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On Classification of Semisimple Algebraic Groups

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In this note we give a survey of the classification theory of semisimple algebraic groups over a number field. As is well known, for a given field F, the F-isomorphism class of such a group G defined over F is determined up to F-isogeny by the "Γ-diagram" $\Sigma_F(G)$ and by the F-isomorphism class of the anisotropic kernel of G (see §2; [Sa1], [T]). On the other hand, if G belongs to an inner type of an F-quasisplit group G_0 with center Z, then the F-equivalence class of an "inner F-form" (G, f) of G_0 corresponds in a one-to-one way to a cohomology class in $H^1(F, G_0/Z)$, which in turn determines an element in $H^2(F, Z)$, denoted by $\gamma_F(G, f)$ (see §1; [Sa2]).

For $F = \mathbb{R}$ (the field of real numbers), it is well known that the $\mathbb{R}$-isogeny class of G is uniquely determined only by the Γ-diagram $\Sigma_{\mathbb{R}}(G)$ (cf. [A], [Sa3], [T]), while for a $\mathfrak{p}$-adic field F, a fundamental result of Kneser [K1] says that the F-equivalence class of an inner F-form (G, f) of a simply connected G_0 is uniquely determined only by the cohomological invariant $\gamma_F(G, f)$. In treating the case of a number field, the key step is in the so-called local-global principle, or Hasse principle, which also plays an important role in the class field theory. The Hasse principle for $H^1(F, G_0)$ (G_0 simply connected) had been established by Kneser and Harder ([K2], [K3], [H1]) except for the case of (E_8), which was recently settled by Chernousov [Cher] (1989). On the other hand, for Γ-diagrams, one can deduce the Hasse principle from a result in [H2] (see §4). Combining these results, one obtains a complete picture of the classification. We can formulate the main result in the following form.

MAIN THEOREM. *Let F be an algebraic number field of finite degree and let $V_{\infty,1}$ denote the set of all real places of F. Let G_0 be an F-quasisplit simply connected semisimple algebraic group over F and let Z be the center of G_0. Suppose there are given a collection of Γ-diagrams $\{\Sigma^{(v)}\ (v \in V_{\infty,1})\}$ over $\mathbb{R}$ and $c \in H^2(F, Z)$ such that, for each $v \in V_{\infty,1}$,*

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there exists an inner F_v-form $(G^{(v)}, f^{(v)})$ of G_0 with $\gamma_{\mathbb{R}}(G^{(v)}, f^{(v)}) = c_v$ and $\Sigma_{\mathbb{R}}(G^{(v)}) = \Sigma^{(v)}$. Then there exists an inner F-form (G, f) of G_0 (uniquely determined up to F-equivalence) such that $\gamma_F(G, f) = c$ and (G, f) is F_v-equivalent to $(G^{(v)}, f^{(v)})$ (hence $\Sigma_{F_v}(G) = \Sigma^{(v)}$) for all $v \in V_{\infty,1}$. (See, §5, Th. 7, 8.)

It should be noted that this result is quite analogous to the classical result of Minkowski [Mi] (1891) on the equivalence of quadratic forms with coefficients in $\mathbb{Q}$. Here we see that the F-equivalence class of (G, f) is uniquely determined by the cohomological invariant $\gamma_F(G, f)$, which is an analogue of the "Hasse invariant", and a collection of Γ-diagrams $\{\Sigma^{(v)}\ (v \in V_{\infty,1})\}$ (or more precisely $\{(G^{(v)}, f^{(v)})\}$) satisfying the above consistency condition, which is an analogue of the "signature(s)" of a quadratic form.

The above main theorem is essentially contained in a result of Sansuc ([San], Cor.4.5), which was generalized quite recently to the case of reductive groups by Borovoi ([Bo2], Th.5.11). In §5 of this note, we give a direct proof of it based on the Hasse principle. An explicit determination of the relevant invariants is given in §6.

§1. Cohomological invariants ([Se], [Sa2]).

Let F be a field of characteristic zero and G_0 an algebraic group defined over F. Let Z denote the center of G_0 and set $\overline{G}_0 = G_0/Z$. Then $\overline{G}_0$ can naturally be identified with the group of inner automorphisms of G_0, $\mathrm{Inn}(G_0)$, by the correspondence $\overline{g} \longleftrightarrow I_g$ $(g \in G_0)$, $\overline{g}$ and I_g denoting the class of g mod Z and the inner automorphism $I_g : x \mapsto gxg^{-1}$ $(x \in G_0)$, respectively.

By an *inner F-form* of G_0 we mean a pair (G, f) formed of an algebraic group G defined over F and an $\overline{F}$-isomorphism $f : G \to G_0$ such that for all $\sigma \in \Gamma = \mathrm{Gal}(\overline{F}/F)$ one has $\varphi_\sigma = f^\sigma \circ f^{-1} \in \mathrm{Inn}(G_0)$, $\overline{F}$ denoting the algebraic closure of F. Two inner F-forms (G, f) and (G', f') are said to be *F-equivalent*, if there exists an F-isomorphism $\varphi : G \to G'$ such that $f' \circ \varphi \circ f^{-1} \in \mathrm{Inn}(G_0)$. Sometimes, G alone is called an inner F-form of G_0, or G and G_0 are said to be in the same inner type over F, if there exists an isomorphism $f : G \to G_0$ such that (G, f) is an inner F-form of G_0 in the above sense. In that case, two isomorphisms f, f' of G onto G_0 satisfying this condition are said to be *F-equivalent* if (G, f) and (G, f') are F-equivalent in the above sense.

Let (G, f) be an inner F-form of G_0. Then in the above notation it is clear that (φ_σ) is a (continuous) 1-cocycle of Γ in $\overline{G}_0 \cong \mathrm{Inn}(G_0)$, i.e., it satisfies the condition $\varphi_\sigma^\tau \varphi_\tau = \varphi_{\sigma\tau}$ for all σ, $\tau \in \Gamma$. We denote the cohomology class of (φ_σ) in $H^1(F,\overline{G}_0)$ by $c(G,f)$, or by $c_F(G,f)$ if F is to be specified. Writing $\varphi_\sigma = I_{g_\sigma}$ with $g_\sigma \in G_0(\overline{F})$, one has

$$c_{\sigma,\tau} = g_\sigma^\tau g_\tau g_{\sigma\tau}^{-1} \in Z,$$

and it is clear that $(c_{\sigma,\tau})$ is a (continuous) 2-cocycle of Γ in Z. The cohomology class of $(c_{\sigma,\tau})$ in $H^2(F,Z)$ is denoted by $\gamma(G,f)$ or $\gamma_F(G,f)$. It is clear that these cohomology classes depend only on the F-equivalence class of the inner F-form (G,f).

