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Local types of singularities of plane curves and the topology of their complements

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§ Introduction

Let B be a reduced plane curve in $\mathbf{P}^2 = \mathbf{P}^2_{\mathbf{C}}$. After Zariski's famous article [37], there have been many results on the topology of $\mathbf{P}^2 \setminus B$(see References of [8], for example). The main purpose of this article is to survey some of recent progress on the topology of $\mathbf{P}^2 \setminus B$ with a special emphasis on the case of $\deg B = 6$, including a new example of a Zariski pair. Throughout this article, our fundamental question is the following:

Problem 0.1. What one can say about the topology of $\mathbf{P}^2 \setminus B$ just from the data of local types of singularities of B?

Hereafter we simply say *the configuration of singularities* in the place of *the data of local topological types of singularities.*

As Problem 0.1 seems to be rather vague, we consider more specific problem:

Problem 0.2. Under what condition on the configuration of singularities of B, can one determine the (non-) commutativity of $\pi_1(\mathbf{P}^2 \setminus B)$?

Even Problem 0.2 is still by no means easy. To know how subtle this problem is, let us recall Zariski's famous example:

Example 0.3 (Zariski [37], [38]). Let (B_1, B_2) be a pair of sextic curves with 6 cusps such that

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(i) there exists a conic, C, passing through the 6 cusps for B_1, while
(ii) there exists no such conic as in (i) for B_2.
For these sextic curves, $\pi_1(\mathbf{P}^2 \setminus B_1) \not\cong \pi_1(\mathbf{P}^2 \setminus B_2)$.

Remark 0.4. More precisely, $\pi_1(\mathbf{P}^2 \setminus B_1) \cong \mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/3\mathbf{Z}$. For B_2, Oka found an explicit example such that $\pi_1(\mathbf{P}^2 \setminus B_2) \cong \mathbf{Z}/6\mathbf{Z}$ in [22]. It is, however, still unknown whether $\pi_1(\mathbf{P}^2 \setminus B_2) \cong \mathbf{Z}/6\mathbf{Z}$ always holds for any sextic curve of second type.

As Zariski's example shows, in general, just the configuration of singularities is not enough to determine whether $\pi_1(\mathbf{P}^2 \setminus B)$ is abelian or non-abelian. Nevertheless, under some particular conditions, we are able to determine it. Let us begin with the cases when $\pi_1(\mathbf{P}^2 \setminus B)$ is abelian. The first statement is

Theorem 0.5 (Deligne-Fulton [7], [12]**).** *If B has only nodes, then $\pi_1(\mathbf{P}^2 \setminus B)$ is abelian.*

After this statement, Nori generalized it for irreducible plane curves having only nodes and cusps.

Theorem 0.6 (Nori, [15]**).** *Suppose that B is an irreducible curve of degree d and has only nodes and cusps. Let a and b be the numbers of nodes and cusps, respectively. If $2a+6b < d^2$, then $\pi_1(\mathbf{P}^2 \setminus B)$ is abelian.*

Note that Example 0.3 shows that the inequality in Theorem 0.6 is sharp. Shimada recently gave another kind of statement as follows:

Theorem 0.7 (Shimada [28]**).** *Under the same notations and assumption as in Theorem* 0.6, *if $2a \geq d^2 - 5d + 8$, then $\pi_1(\mathbf{P}^2 \setminus B)$ is abelian.*

All of these statements assure that $\pi_1(\mathbf{P}^2 \setminus B)$ is abelian. Although there are many results on reduced plane curves whose complements have non-abelian fundamental groups (see References of [8], for example), most of them are given by explicit equations; and the defining equations give much more information on curves than just the configuration of singularities does. Our main concern in this article is:

(i) To find some condition on the configuration of singularities which assures that $\pi_1(\mathbf{P}^2 \setminus B)$ is *non-abelian.*

(ii) To look into how good the given condition in (i) is.

To state our result, let us introduce some notation as follows:

(a) For $x \in \mathrm{Sing}(B)$, we denote its Milnor number by μ_x. We define the total Milnor number of B by

$$\mu_B = \sum_{x \in \mathrm{Sing}(B)} \mu_x.$$

(b) Let p be an odd prime. For B, we define a non-negative integer l_p as follows:

If $p = 3$, $l_3 =$ the number of singularities of type A_{3k-1} $(k \geq 1)$ and E_6.

If $p \geq 5$, $l_p =$ the number of singularities of type A_{pk-1} $(k \geq 1)$.

