_0^4 + x_1^4 = x_2^4 + x_3^4.$$

Put $K = \mathbb{Q}(\sqrt{-1})$, $F_K = F \otimes_{\mathbb{Q}} K$, and let $p \nmid 6$ be a prime number. Then, $CH^2(F)\{p\}$ and $CH^2(F_K)\{p\}$ are finite.

Remark 3.5. Let Y be a projective smooth variety over a p-adic field k' (i.e. $[k' : \mathbb{Q}_p] < \infty$) which has a projective smooth model $\mathfrak{Y}$ over the integer ring. Then, if $H^2_{\mathrm{Zar}}(Y, \mathcal{O}_Y) = 0$, $CH^2(Y)_{\mathrm{tor}}$ is known to be finite (cf. [CT2])

If $\mathrm{Ker}(CH^2(\mathfrak{Y}) \to CH^2(Y))$ is torsion, then this is finite, and the prime-to-p part of $CH^2(Y)_{\mathrm{tor}}$ is finite because $CH^2(\mathfrak{Y})_{\mathrm{tor}}$ has the property [Ra]. For X and p as in Theorems 3.1, 3.2 and 3.4, the first step of their proofs show the finiteness of $\mathrm{Ker}(CH^2(\mathfrak{Y}) \to CH^2(Y))$ for $Y = X \otimes_{\mathbb{Q}} k'$.

Other examples over p-adic fields are the product of two (possibly different) elliptic curves [Sp] and a class of Hilbert-Blumenthal surfaces [L2].

4. Now let us recall the outline of the method of Langer-Saito[L-S] which is also used in [L1] and [O1]. Let X be one of the surfaces of Theorems 3.1, 3.2 or 3.4 over $\mathbb{Q}$, and p be a prime number satisfying the assumption. The proof for X_K is parallel. By (2.1), since $CH^2(X)\{p\}$ is cofinitely generated by the above result of Mildenhall and the corresponding results for the Kummer surface [L1] [O1], and for the Fermat quartic surface [O1], it is enough to show that $H^1_{\mathrm{Zar}}(X, \mathcal{K}_2) \otimes \mathbb{Q}_p/\mathbb{Z}_p$ is the maximal divisible subgroup of $NH^3_{\mathrm{et}}(X, \mathbb{Q}_p/\mathbb{Z}_p(2))$. We modify

$NH^3_{\mathrm{et}}(X, \mathbb{Q}_p/\mathbb{Z}_p(2))$ by the subgroup

$$\begin{aligned}&K_N H^3_{\mathrm{et}}(X, \mathbb{Q}_p/\mathbb{Z}_p(2))\\ &\qquad := \mathrm{Ker}(NH^3_{\mathrm{et}}(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \to H^3_{\mathrm{et}}(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))),\end{aligned}$$

and reduce to prove

$$H^1_{\mathrm{Zar}}(X, \mathcal{K}_2) \otimes \mathbb{Q}_p/\mathbb{Z}_p = K_N H^3_{\mathrm{et}}(X, \mathbb{Q}_p/\mathbb{Z}_p(2))_{\mathrm{div}}.$$

Then these groups are embedded by using Hochschild-Serre spectral sequence into the Galois cohomology group $H^1(\mathbb{Q}, A)$ where $A := H^2_{\mathrm{et}}(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))$. Taking further the localizations with local conditions we obtain

$$\begin{aligned}(4.1)\qquad H^1_{\mathrm{Zar}}(X, \mathcal{K}_2) \otimes \mathbb{Q}_p/\mathbb{Z}_p &\hookrightarrow K_N H^3_{\mathrm{et}}(X, \mathbb{Q}_p/\mathbb{Z}_p(2))_{\mathrm{div}}\\ &\hookrightarrow H^1(\mathbb{Q}, A) \xrightarrow{\alpha} \bigoplus_{\text{all } l} \frac{H^1(\mathbb{Q}_l, A)}{H^1_f(\mathbb{Q}_l, A)}\end{aligned}$$

where $H^1_f(\mathbb{Q}_l, A)$ is the unramified part for $l \neq p$, and defined by using the Fontaine ring B_{cris} for $l = p$ (see [B-K]). The Selmer group of A is defined by

$$S(\mathbb{Q}, A) := \mathrm{Ker}(\alpha),$$

and its analogue

$$H^1_f(\mathbb{Q}, V) := \mathrm{Ker}\left(H^1(\mathbb{Q}, V) \to \bigoplus_{\text{all } l} \frac{H^1(\mathbb{Q}_l, V)}{H^1_f(\mathbb{Q}_l, V)}\right)$$

is similarly defined.