From the exact sequence

$$1 \to Z \to G_0 \to \overline{G}_0 \to 1$$

one obtain an exact sequence

$$\text{(1)} \qquad \cdots \to H^1(F,Z) \xrightarrow{\alpha} H^1(F,G_0) \xrightarrow{\beta} H^1(F,\overline{G}_0) \xrightarrow{\delta} H^2(F,Z).$$

By the definition one has $\gamma(G,f) = \delta(c(G,f))$. Note that, since Z is abelian, $H^1(F,Z)$ and $H^2(F,Z)$ have a structure of abelian group, while $H^1(F,G_0)$ and $H^1(F,\overline{G}_0)$ are just a set with a distinguished element 1.

Now, conversely, suppose there is given an element $\xi \in H^1(F,\overline{G}_0)$. Let (φ_σ) be a 1-cocycle representing ξ and let $\varphi_\sigma = I_{g_\sigma}$. Then one can define a new action of Γ on $G_0(\overline{F})$ by

$$\text{(2)} \qquad x^{[\sigma]} = g_\sigma^{-1} x^\sigma g_\sigma \qquad \text{for} \quad x \in G_0(\overline{F}),$$

which defines an F-form of G_0, denoted by $(G_0)_\xi$. Then, writing f for the identity map $(G_0)_\xi \to G_0$, one has an inner F-form $((G_0)_\xi, f)$ of G_0, whose F-equivalence class depends only on the cohomology class ξ, and one has $c((G_0)_\xi, f) = \xi$. Thus we see that the set of F-equivalence classes of inner F-forms of G_0 is in one-to-one correspondence with the cohomology set $H^1(F,\overline{G}_0)$. Clearly, one has $c(G,f) = 1$ if and only if f is F-equivalent to an F-isomorphism.

The following lemma ([Se], Ch.I, 5.7) will be useful later.

Lemma 1. *Let (φ_σ) and (ψ_σ) be 1-cocycles representing ξ, $\eta \in H^1(F,\overline{G}_0)$, respectively, and set $G = (G_0)_\xi$ and $\overline{G} = G/(center)$. Then $(\varphi_\sigma^{-1}\psi_\sigma)$ is a 1-cocycle of Γ in $\overline{G}(\overline{F})$ and, denoting its cohomology class by $\xi^{-1}\eta$, one has (for a fixed ξ) a bijective map*

$$\eta \in H^1(F,\overline{G}_0) \mapsto \xi^{-1}\eta \in H^1(F,\overline{G}).$$

The proof is straightforward. It is clear that, if (G',f') is an inner F-form of G_0 corresponding to η, then $(G', f^{-1}f')$ is an inner F-form of

G corresponding to $\xi^{-1}\eta$. If one identifies the center of G with Z by f, then one has

$$\delta(\xi^{-1}\eta) = \delta(\xi)^{-1}\delta(\eta)$$

in $H^2(F, Z)$. Since the sequence (1) (for G) is exact, it follows that

$$\xi^{-1} \cdot \delta^{-1}(\delta(\xi)) = \mathrm{Im}(H^1(F, G) \to H^1(F, \overline{G})). \tag{3}$$

§2. Γ-diagrams ([Sa3], [T]).

From now on, we assume that G_0 (and hence G, G', etc.) is a (connected) simply connected semisimple algebraic group defined over F. Let T be a maximal torus in G defined over $\overline{F}$ and let $X = X(T)$ denote the character module of T. Then one has

$$X \cong \mathbb{Z}^l, \ l = \dim T = \text{rank } G.$$

Let $\Phi = \Phi(G, T) \subset X$ be a root system of G relative to T and let Δ be a basis of Φ; we call such a pair (T, Δ) a "coordinate" (defined over $\overline{F}$) in G. Let (T', Δ') be another coordinate in G. Then, as is well known, there exists $\varphi \in \mathrm{Inn}(G)$ such that one has $\varphi(T) = T'$, $\varphi^*(\Delta) = \Delta'$, where $\varphi^* \in {}^t(\varphi|T)^{-1}$; for simplicity, we write

$$\varphi : (T, \Delta) \to (T', \Delta').$$

The inner automorphism φ with this property is uniquely determined up to a right multiplication by I_g with $g \in T$; hence $\varphi \mid T$ and φ^* are uniquely determined.

Now, let $\Gamma = \mathrm{Gal}(\overline{F}/F)$. Then for every $\sigma \in \Gamma$ there exists $\psi_\sigma \in \mathrm{Inn}(G)$ such that

$$\psi_\sigma : (T, \Delta) \to (T^\sigma, \Delta^\sigma).$$

We set

$$\chi^{[\sigma]} = \psi_\sigma^{*-1}(\chi^\sigma) \qquad \text{for all} \quad \chi \in X, \tag{4}$$

which is well defined and gives a new action of Γ on X leaving Δ invariant (as a whole). Moreover, this Galois action, called a $[\Gamma]$-*action* (or "$*$-action" in [T]), is defined intrinsically, independently of the choice of coordinates (defined over $\overline{F}$); it is also inherited to all groups in the same inner type. In fact, let (G', f') be another F-form of G_0, (T', Δ') a coordinates (defined over $\overline{F}$) in G', and let

$$\psi'_\sigma : (T', \Delta') \to (T'^\sigma, \Delta'^\sigma)$$

with $\psi'_\sigma \in \mathrm{Inn}(G')$. Then there exists an $\overline{F}$-isomorphism $\varphi : G \to G'$ such that one has $\varphi \circ f^{-1} \circ f' \in \mathrm{Inn}(G')$ and $\varphi : (T, \Delta) \to (T', \Delta')$. If (G', f') is an inner F-form of G, then from $\varphi^\sigma \circ \varphi^{-1} \in \mathrm{Inn}(G')$, one has $\psi'_\sigma \circ \varphi = \varphi^\sigma \circ \psi_\sigma$ on T, whence follows that

$$\varphi^*(\chi)^{[\sigma]'} = \varphi^*(\chi^{[\sigma]}) \qquad \text{for all} \quad \chi \in X,\ \sigma \in \Gamma, \tag{5}$$

i.e., φ^* is a $[\Gamma]$-isomorphism of X onto $X' = X(T')$ (and the converse is also true).