Now we are in position to state our result.

Theorem 0.8. *Let B be a reduced plane curve of even degree with at most simple singularities. Suppose that there exists an odd prime p such that*

$$l_p + \mu_B > d^2 - 3d + 3.$$

Then $\pi_1(\mathbf{P}^2 \setminus B)$ is non-abelian.

A straightforward, but interesting corollary to Theorem 0.8 is:

Corollary 0.9. *Let B be a plane curve of even degree with only nodes and cusps. Let a and b be the number of nodes and cusps, respectively. If $a + 3b > d^2 - 3d + 3$, then $\pi_1(\mathbf{P}^2 \setminus B)$ is non-abelian.*

Note that Corollary 0.9 gives a nice contrast to Theorem 0.6. In fact, the inequality Corollary 0.9 is equivalent to $2a + 6b > 2d^2 - 6d + 6$; and the left hand side is the same as that of the inequality in Theorem 0.6. We give examples of plane curves satisfying the conditions in Theorem 0.8 in §3.

Now our next question is:

Question 0.10. *Is the inequality in Theorem* 0.8 *best possible?*

As we see in §2, our proof for Theorem 0.8 is based on the existence of non-abelian Galois covering branched along B. Hence the inequality does not seem to be sharp. Nevertheless, it is best possible when $d = 6$. In fact, Oka proved the following result in [23].

Theorem 0.11 (Oka [23]). *There exists a pair of irreducible sextic curves (B_1, B_2) satisfying the following conditions:*

(i) *The configuration of singularities of B_1 and B_2 are the same; and they are either $3E_6$ or $3A_1 + 6A_2$.*

(ii) *$\pi_1(\mathbf{P}^2 \setminus B_1) \cong \mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/3\mathbf{Z}$, while $\pi_1(\mathbf{P}^2 \setminus B_2) \cong \mathbf{Z}/6\mathbf{Z}$.*

A pair of plane curves as in Example 0.3 and Theorem 0.9 is called a Zariski pair, precise definition of which is as follows:

Definition 0.12 (cf. [1]). A pair of irreducible plane curves of the same degree, (B_1, B_2), is called a Zariski pair if (i) the configuration of singularities of B_1 and B_2 are the same, and (ii) $\mathbf{P}^2 \setminus B_1$ is not homeomorphic to $\mathbf{P}^2 \setminus B_2$.

As we see in §4, there are several examples for Zariski pairs of sextic curves satisfying the equality $l_3 + \mu_B = d^2 - 3d + 3$ (Theorem 4.1). All of these are possible candidates showing that the inequality in Theorem 0.8 is sharp. It might be interesting to determine the fundamental groups of the complements of such curves.

This article consists of five sections. In §1, we give a summary on Galois coverings. In §2, we explain how we prove Theorem 0.8. §3 and §4 are devoted to examples. In §5, we give a method to obtain sextic curves with the desired properties.

Notations and conventions

Throughout this article, the ground field is always the complex number field $\mathbf{C}$. We always understand (unless otherwise explicitly stated) by *variety* (*resp. surface*) a smooth projective variety (resp. surface) defined over $\mathbf{C}$. We denote the rational function field of X by $\mathbf{C}(X)$.

Let X be a normal variety, and let Y be a variety. Let $\pi : X \to Y$ be a finite morphism from X to Y. We define the branch locus of f, which we denote by $\Delta(X/Y)$ or $\Delta(f)$, as follows:

$$\Delta(X/Y) = \{y \in Y | \sharp(\pi^{-1}(y)) < \deg \pi\}.$$

For a divisor D on Y, $\pi^{-1}(D)$ denotes the set-theoretic inverse image of D, while $\pi^*(D)$ denotes the ordinary pullback. Also, $\operatorname{Supp} D$ means the supporting set of D.

Let $\pi : X \to Y$ be a $\mathcal{D}_{2p}$ covering of Y. Morphisms, β_1 and β_2, and the variety $D(X/Y)$ always mean those defined in §1.

Let W be a finite double covering of a surface Σ. The "*canonical resolution*" of W always means the resolution given by Horikawa in [13].

Let S be an elliptic surface over B. We call S *minimal* if the fibration is relatively minimal. In this paper, we always assume that an elliptic surface is minimal. For singular fibers of an elliptic surface, we use the notation of Kodaira [14], and for its configuration, we use the notation as in [25].