Then we are to prove:

(i) In (4.1), the image of the first group in the final direct sum coincides with that of the second one;

(ii) The Selmer group $S(\mathbb{Q}, A)$ is finite.

The key to show (i) is the following commutative diagram with the vertical isomorphism for $l \nmid N$. Define $V := H^2_{\mathrm{et}}(\overline{X}, \mathbb{Q}_p(2))$, and let the subspace $H^1_g(\mathbb{Q}_l, V) \subset H^1(\mathbb{Q}_l, V)$ be the whole space for $l \neq p$ and the one defined in [B-K] using the Fontaine ring B_{dR} for $l = p$. Then we have

$$\begin{array}{ccc} H^1_{\mathrm{Zar}}(X,\mathcal{K}_2)\otimes\mathbb{Q}_p & \longrightarrow & H^1_g(\mathbb{Q}_l,V)/H^1_f(\mathbb{Q}_l,V) \\ & {\scriptstyle \partial_l}\searrow & \downarrow\wr \\ & & \mathrm{Pic}(X_l)\otimes\mathbb{Q}_p \end{array}$$

where ∂_l is the l-part of the boundary map of (3.1) tensored with $\mathbb{Q}_p$. The proof of its $l=p$ part requires recent results in p-adic Hodge theory. Since we know that ∂_l is surjective for $l \nmid N$, the composition of (4.1) is "almost" surjective modulo many delicate arguments such as the difference between $\mathbb{Q}_p$-coefficients and $\mathbb{Q}_p/\mathbb{Z}_p$-coefficients or the bad reduction primes.

The part (ii) is more arithmetic in nature. When $X = E\times E$ for a semi-stable E, the Selmer group is studied in [Fl2]. When E has CM, there are results [Fl1], [W], [D], all of which are based on the two-variable Iwasawa main conjecture proved by Rubin[Ru].

5. Once the result for $E\times E$ (or $E_K \times E_K$) is obtained, it is not so difficult to prove the statement for the associated Kummer surface. One should notice, however, that the proof will be more complicated if the 2-torsion points of E is not defined over $\mathbb{Q}$ (or the CM field K), in which case we have possibly infinite Selmer group and we need some tricks as in [O1].

Finally, we consider the Fermat quartic surface F. The key is a geometric construction connecting F to a Kummer surface which enables us to use the results on $E\times E$. It is known [Ka-Sh] that F is constructed from the product of two copies of the Fermat quartic curve C, by taking blowing-up, quotient by a finite group and blowing-down. We can find a finite morphism $C \longrightarrow E$ where E is an elliptic curve, such that it induces a finite morphism $\widetilde{F} \longrightarrow \mathrm{Km}(E\times E)$ of degree 2 where $\widetilde{F}$ is the blowing-up of F at certain eight points. This E has complex multiplication by $\mathbb{Z}[\sqrt{-1}]$.

Pull-backs of this morphism induce a commutative diagram:

$$\begin{array}{ccc} \mathrm{Pic}(\overline{\mathrm{Km}(E\times E)})\otimes\mathbb{Q}_p & \hookrightarrow & \mathrm{Pic}(\overline{\widetilde{F}})\otimes\mathbb{Q}_p \\ \downarrow & & \downarrow \\ H^2_{\mathrm{et}}(\overline{\mathrm{Km}(E\times E)},\mathbb{Q}_p(1)) & \hookrightarrow & H^2_{\mathrm{et}}(\overline{\widetilde{F}},\mathbb{Q}_p(1)) \end{array}$$

where the vertical maps are the cycle maps. We can define explicitly eight divisors on $\widetilde{F}$ whose divisor (resp. cohomology) classes generate the cokernel of the upper (resp. lower) map. This means that both the Picard group and the second cohomology group of $\widetilde{F}$ (then, of F) are described by those of $\mathrm{Km}(E\times E)$ and the classes of the explicit divisors.

The crucial difference from the other cases is the fact that the Selmer group for F or F_K is not finite.