We call a coordinate (T, Δ) in G F-*admissible* if the following two conditions are satisfied.

(i) T is defined over F and contains a maximal F-split torus A in G.

(ii) Let X_0 denote the annihilater of A in X. Then the basis Δ is "adapted to X_0" in the sense that there exists a linear order in X for which all $\alpha_i \in \Delta$ are positive and the following condition is satisfied:

$$\chi, \chi' \in X,\ \chi > 0,\ \chi \equiv \chi' \not\equiv 0 \pmod{X_0} \Rightarrow \chi' > 0.$$

Let (T, Δ) be an F-admissible coordinate in G and set

$$\Phi_0 = \Phi \cap X_0,\ \Delta_0 = \Delta \cap X_0,$$

$$\overline{\Phi} = \pi(\Phi - \Phi_0),\ \overline{\Delta} = \pi(\Delta - \Delta_0),$$

π denoting the projection $X \to \overline{X} = X/X_0 = X(A)$. Then it is known (e.g. [Sa3]) that Φ_0 is a (closed) subsystem of Φ, of which Δ_0 is a basis, and that $\overline{\Phi}$ is a system of F-roots of G relative to A (which becomes a root system in a wider sense) and $\overline{\Delta}$ is a basis of $\overline{\Phi}$. The closed (semisimple) subgroup of G corresponding to Δ_0, denoted by $G(\Delta_0)$, coincides with the semisimple part of $Z(A)$ (centralizer of A) and is called the (semisimple) "anisotropic kernel" of G over F (relative to (T, Δ)). Moreover it is known that, for $\varphi = I_g$ with $g \in N(T)$ (normalizer of T), the coordinate $(T, \varphi^*(\Delta))$ is F-admissible if and only if one has $g \in N(A)T$ and that, in particular, for $\varphi = \psi_\sigma$ one has $g \in Z(A)T$. It follows that Δ_0 is $[\Gamma]$-invariant and the $[\Gamma]$-orbit decomposition of $\Delta - \Delta_0$ is given by

$$\Delta - \Delta_0 = \bigcup_{\gamma_i \in \overline{\Delta}} \pi^{-1}(\gamma_i) \cap \Delta. \tag{6}$$

Note that, if (T', Δ') is another F-admissible coordinate in G with a maximal F-split torus A' and if $\varphi \in \mathrm{Inn}(G)$ and $\varphi : (T, \Delta) \to (T', \Delta')$, then one has automatically $\varphi(A) = A'$ (see Lem. 2 in §4). Thus Δ_0-part of Δ is also intrinsically determined, independently of the choice of F-admissible coordinate (T, Δ).

As usual, the basis Δ is expressed by a Dynkin diagram. The system $\Sigma = (\Delta, \Delta_0, [\Gamma])$ formed of a Dynkin diagram Δ, a $[\Gamma]$-action on Δ, and Δ_0 will be called a *Γ-diagram* (or "Tits index", or "Satake diagram") of G relative to (T, Δ). We express $\alpha \in \Delta_0$ by a *black* vertex and $\alpha \in \Delta - \Delta_0$ by a *white* vertex. As noted above, the Γ-diagram of G is uniquely determined up to "congruence" (in an obvious sense) only by the F-structure of G. Hence we write $\Sigma = \Sigma(G)$ or $\Sigma_F(G)$.

One has the following "isomorphism theorem" due to Tits and independently to the author (cf. [B-T], [T], [Sa1], [Sa3]).

Theorem 1. *Let G and G' be two simply connected semisimple algebraic groups over a field F of characteristic zero. Let (T, Δ) and (T', Δ') be F-admissible coordinates in G and G', respectively, and let*

$$\Sigma = (\Delta, \Delta_0, [\Gamma]) \quad \textit{and} \quad \Sigma' = (\Delta, \Delta_0', [\Gamma]')$$

be the corresponding Γ-diagrams. Then G and G' are F-isomorphic if and only if one has a congruence $\varphi^* : \Sigma \to \Sigma'$ and an F-isomorphism $\varphi_0 : G(\Delta_0) \to G'(\Delta_0')$ such that $\varphi^* \mid \Delta_0$ coincides with φ_0^*.

In the notation of the above theorem, suppose one has an F-isomorphism $\varphi : G \to G'$. Then φ^* is a congruence of Σ onto a Γ-diagram $(\varphi^*(\Delta), \varphi^*(\Delta_0), \varphi^*[\Gamma]{\varphi^*}^{-1})$ of G', which in turn is congruent to Σ'. Hence, combining these two congruence, one obtains a congruence $\Sigma \to \Sigma'$, which we call a congruence *induced by* φ.

For convenience, we recall here some well-known definitions. G is called "F-split" (or of Chevalley type), if there is an F-split maximal torus $T = A$ in G. For such a T, the coordinate (T, Δ) (with any basis Δ) is F-admissible and the corresponding Γ-diagram Σ has the property that $\Delta_0 = \emptyset$ and the $[\Gamma]$-action is trivial. Conversely, if $\Sigma = \Sigma_F(G)$ has this property, then G is F-split. G is called "F-quasisplit" (or of Steinberg type) if one has $T = Z(A)$, or equivalently $\Phi_0 = \emptyset$. In this case, (T, Δ) is F-admissible if and only if Δ is Γ-invariant (as a whole); and of course one then has $\Delta_0 = \emptyset$. Conversely, if $\Delta_0 = \emptyset$ in $\Sigma_F(G)$, then G is F-quasisplit. It should also be noted that G is "F-anisotropic" (i.e., F-rank $G = 0$) if and only if one has $\Delta = \Delta_0$ in $\Sigma_F(G)$.

§3. Classification over a local field.

It was shown by Chevalley [Ch1,2] that for any field F (of any characteristic) and for any Dynkin diagram Δ there exists uniquely (up to

F-isomorphism) an F-split semisimple algebraic group of adjoint type defined over F (the so-called Chevalley group). When F is algebraically closed, this gives a complete classification of (simply connected) semisimple algebraic group over F. It follows also that for any field F, any Dynkin diagram Δ, and for any action of Γ on Δ, there exists uniquely (up to F-isomorphism) an F-quasisplit simply connected semisimple algebraic group G_0 defined over F with $\Sigma_F(G) = (\Delta, \emptyset, \Gamma)$ (the uniqueness follows from Th.1). Therefore, for the classification theory over F of characteristic zero), it is enough to fix an F-quasisplit simply connected semisimple algebraic group G_0 over F and to determine all inner F-forms of G_0.