Let D_1, D_2 be divisors.

$D_1 \sim D_2$: linear equivalence of divisors.

$D_1 \approx D_2$: algebraic equivalence of divisors.

$D_1 \approx_{\mathbf{Q}} D_2$: $\mathbf{Q}$-algebraic equivalence of divisors.

For simple singularities of a plane curve, we use the same notation as that in [2].

§1. Preliminaries

1. Galois coverings of algebraic varieties

Let Y be a normal projective variety, and let X be a normal variety with a finite morphism $\pi : X \to Y$. Then $\mathbf{C}(X)$ is a finite extension of $\mathbf{C}(Y)$.

Definition 1.1. We call $\pi : X \to Y$ a Galois covering if $\mathbf{C}(X)$ is a Galois extension of $\mathbf{C}(Y)$.

Remark 1.2. Let X' be the $\mathbf{C}(X)$-normalization of Y. Then $X \cong X'$ over Y by the uniqueness for the $\mathbf{C}(X)$-normalization of Y.

The following proposition is fundamental in connecting branched coverings with $\pi_1(Y \setminus B)$. For its proof, see [30].

Proposition 1.3. *Let Y be a variety, X be a normal variety with a finite morphism $\pi : X \to Y$, and let B be the branch locus of π. If $\mathbf{C}(X)$ is a Galois extension of $\mathbf{C}(Y)$ with the Galois group, G, then there exists a surjective homomorphism $\pi_1(Y \setminus B) \to G$.*

Corollary 1.4. *Let Y be a variety, and let B be a reduced divisor on Y. If there exists a Galois covering $\pi : X \to Y$ branched along B with non-abelian Galois group, then $\pi_1(Y \setminus B)$ is non-abelian.*

2. $\mathcal{D}_{2p}$ coverings

Let p be an odd prime. Let $\pi : X \to Y$ be a Galois covering. We call X a $\mathcal{D}_{2p}$ covering if $\mathrm{Gal}(\mathbf{C}(X)/\mathbf{C}(Y))$ is a dihedral group of order $2p$. In this subsection, we give a summary on $\mathcal{D}_{2p}$ coverings. For details, see [29] and [33].

Let $\pi : X \to Y$ be a $\mathcal{D}_{2p}$ covering of a variety Y. Put $\mathcal{D}_{2p} = \langle \sigma, \tau | \sigma^2 = \tau^p = (\sigma\tau)^2 = 1 \rangle$. The invariant subfield, $\mathbf{C}(X)^\tau$, of $\mathbf{C}(X)$ is a quadratic extension of $\mathbf{C}(Y)$. Let $D(X/Y)$ be the $\mathbf{C}(X)^\tau$-normalization of Y. Then $D(X/Y)$ is a double covering of Y satisfying the following commutative diagram:

$$\begin{tikzcd} X \arrow[dr, "\beta_2"] \arrow[dd, "\pi"'] & \\ & D(X/Y) \arrow[dl, "\beta_1"] \\ Y & \end{tikzcd}$$

where $\beta_1 : D(X/Y) \to Y$ is a double covering of Y and $\beta_2 : X \to D(X/Y)$ is a p-fold cyclic covering of $D(X/Y)$.

With these notation, we have the following result in constructing a $\mathcal{D}_{2p}$ covering of Y.

Proposition 1.5. *Let $f : Z \to Y$ be a smooth finite double covering of a smooth projective variety Y. Let σ be the involution determined by the covering transformation of f. Suppose that there exist three effective divisors D_1, D_2, and D_3 on Z satisfying the following conditions*:

(i) *D_1 is positive. D_1 and $\sigma^* D_1$ have no common component.*

(ii) *If $D_1 = \sum_i a_i D_i^{(1)}$ denotes the decomposition into irreducible components, then $0 < a_i \leq (p-1)/2$ for every i.*

(iii) *$D_1 + pD_2 \sim \sigma^* D_1 + pD_3$.*

Then there exists a $\mathcal{D}_{2p}$ covering, X, of Y such that (i) $Z = D(X/Y)$ *and* (ii) $\Delta(X/Y) = \Delta(Z/Y) \cup f(\mathrm{Supp}(D_1))$.