Theorem 5.1 ([O1]). *Let* $A = H^2_{\mathrm{et}}(\overline{F}, \mathbb{Q}_p/\mathbb{Z}_p(2))$. *Then we have*

$$\mathrm{corank}_{\mathbb{Z}_p}(S(\mathbb{Q}, A)) = 2, \ \mathrm{corank}_{\mathbb{Z}_p}(S(K, A)) = 4.$$

This breaks the part (ii), but we can separate from $A = H^2_{\mathrm{et}}(\overline{F}, \mathbb{Q}_p/\mathbb{Z}_p(2))$ a part which causes the infinite Selmer groups and treat this directly without taking localizations. This part is controlled by the classes of the eight divisors mentioned above and the multiplicative group of $\mathbb{Q}(\zeta_8)$ because the divisors are defined only over $\mathbb{Q}(\zeta_8)$. In view of the exact sequence

$$0 \longrightarrow \mathcal{O}_k^* \otimes \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow S(k, \mathbb{Q}_p/\mathbb{Z}_p(1)) \longrightarrow \mathrm{Pic}(\mathcal{O}_k)\{p\} \longrightarrow 0$$

for a number field k, this explains why our Selmer groups are infinite.

By a conjecture of Bloch-Kato[B-K], the $\mathbb{Z}_p$-corank of the Selmer group of a general motive should coincide with the rank of a certain motivic cohomology group defined by K-theory, and then with the order of vanishing at an integer point of the L-function by Beilinson's conjecture [Be]. These are wide generalization of Conjecture 1.3. Note that for a $\mathbb{Q}_p$-representation V of $\mathrm{Gal}(\overline{k}/k)$ of geometric origin, its Galois stable $\mathbb{Z}_p$-lattice T and $A = V/T$, we have $\mathrm{corank}_{\mathbb{Z}_p}(S(k, A)) = \dim_{\mathbb{Q}_p}(H^1_f(k, V))$.

In our situation the desired equalities are

$$\begin{aligned}\mathrm{corank}_{\mathbb{Z}_p}(S(\mathbb{Q}, A)) &= \mathrm{rank}(H^3_{\mathcal{M}}(F, \mathbb{Q}(2))_{\mathbb{Z}}) &&= \mathrm{ord}_{s=1} L(H^2(F), s),\\ \mathrm{corank}_{\mathbb{Z}_p}(S(K, A)) &= \mathrm{rank}(H^3_{\mathcal{M}}(F_K, \mathbb{Q}(2))_{\mathbb{Z}}) &&= \mathrm{ord}_{s=1} L(H^2(F_K), s).\end{aligned}$$

Remark 5.2. We have $H^3_{\mathcal{M}}(-, \mathbb{Q}(2)) \simeq H^1_{\mathrm{Zar}}(-, \mathcal{K}_2) \otimes \mathbb{Q}$. The subscript $\mathbb{Z}$ means the integral part, that is, the elements extending to an integral model over the whole integer ring.

We have in fact

Theorem 5.3 ([O1]).
(i) $\mathrm{ord}_{s=1} L(H^2(F), s) = 2$, $\mathrm{ord}_{s=1} L(H^2(F_K), s) = 4$.
(ii) *There exist two (resp. four) elements in* $H^3_{\mathcal{M}}(F, \mathbb{Q}(2))_{\mathbb{Z}}$ *(resp.*

$H^3_{\mathcal{M}}(F_K, \mathbb{Q}(2))_{\mathbb{Z}})$ *whose image by the Chern class map generate the maximal divisible subgroup of the Selmer group.*

Since we have explicit description of the Picard group, (i) follows from the functional equation and the Tate conjecture ([Fa1] using [Ka-Sh]). The elements of (ii) are constructed using the eight specific divisors mentioned above and certain units of $\mathbb{Q}(\zeta_8)$. These are decomposable over $\mathbb{Q}(\zeta_8)$, but not over $\mathbb{Q}$ nor K.

Remark 5.4. This method is generalized in [O2] to construct elements in $H^{2m+1}_{\mathcal{M}}(X, \mathbb{Q}(m+r))_{\mathbb{Z}}$ $(r \geq 1)$, for any projective smooth variety X over a number field. The image of these elements under the Chern class map generate the Selmer group of $V'(r) \subset H^{2m}_{\text{et}}(\overline{X}, \mathbb{Q}_p(m+r))$ where V' is the sub-representation of $H^{2m}_{\text{et}}(\overline{X}, \mathbb{Q}_p(m))$ generated by the classes of cycles on $\overline{X}$ of codimension m.

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Added in Proof. After the first manuscript was written, Langer [L3] axiomatized the method for the finiteness of $CH^2(X)_{\rm tor}$ including the case where the Selmer group is not finite, in which case the generalization of Theorem 5.3 (see Remark 5.4) should be useful.

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