For $F = \mathbb{R}$, one has the following theorem.

Theorem 2. *Let G and G' be simply connected semisimple algebraic groups defined over $\mathbb{R}$. Then G and G' are $\mathbb{R}$-isomorphic if and only if the Γ-diagrams $\Sigma_{\mathbb{R}}(G)$ and $\Sigma_{\mathbb{R}}(G')$ are congruent.*

This follows from Theorem 1 and from the fact that a compact (i.e., $\mathbb{R}$-anisotropic) $\mathbb{R}$-form G is uniquely determined (up to $\mathbb{R}$-isomorphism) only by its (unmarked) Dynkin diagram Δ (Weyl's theorem). A direct method of classifying Γ-diagrams over $\mathbb{R}$ was given by Araki. (See [A] or [Sa3], Appendix by Sugiura. For a more general method of classifying "Tits indices", see [T]. For the classification over $\mathbb{R}$, cf. also [Mu], [Bo1]). For the determination of the invariant γ over $\mathbb{R}$, see §6.

For a $\mathfrak{p}$-adic field F (i.e., a finite extension of $\mathbb{Q}_p$) the following theorem of M. Kneser is fundamental. (For a uniform proof of it, see [Br-T]).

Theorem 3 ([K1]). *Let F be a $\mathfrak{p}$-adic field and G a simply connected semisimple algebraic group defined over F. Then $H^1(F, G) = 1$.*

In view of the exact sequence (1), this implies the following

Theorem 4 ([K1]). *Let F be a $\mathfrak{p}$-adic field. Let G_0 be a simply connected semisimple algebraic group defined over F and let Z be the center of G_0. Then the map $(G, f) \mapsto \gamma(G, f)$ gives rise to a bijective correspondence between the set of F-equivalence classes of inner F-forms (G, f) of G_0 and $H^2(F, Z)$.*

In fact, it is enough to show that the map δ in the sequence (1) is bijective. It is known (Lem. 4 in §4) that when F is a $\mathfrak{p}$-adic field δ is surjective. The injectivity follows from (3) and Theorem 3.

Theorem 4 shows that over a $\mathfrak{p}$-adic field F the simply connected semisimple algebraic groups are completely classified by the F-quasisplit group G_0 (i.e., by the $[\Gamma]$-action on Δ) and the cohomological invariant $\gamma \in H^2(F, Z)$. From the result of classification, one sees that over a $\mathfrak{p}$-adic field F an absolutely simple "anisotropic" F-form G occurs only for the type $({}^1\mathrm{A}_l)$. Consequently, the cohomological invariant $\gamma(G, f)$ reduces essentially to the classical Hasse invariant of central simple algebras (cf. [K1], [Sa3], and §6).

§4. Scalar extensions and Hasse principles.

Let G be a simply connected semisimple algebraic group defined over a field F of characteristic zero. We use the notation introduced in §§1, 2.

Let F' be an extension of F and let $\Gamma' = \mathrm{Gal}(\overline{F}'/F')$, $\overline{F}'$ being an algebraic closure of F'. Identifying $\overline{F}$ with the algebraic closure of F in $\overline{F}'$, we denote the restriction of $\sigma' \in \Gamma'$ on $\overline{F}$ by σ'_F.

The scalar extension F'/F gives rise in a natural manner to canonical maps (homomorphisms) between cohomology sets (groups), which make the following diagram commutative:

$$\begin{array}{ccccccccc} \cdots & \to & H^1(F,Z) & \to & H^1(F,G) & \to & H^1(F,\overline{G}) & \to & H^2(F,Z) \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \to & H^1(F',Z) & \to & H^1(F',G) & \to & H^1(F',\overline{G}) & \to & H^2(F',Z) \end{array}$$

For instance, for $\xi \in H^1(F, \overline{G})$ we denote by $\xi_{F'}$ the corresponding element in $H^1(F', \overline{G})$. Then, in the notation of §1, for $\xi = c_F(G, f) \in H^1(F, \overline{G}_0)$ one has $\xi_{F'} = c_{F'}(G, f)$.

Let (T, Δ) be an F-admissible coordinate in G and let

$$\Sigma = \Sigma_F(G) = (\Delta, \Delta_0, [\Gamma])$$

be the corresponding Γ-diagram. Similarly, let (T', Δ') be an F'-admissible coordinate in G with

$$\Sigma' = \Sigma_{F'}(G) = (\Delta', {\Delta'}_0, [\Gamma']).$$

Then there exists $\varphi \in \mathrm{Inn}(G_0)(\overline{F}')$ such that $\varphi : (T, \Delta) \to (T', \Delta')$; then one has automatically $\varphi(A) \subset A'$, where A and A' are maximal F-split resp. F'-split tori contained in T and T' (see Lemma 2 below). Therefore the induced isomorphism φ^* has the following properties:

(7) $$\varphi^*(\Delta) = \Delta', \ \varphi^*(\Delta_0) \supset {\Delta'}_0, \quad \text{and}$$

$$\varphi^*(\chi^{[\sigma'_F]}) = \varphi^*(\chi)^{[\sigma']} \qquad \text{for all} \quad \chi \in X,\ \sigma' \in \Gamma'.$$

Note that the map $\varphi^* : \Sigma \to \Sigma'$ is determined intrinsically, independently of the choice of coordinates (T, Δ), (T', Δ'). The image by φ^* of a $[\Gamma]$-orbit in Σ is a union of a finite number of $[\Gamma']$-orbits in Σ'. In particular, the image of a white $[\Gamma]$-orbit is always a union of white $[\Gamma']$-orbits.