We modify Proposition 1.5 slightly so that it is rather convenient for our purpose. Let B be as in Theorem 0.8. Let $f' : Z' \to \mathbf{P}^2$ be a double covering with $\Delta(f') = B$. Since B has at most simple singularities, Z' has at most rational double points. Let $\mu : Z \to Z'$ be the canonical resolution of Z' (see [2] III, §7 or [13] §2 for its definition). By the definition, we have the following diagram:

$$\begin{array}{ccc} Z' & \xleftarrow{\mu} & Z \\ f' \downarrow & & \downarrow f \\ \mathbf{P}^2 & \xleftarrow{q} & \Sigma \end{array}$$

where q is a sequence of blowing-ups and f is a double covering branched along the proper transform of B and (possibly empty) some irreducible component of the exceptional divisor of q. We put $\tilde{f} = q \circ f$. Then:

Proposition 1.6. *Let $f : Z \to \Sigma$ be as above, and let σ be the covering transformation. Suppose that there exists a pair of a positive divisor D and a line bundle $\mathcal{L}$ satisfying the condition as follows*:

(i) *If we let $D = \sum_i a_i D_i$ be the irreducible decomposition, then* $\gcd(\{a_i\}, p) = 1$; *and D and $\sigma^* D$ have no common component.*

(ii) *$D - \sigma^* D \sim p\mathcal{L}$.*

Then there exists a p-cyclic covering $g : S \to Z$ such that

(i) $\Delta(g) \subset \mathrm{Supp}(D + \sigma^* D)$ *and*

(ii) *the composition $f \circ g$ gives rise to a $\mathcal{D}_{2p}$ covering of Σ.*

For a proof, see [33] Proposition 1.1.

Corollary 1.7. *With the same notation as in Proposition* 1.6, *if* $\mathrm{Supp}(D + \sigma^* D)$ *is contained in the supporting set of the exceptional* *divisor of μ, then there exists a $\mathcal{D}_{2p}$ covering, S', of $\mathbf{P}^2$ branched along B.*

Proof. Let S' be the Stein factorization of $q \circ f$; and we denote the induced morphism by $\pi : S' \to \mathbf{P}^2$. Since $\mathbf{C}(S') \cong \mathbf{C}(S)$ and $\mathbf{C}(\mathbf{P}^2) \cong \mathbf{C}(\Sigma)$, π is a $\mathcal{D}_{2p}$ covering of $\mathbf{P}^2$. Hence it is enough to show $\Delta(\pi) = B$. By the assumption in the construction of S, the branch locus of $f \circ g$ is contained in the supporting set of the proper transform, $\bar{B}$, of B and the exceptional divisor of μ. As the the image of the exceptional set of q is a subset of $\mathrm{Sing}(B)$, we have our statement. Q.E.D.

§2. A sketch of a proof of Theorem 0.8

We keep the same notation as those in §1. The goal of this section is to show the following theorem.

Theorem 2.1. *Let B be as Theorem 0.8. Suppose that there exists an odd prime p such that*

$$l_p + \mu_B > d^2 - 3d + 3.$$

Then there exists a $\mathcal{D}_{2p}$ covering branched along B.

Note that Theorem 0.8 easily follows from Theorem 2.1 and Corollary 1.4. To prove Theorem 2.1, it is enough to show that the inequality assures the existence of a pair of a divisor and a line bundle, $(D, \mathcal{L})$, on Z satisfying the conditions in Proposition 1.6 and Corollary 1.7. The rest of this section is devoted to it.

Let $\mathrm{NS}(Z)$ be the Néron-Severi group of Z. As $\pi_1(\mathbf{P}^2) = \{1\}$ and B has at most simple singularities, by [3], [4] and [6], $\pi_1(Z) = \{1\}$. Hence $H^2(Z, \mathbf{Z})$ is a unimodular lattice with respect to the intersection pairing. In particular, $\mathrm{NS}(Z) = \mathrm{Pic}(Z)$ and it is a sublattice of $H^2(Z, \mathbf{Z})$. Let T be the subgroup of $\mathrm{NS}(Z)$ generated by the pull-back of a line of $\mathbf{P}^2$ and irreducible components of the exceptional divisor of μ. As we can easily see, T has a direct decomposition

$$T = \mathbf{Z}L \oplus \bigoplus_{x \in \mathrm{Sing}(B)} T_x,$$

where L is the pull-back of a line of $\mathbf{P}^2$ and T_x is the subgroup of $\mathrm{NS}(Z)$ generated by irreducible components of the exceptional divisor arising from the singularity ${f'}^{-1}(x)$. Note that the direct decomposition as above is orthogonal with respect to the intersection pairing.