Lemma 2. *The notation being as above, let (T, Δ) (resp. (T', Δ')) be an F- (resp. F'-)admissible coordinate in G and let $\varphi \in \mathrm{Inn}(G)$ be such that $\varphi : (T, \Delta) \to (T', \Delta')$. Then, for maximal F-split resp. F'-split tori A and A' contained in T and T', one has $\varphi(A) \subset A'$.*

Proof. First there exists $\varphi_1 \in \mathrm{Inn}(G)(F')$ such that $\varphi_1(A) \subset A'$. Then there exists $\varphi_2 = I_{g_2}$, $g_2 \in Z(\varphi_1(A))(\overline{F'})$ such that $\varphi_2\varphi_1(T) = T'$. Then one has $X'_0 \subset \varphi_2^*\varphi_1^*(X_0)$. Let Δ_1 be a basis of Φ adapted to both $(\varphi_2^*\varphi_1^*)^{-1}(X'_0)$ and X_0; then $\varphi_2^*\varphi_1^*(\Delta_1)$ is a basis of Φ' adapted to X'_0. Therefore there exist

$$g_3 \in N(A) \cap N(T)(\overline{F}) \text{ and } g_4 \in N(A') \cap N(T')(\overline{F}')$$

such that, for $\varphi_3 = I_{g_3}$ and $\varphi_4 = I_{g_4}$, one has $\varphi_3^*\Delta = \Delta_1$ and $\varphi_4^*\varphi_2^*\varphi_1^*\Delta_1 = \Delta'$. Then one has

$$\varphi_4\varphi_2\varphi_1\varphi_3 : (T, \Delta) \to (T', \Delta').$$

By the uniqueness of such a map, one has $\varphi = \varphi_4\varphi_2\varphi_1\varphi_3$ on T; hence, in particular, one has $\varphi(A) \subset A'$, q.e.d.

Now let F be a number field (i.e., a finite extension of $\mathbb{Q}$) and let $V = V^F$ denote the set of all places (i.e., equivalence classes of valuations) of F, and let $V_{\infty,1} = V^F_{\infty,1}$ denote the set of all real places. For $v \in V$ we denote by F_v the completion of F with respect to the place v. In the above notation, we write ξ_v for ξ_{F_v}; similarly, when $\Sigma = \Sigma_F(G)$ we write $\Sigma_v = \Sigma_{F_v}(G)$.

For our purpose it is important to consider the canonical map

$$\theta : H^1(F, G) \to \prod_{v \in V} H^1(F_v, G). \tag{8}$$

Since, by Theorem 3, $H^1(F_v, G)$ is trivial except for $v \in V_{\infty,1}$, the map θ can also be written as

$$\theta : H^1(F, G) \to \prod_{v \in V_{\infty,1}} H^1(F_v, G). \tag{8'}$$

Then the "Hasse principle" for H^1, established by Kneser [K2], [K3], Harder [H1], and Chernousov [Cher], can be stated as follows.

Theorem 5. *Let G be a simply connected semisimple algebraic group defined over a number field F. Then the canonical map θ in* (8′) *is bijective.*

For the proof, see [P-R] (Th. 6.6); the proof for the surjectivity of θ (due to Kneser) is relatively easy. (It seems that no uniform proof for the injectivity of θ is yet known.) For the Galois cohomology of the center Z, one has the following

Lemma 3. (i) *The canonical map*

$$H^1(F, Z) \to \prod_{v \in V_{\infty,1}} H^1(F_v, Z) \tag{9}$$

is surjective.

(ii) *The canonical map*

$$H^2(F, Z) \to \prod_{v \in V} H^2(F_v, Z) \tag{10}$$

is injective.

(Cf. [P-R], Prop. 7.8, Cor. 2 and Lemma 6.19.)

Lemma 4. *If F is a $\mathfrak{p}$-adic field or a number field, then the map $\delta : H^1(F, \overline{G}) \to H^2(F, Z)$ in the sequence* (1) *is surjective.*

(Cf. [P-R], Th. 6.20.)

In order to formulate another type of Hasse principle concerning the Γ-diagrams, let G be a connected semisimple algebraic group defined over F. (Note that here the simply connectedness is irrelevant.) Let (T, Δ) be an F-admissible coordinate in G and let $B = B(\Delta)$ be the corresponding Borel subgroup of G. For a subset Δ_1 of Δ we denote by $G(\Delta_1)$ the corresponding (connected) semisimple closed subgroup of G and set $P(\Delta_1) = G(\Delta_1)B$. Then it is known that $P(\Delta_1)$ is a parabolic subgroup of G and all parabolic subgroup of G is conjugate to a subgroup of this form. We denote by $\mathcal{P}(\Delta_1)$ the conjugacy class of $P(\Delta_1)$, which can be identified with $G/P(\Delta_1)$; thus $\mathcal{P}(\Delta_1)$ has a natural structure of a projective variety.

Now, for $\sigma \in \Gamma$ one has $B^\sigma = B(\Delta^\sigma) = \psi_\sigma B \psi_\sigma^{-1}$ and hence

$$P(\Delta_1)^\sigma = G(\Delta_1^\sigma) B^\sigma = \psi_\sigma P(\Delta_1^{[\sigma]}) \psi_\sigma^{-1}. \tag{11}$$

It follows that $\mathcal{P}(\Delta_1)$ is Γ-invariant if and only if Δ_1 is $[\Gamma]$-invariant. Thus, in this case, $\mathcal{P}(\Delta_1)$ is a variety defined over F.

We call a parabolic subgroup P of G *F-parabolic* if it is defined over F. From (11) it can be seen that, if Δ_1 is $[\Gamma]$-invariant and contains Δ_0, then $P(\Delta_1)$ is F-parabolic. It is known that all F-parabolic subgroup of G is conjugate (with respect to an element in $G(F)$) to a $P(\Delta_1)$ with Δ_1 having this property. Thus one obtains

Lemma 5 ([T]). *The notation being as above, suppose that Δ_1 is $[\Gamma]$-invariant. Then the variety $\mathcal{P}(\Delta_1)$ is defined over F. It contains an F-rational point if and only if Δ_1 contains Δ_0.*

Now, one has the following Hasse principle due to Harder ([H2], Satz 4.3.3).

Theorem 6. *Let G be a connected semisimple algebraic group defined over a number field F. Let Δ_1 be a subset of Δ invariant under $[\Gamma]$ and let $\mathcal{P}(\Delta_1)$ denote the variety (defined over F) of parabolic subgroup of G conjugate to $P(\Delta_1)$. Then $\mathcal{P}(\Delta_1)$ has an F-rational point if and only if it has an F_v-rational point for all $v \in V^F$.*

By the above observation, one can rephrase this theorem in the following form.