Suppose that the effective divisor D as in Corollary 1.7 exists. Then this implies that $\mathrm{NS}(Z)/T$ has a p-torsion. In constructing $\mathcal{D}_{2p}$ coverings, what is important is that the converse of this holds.

Theorem 2.2. *If* $\mathrm{NS}(Z)/T$ *has a p-torsion, then there exists an effective divisor D and a line bundle on* $\mathcal{L}$ *satisfying the conditions in Proposition* 1.6 *and Corollary* 1.7.

We give here a rough explanation. For details, see [33].

Let $T^{\sharp} = \{D \in \mathrm{NS}(Z) | nD \in T \text{ for some } n \in \mathbf{N}\}$ and let $T^{\vee} = \mathrm{Hom}_{\mathbf{Z}}(T, \mathbf{Z})$. Then:

(i) $T^{\perp\perp} = T^{\sharp}$ and $T^{\sharp}/T \cong (\mathrm{NS}(Z)/T)_{\mathrm{tor}}$. Here for a subgroup, M, of $H^2(Z, \mathbf{Z})$, we denote its orthogonal complement with respect to the intersection pairing by $M^{\perp}$.

(ii) By using intersection pairing, one can identify $T^{\sharp}$ with a subgroup of $T^{\vee}$. Hence $T^{\sharp}/T \subset T^{\vee}/T \cong \mathbf{Z}/2\mathbf{Z} \oplus \oplus_{x\in \mathrm{Sing}(B)} T_x^{\vee}/T_x$. Also, as $T^{\vee} \otimes \mathbf{Q} = T \otimes \mathbf{Q}$, we can use a $\mathbf{Q}$-divisor in $T_x \otimes \mathbf{Q}$ as a representative for an element in $T^{\vee}/T$. For example, if the singularity x is of A_n type, then $T^{\vee}/T \cong \mathbf{Z}/n\mathbf{Z}$ and we can choose a representative of a generater of $T^{\vee}/T$ as follows:

$$\frac{1}{n+1} D_x,$$

where if n is even,

$$D_x = n(\Theta_1 - \Theta_n) + (n-1)(\Theta_2 - \Theta_{n-1}) + \cdots + \frac{n}{2}(\Theta_{n/2} - \Theta_{n/2+1}),$$

and if n is odd,

$$\begin{aligned} D_x &= n(\Theta_1 - \Theta_n) + (n-1)(\Theta_2 - \Theta_{n-1}) + \cdots \\ &+ \frac{n-1}{2}(\Theta_{(n-1)/2} - \Theta_{(n+3)/2)}) + \frac{n+1}{2}\Theta_{(n+1)/2}, \end{aligned}$$

where Θ_i's are irreducible components of the exceptional divisor labeled in such way that $\Theta_i\Theta_{i+1} = 1$ $(1 \le i \le n-1)$. Note that $\sigma^*\Theta_i = \Theta_{n+1-i}$ with respect to the covering transformation of f.

Let $\mathcal{L}'$ be any element of $\mathrm{NS}(Z)$ that gives rise to a p-torsion element, α, in $\mathrm{NS}(Z)/T$. Then we may assume that $\mathcal{L}' \in T^{\sharp}$; and we have

$$\alpha = \left(\alpha_L, (\alpha_x)_{x\in \mathrm{Sing}(B)}\right), \quad \alpha_L \in \mathbf{Z}/2\mathbf{Z}, \ \alpha_x \in T_x^{\vee}/T_x.$$

Since $p | \sharp(T_x^{\vee}/T_x)$ if and only if either x is of type A_{pk-1} or x is of type E_6 and $p = 3$, we may assume that $\alpha_x = 0$ for other type of singularities. For x with type A_{pk-1}, by (ii) as above, we may assume that $\alpha_x = i/pD_x \bmod T$ for some $0 < i < p$. By the above explicit formula, we can show $\alpha_x = 1/p(D' - \sigma^*D') \bmod T$, where D' is an effective divisor satisfying the condition (i) in Proposition 2.1. For x with type E_6, the situation is similar (see [30] or [33]). Thus, by replacing

$\mathcal{L}'$ if necessary, we can see there exists an effective divisor D and a line bundle $\mathcal{L}$ on Z satisfying the conditions in Proposition 1.6 and Corollary 1.7.

We now go on to show that the inequality in Theorem 0.8 implies the existence of p-torsion.