Theorem 6′ . *Let G be a connected semisimple algebraic group defined over a number field F and let $\Sigma = (\Delta, \Delta_0, [\Gamma])$ and $\Sigma_v = (\Delta, \Delta_0^{(v)}, [\Gamma^{(v)}])$ $(v \in V^F)$ be the Γ- resp. $\Gamma^{(v)}$-diagrams of G over F and F_v. Then Δ_0 is the smallest $[\Gamma]$-invariant subset of Δ containing all $\Delta_0^{(v)}$ $(v \in V^F)$.*

Otherwise expressed, one has the following Hasse principle for the Γ-diagrams: a $[\Gamma]$-orbit in a Γ-diagram Σ is white if and only if it decomposes in Σ_v into a union of white $[\Gamma^{(v)}]$-orbit for all $v \in V^F$.

§5. Classification over a number field.

In this section, let F be a number field. We fix a simply connected semisimple algebraic group G_0 defined over F. (In this section, the assumption for G_0 to be F-quasisplit is irrelevant.) The main results on the classification of inner F-forms of G_0 can be formulated as follows.

Theorem 7. *Let (G, f) and (G', f') be two inner F-forms of a simply connected semisimple algebraic group G_0 over a number field*

F. Then (G,f) *and* (G',f') *are* F*-equivalent (i.e., there exists an* F*-isomorphism* $\varphi : G \to G'$ *such that* $\varphi \circ f^{-1} \circ f' \in \mathrm{Inn}(G')$*) if and only if the following two conditions are satisfied.*

(i) *One has* $\gamma(G,f) = \gamma(G',f')$.

(ii) (G,f) *and* (G',f') *are* F_v*-equivalent for all* $v \in V_{\infty,1}$.

Proof. The "only if" part is obvious. To prove the "if" part assume that the conditions (i),(ii) are satisfied. Then, by (i) the 1-cohomology classes $\xi = c(G,f)$ and $\xi' = c(G',f')$ are in the same fiber of the map $\delta : H^1(F,\overline{G}_0) \to H^2(F,Z)$. Therefore, by the formula (3) there exists $\eta \in H^1(F,G)$ such that $\beta(\eta) = \xi^{-1}\xi'$. By the condition (ii) one has $\xi_v = \xi'_v$ for all $v \in V_{\infty,1}$, which implies that $\beta(\eta_v) = \xi_v^{-1}\xi'_v = 1$. Hence, for each $v \in V_{\infty,1}$, by the exactness of the sequence (1) (over F_v), one has $\alpha(\zeta^{(v)}) = \eta_v$ for some $\zeta^{(v)} \in H^1(F_v,Z)$. By Lemma 3, (i), there exists $\zeta \in H^1(F,Z)$ such that $\zeta_v = \zeta^{(v)}$ for all $v \in V_{\infty,1}$; then one has $\alpha(\zeta)_v = \alpha(\zeta_v) = \eta_v$. Hence by Theorem 5 (injectivity of θ) one has $\alpha(\zeta) = \eta$, whence $\beta(\eta) = 1$ and so $\xi = \xi'$, q.e.d.

It is clear that the condition (ii) in Theorem 7 can also be stated in the following form:

(ii′) For $v \in V^F_{\infty,1}$ let $\Sigma_v = \Sigma_{F_v}(G), \Sigma'_v = \Sigma_{F_v}(G')$. Then for each v one has a congruence $\Sigma_v \to \Sigma'_v$ induced by an F_v-isomorphism $\varphi^{(v)} : G \to G'$ such that $\varphi^{(v)} \circ f^{-1} \circ f' \in \mathrm{Inn}(G')$.

An "existence theorem" for inner F-forms is given as follows:

Theorem 8. *Let* G_0 *be a simply connected semisimple algebraic group defined over a number field* F*. Suppose there are given* $\gamma \in H^2(F,Z)$ *and, for each* $v \in V_{\infty,1}$*, an inner* F_v*-forms* $(G^{(v)}, f^{(v)})$ *of* G_0 *such that the following consistency condition* (C) *is satisfied:*

(C) *One has* $\gamma_v = \gamma_{F_v}(G^{(v)}, f^{(v)})$ *for all* $v \in V_{\infty,1}$.

Then there exists uniquely (up to an F*-equivalence) an inner* F*-form* (G,f) *of* G_0 *such that* $\gamma(G,f) = \gamma$ *and that* (G,f) *is* F_v*-equivalent to* $(G^{(v)}, f^{(v)})$ *for all* $v \in V_{\infty,1}$.

Proof. By Lemma 4 the map $\delta : H^1(F,\overline{G}_0) \to H^2(F,Z)$ in the sequence (1) is surjective. Hence there exists an inner F-form (G,f) of G_0 such that $\gamma_F(G,f) = \delta(c_F(G,f)) = \gamma$. Then by the condition (C) one has $\gamma_{F_v}(G,f) = \gamma_{F_v}(G^{(v)}, f^{(v)})$ for all $v \in V_{\infty,1}$; this means that, if one puts $\xi = c_F(G,f)$, $\xi^{(v)} = c_{F_v}(G^{(v)}, f^{(v)})$, then ξ_v and $\xi^{(v)}$ are in the same fiber of the map δ in the sequence (1) over F_v. Hence by the formula (3) one has $\beta(\eta^{(v)}) = \xi_v^{-1}\xi^{(v)}$ for some $\eta^{(v)} \in H^1(F_v,G)$. By Theorem 5 (surjectivity of θ) there exists $\eta \in H^1(F,G)$ such that one has $\eta_v = \eta^{(v)}$ for all $v \in V_{\infty,1}$. Then, putting

$$\xi' = \xi\beta(\eta) \in H^1(F, G_0), \quad \xi' = c_F(G', f'),$$

one has

$$\gamma_F(G', f') = \delta(\xi') = \delta(\xi) = \gamma,$$
$$c_{F_v}(G', f') = \xi'_v = \xi_v\beta(\eta_v) = \xi^{(v)}.$$

Thus (G', f') is an inner F-form of G_0 satisfying all the requirements. The uniqueness follows from Theorem 7, q.e.d.

Remark 1. As will be shown in §6, one has $H^2(F_v, Z) = 1$ for all $v \in V_{\infty,1}$, if G_0 is absolutely simple, F-quasisplit and of one the types (A_l) (l even), (E_6), (E_8), (F_4), (G_2). Hence in these cases, the above consistency condition (C) is automatically satisfied.

Remark 2. If F is totally imaginary, one has (analogously to Th.4) that the map $\delta : H^1(F, \overline{G}_0) \to H^2(F, Z)$ is bijective. (For a similar result in the function field case, see [H3].)

Remark 3. The list of all possible Γ-diagrams ("Tits indices") $\Sigma(G)$ over a number field F was given in [T]. From our point of view, the same result can also be obtained by Theorems 6′ and 8, using the classification over local fields. For groups of exceptional type, a method of explicit construction of F-forms was also given by Tits (see e.g. [Sc]).

§6. Determination of the invariant.

In this section, G_0 is an F-quasisplit simply connected absolutely simple algebraic group over a number field F. We give an explicit determination of $H^2(F, Z)$. At the end, we also give a list of $\gamma(G)$ for all $\mathbb{R}$-forms G of G_0. (Note that except for the case where G_0 is of type (D_l) (l even) the invariant $\gamma(G, f)$ is actually independent of f; hence we omit f.) For convenience, we treat the case of groups of type (D_l) (l even) separately.

I) The case where G_0 is F-split (except the case ($^1\mathrm{D}_l$), l even).

We denote by μ_n the group of n-th roots of unity in $\bar{F}$ viewed as a group on which Γ is acting. Then, in the case of F-split G_0 (not of type ($^1\mathrm{D}_l$), l even), one has

(12) $$Z \cong \mu_n,$$

where n is given as follows:

G_0	$=$	${}^1\mathrm{A}_l$,	B_l,	C_l,	${}^1\mathrm{D}_l$ (l odd),	${}^1\mathrm{E}_6$,	E_7,	E_8,	F_4,	G_2
n	$=$	$l+1$,	2,	2,	4,	3,	2,	1,	1,	1

It follows that

$$H^1(F,Z) \cong F^*/(F^*)^n, \ H^2(F,Z) \cong Br(F)_n, \tag{13}$$

where $Br(F)$ is the Brauer group of F and $Br(F)_n$ denotes the subgroup of $Br(F)$ consisting of those elements ξ with $\xi^n = 1$ (see [P-R], p.73, Lem. 2.6). Therefore over the local fields F_v $(v \in V^F)$ one has

$$H^2(F_v,Z) \cong Br(F_v)_n \cong \begin{cases} (1/n)\mathbb{Z}/\mathbb{Z} & (v \notin V_\infty) \\ (1/2)\mathbb{Z}/\mathbb{Z} & (v \in V_{\infty,1},\ n \text{ even}) \\ 1 & (\text{otherwise}). \end{cases} \tag{13a}$$

For the case n even, the invariant $\gamma_{\mathbb{R}}(G)$ for all inner $\mathbb{R}$-forms G of G_0 is given in the list at the end of the section. For classical groups, the determination of this invariant is well known. For the case $G_0 = \mathrm{E}_7$, this can be done, e.g., by using the results in [Mu], [Sa2].

II) The case where G_0 in not F-split (except the case $({}^2\mathrm{D}_l)$, l even). There are three cases

$$G_0 = {}^2\mathrm{A}_l,\ {}^2\mathrm{D}_l\ (l \text{ odd} \geq 3),\ {}^2\mathrm{E}_6.$$

In these cases, there is a quadratic extension F'/F such that G_0 is split over F'. Then one has

$$Z \cong R^{(1)}_{F'/F}(\mu_n) = \{\zeta = (\zeta_1,\zeta_2) \in R_{F'/F}(\mu_n) \mid \zeta_1\zeta_2 = 1\}. \tag{14}$$

and an exact sequence

$$1 \to F^*/(F^*)^n N_{F'/F}(F'^*) \to H^2(F,Z) \to \tag{15}$$

$$\to \mathrm{Ker}(Br(F')_n \xrightarrow{N} Br(F)_n) \to 1,$$

where N stands for $N_{F'/F}$ (see [P-R] , p.332, (6.31)).

When n is odd (i.e., $G_0 = {}^2\mathrm{A}_l$ (l even), ${}^2\mathrm{E}_6$), one has

$$H^2(F,Z) \cong \mathrm{Ker}(Br(F')_n \xrightarrow{N} Br(F)_n), \tag{15'}$$

$$H^2(F',Z) \cong Br(F')_n.$$

Therefore, if $v \in V(F)$ dose not decompose in F'/F (i.e., if v has a unique extension to F', denoted again by v, $F' \otimes F_v = F'_v$), one has $N : Br(F'_v)_n \cong Br(F_v)_n$ and hence

$$(15'a) \qquad H^2(F_v, Z) = 1.$$

If v decomposes in F'/F (i.e., if v has two extensions w, w' in F', $F' \otimes F_v = F'_w \oplus F'_{w'}$), then one has

$$(15'b) \qquad H^2(F_v, Z) \cong Br(F_v)_n.$$

In either case, one has $H^2(F_v, Z) = 1$ for $v \in V_{\infty,1}$.

When n is even (i.e., $G_0 = {}^2\mathrm{A}_l$ (l odd), ${}^2\mathrm{D}_l$ (l odd)), one has an exact sequence

$$(15'') \qquad 1 \to F^*/N_{F'/F}(F'^*) \to H^2(F, Z) \to$$

$$\to \mathrm{Ker}(Br(F')_n \xrightarrow{N} Br(F)_n) \to 1,$$

and

$$H^2(F', Z) \cong Br(F')_n.$$

Therefore, if v does not decompose in F'/F, then one has

$$(15''a) \qquad H^2(F_v, Z) \cong F_v^*/N_{F'_v/F_v}(F'^*_v) \cong Br(F_v)_2.$$

If v decomposes in F'/F, then one has

$$(15''b) \qquad H^2(F_v, Z) \cong Br(F_v)_n.$$

Thus in view of Lemma 3, (ii) one has actually (instead of (15''))

$$(16) \qquad H^2(F_v, Z) \cong (F^*/N_{F'/F}(F'^*)) \times \mathrm{Ker}(Br(F')_n \xrightarrow{N} Br(F)_n).$$

For the case n even, the invariant $\gamma_{\mathbb{R}}(G)$ for all inner $\mathbb{R}$-forms of G_0 is given in the list below.

III) The case where G_0 is of type (D_l) (l even)

Let F' be the smallest Galois extension of F such that G_0 is split over F' and let $[F' : F] = m$; we write $G_0 = {}^m\mathrm{D}_l$. Then there are the following four case:

$$G_0 = {}^1\mathrm{D}_l,\ {}^2\mathrm{D}_l\ (l \text{ even} \geq 4),\ {}^3\mathrm{D}_4,\ {}^6\mathrm{D}_4.$$

When $G_0 = {}^1\mathrm{D}_l$, one has

$$(17) \qquad Z = \mu_2 \times \mu_2,$$

$$H^2(F, Z) \cong Br(F)_2 \times Br(F)_2, \tag{18}$$

$$H^2(F_v, Z) \cong Br(F_v)_2 \times Br(F_v)_2. \tag{18a}$$

When $G_0 = {}^2\mathrm{D}_l$, one has

$$Z = R_{F'/F}(\mu_2), \tag{19}$$

$$H^2(F, Z) \cong Br(F')_2, \quad H^2(F', Z) \cong Br(F')_2 \times Br(F')_2. \tag{20}$$

$$H^2(F_v, Z) \cong Br(F'_v)_2 \quad \text{for } v \text{ not decomp. in } F'/F, \tag{20a}$$

$$H^2(F_v, Z) \cong Br(F_v)_2 \times Br(F_v)_2 \quad \text{for } v \text{ decomp. in } F'/F. \tag{20b}$$

When $G_0 = {}^3\mathrm{D}_4$, one has

$$Z = R^{(1)}_{F'/F}(\mu_2), \tag{21}$$

$$\begin{gathered} H^2(F, Z) \cong \mathrm{Ker}(Br(F')_2 \longrightarrow Br(F)_2), \\ H^2(F', Z) \cong Br(F')_2 \times Br(F')_2. \end{gathered} \tag{22}$$

$$H^2(F_v, Z) = 1 \quad \text{for } v \text{ not decomp. in } F'/F, \tag{22a}$$

$$H^2(F_v, Z) \cong Br(F_v)_2 \times Br(F_v)_2 \quad \text{for } v \text{ decomp. in } F'/F. \tag{22b}$$

When $G_0 = {}^6\mathrm{D}_4$, we take an intermediate field F_1 such that $F \subset F_1 \subset F'$ and $[F_1 : F] = 3$. Then one has

$$Z = R^{(1)}_{F_1/F}(\mu_2), \tag{23}$$

$$\begin{gathered} H^2(F, Z) \cong \mathrm{Ker}(Br(F_1)_2 \longrightarrow Br(F)_2), \\ H^2(F_1, Z) \cong Br(F')_2, \quad H^2(F', Z) \cong Br(F')_2 \times Br(F')_2. \end{gathered} \tag{24}$$

If v does not decompose in F_1/F, then one has

$$H^2(F_v, Z) = 1. \tag{24a}$$

If v decomposes in F_1/F but does not decompose completely in F'/F, then one has

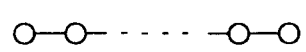
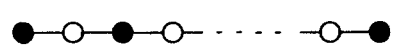
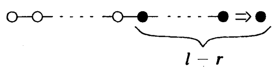
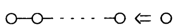
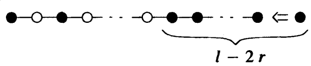
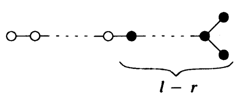
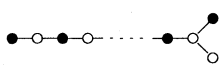
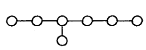
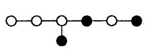
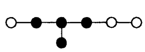
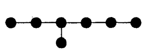
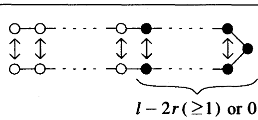
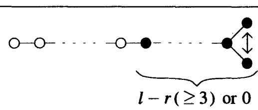
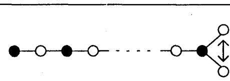
$$H^2(F_v, Z) \cong Br(F'_v)_2. \tag{24b}$$

If v decomposes completely in F'/F, one has

$$H^2(F_v, Z) \cong Br(F_v)_2 \times Br(F_v)_2. \tag{24c}$$

In all cases, one has $H^2(F_v, Z) = 1$ for $v \in V_{\infty,1}$ except for the case where v decomposes completely in F'/F. Hence for the determination of $\gamma_{\mathbb{R}}(G)$ it is enough to consider only inner $\mathbb{R}$-forms of $G_0 = {}^1\mathrm{D}_l$ (l even), which is given in the list below.

In the following list, one has l =rank G, $r = \mathbb{R}$-rank G (which equals the number of white $[\Gamma]$-orbits in $\Sigma_{\mathbb{R}}(G)$), and the type of G over $\mathbb{R}$ is expressed by Cartan's symbol. An element in $Br(\mathbb{R})$ is expressed by the corresponding Hasse invariant $0,\ 1/2 \in (1/2)\mathbb{Z}/\mathbb{Z}$. As remarked above, for all G_0 not included in this list, one has $H^2(\mathbb{R}, Z) = 1$. (For a complete list of Γ-diagrams over local fields, see [A], [Sa3], or [T].)

$G_0/\mathbb{R}$	$G/\mathbb{R}$	Γ-diagram of $G/\mathbb{R}$	$\gamma_{\mathbb{R}}(G)$
${}^1\mathrm{A}_l$	AI		0
	AII (l odd)		$\frac{1}{2}$
B_l	BI		0 if $l-r \equiv 0,3\ (4)$ $\frac{1}{2}$ if $l-r \equiv 1,2\ (4)$
C_l	CI		0
	CII		$\frac{1}{2}$
${}^1\mathrm{D}_l$	DI $\left(\begin{array}{c} l \text{ odd} \\ l-r \text{ even}\end{array}\right)$		0 if $l-r \equiv 0\ (4)$ $\frac{1}{2}$ if $l-r \equiv 2\ (4)$
	DI $\left(\begin{array}{c} l \text{ even} \\ l-r \text{ even}\end{array}\right)$	same	0 if $l-r \equiv 0\ (4)$ $(\frac{1}{2},\frac{1}{2})$ if $l-r \equiv 2\ (4)$
	DIII (l even)		$(\frac{1}{2},0)$ or $(0,\frac{1}{2})$
E_7	EV		0
	EVI		$\frac{1}{2}$
	EVII		0
	compact		$\frac{1}{2}$
${}^2\mathrm{A}_l$	AIII (l odd)		0 if $l-2r \equiv 3\ (4)$ $\frac{1}{2}$ if $l-2r \equiv 1\ (4)$
${}^2\mathrm{D}_l$	DI ($l-r$ odd)		0
	DIII (l odd)		$\frac{1}{2}$

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