Lemma 2.3. *Let $b_i(Z)$ be the i-th Betti number of Z. Then we have*

$$b_2(Z) = d^2 - 3d + 4.$$

Proof. The statement easily follows from Lemma 6, [13] and the Noether formula.

In the following, we make use of some Nikulin theory ([21]). This argument is a modification of Miranda-Persson's in §4, [18]. A similar argument is also found in [35].

Suppose that there exists no p-torsion in $T^{\perp\perp}/T$. Then

$$S_p(T^\vee/T) \cong S_p\left((T^{\perp\perp})^\vee/T^{\perp\perp}\right),$$

where $S_p(G)$ denote the p-Sylow group of G. On the other hand, by Proposition 1.2 in [11], we have

$$(T^\perp)^\vee/T^\perp \cong (T^{\perp\perp})^\vee/T^{\perp\perp}.$$

Hence the number of generators, l_1, of $S_p(G_{T^{\perp\perp}}) \leq \operatorname{rank} T^\perp = b_2(Z) - \operatorname{rank} T = d^2-3d+4-(\mu_B+1)$. On the other hand, by the assumption we have $l_1 \geq l_p > d^2-3d+3-\mu_B$. This leads us to a contradiction. Q.E.D.

§3. Examples

In this section, we give some examples of plane curves satisfying the inequality in Theorem 0.8. Since $\pi_1(\mathbf{P}^2 \setminus B)$ is always abelian for conics, we start with the case of $\deg B = 4$.

Example 3.1. $\deg B = 4$. In this case, $d^2 - 3d + 3 = 7$.

(i) Let B be a quartic curve with $3A_2$ singularities. Then $\mu_B = 6$, $l_3 = 3$. Hence the inequality in Theorem 0.8 is satisfied. This implies that $\pi_1(\mathbf{P}^2 \setminus B)$ is non-abelian.

As it is well-known, $\pi_1(\mathbf{P}^2 \setminus B)$ is a finite non-abelian group of order 12. In fact, $\pi_1(\mathbf{P}^2 \setminus B) \cong B_3(\mathbf{P}^1)$ (see [37] or [8])

(ii) Let B be a quartic curve having two irreducible components; one is a cuspidal cubic, C, and the other is a tangent line, l, at an inflection point of C. In this case, the singularities of B are of type A_5 and A_2. Hence $\mu_B = 7$, $l_3 = 2$. Hence the inequality in Theorem 0.8

is satisfied; and $\pi_1(\mathbf{P}^2 \setminus B)$ is non-abelian. In [19], one can find more detailed description.

We now go on to the case of $\deg B = 6$.

Example 3.2. $\deg B = 6$. In this case, $d^2 - 3d + 3 = 21$. There exists a sextic curve B for every case in the following table. In each case, the inequality in Theorem 0.8 is satisfied. Hence $\pi_1(\mathbf{P}^2 \setminus B)$ is non-abelian.

	types of singularities of B	μ_B	l_3
1	$6A_2 + 4A_1$	16	6
2	$3E_6 + A_1$	19	3
3	$2E_6 + 2A_2 + 2A_1$	18	4
4	$E_6 + 4A_2 + 3A_1$	17	5
5	$E_6 + A_5 + 4A_2$	19	6
6	$E_6 + A_{11} + A_2$	19	3
7	$E_6 + A_8 + A_3 + A_2$	19	3
8	$E_6 + A_8 + 2A_2 + A_1$	19	4
9	$E_6 + A_5 + A_4 + 2A_2$	19	3
10	$D_5 + A_8 + 3A_2$	19	4
11	$E_6 + A_5 + A_3 + 2A_2 + A_1$	19	4
12	$E_6 + 2A_5 + A_3$	19	3
13	$D_5 + 2A_5 + 2A_2$	19	4
14	$D_4 + 3A_5$	19	3
15	$D_4 + A_{11} + 2A_2$	19	3
16	$3A_5 + 4A_1$	19	3

What is problem here is the existence of curves as above. We here explain it for No. 1, 2 and 15. For the others, we give a sketch how we show it in §5. Also, for those with $\mu_B = 19$, one can check it in [36]

No. 1: $6A_2 + 4A_1$. One obtains such a sextic curve as a generic plane section of the discriminant variety, Disc $\left(H^0(\mathbf{P}^1, \mathcal{O}(4)), \mathcal{O}(4)\right)$ (see [10] for details).

No. 2, 16: $3E_6 + A_1$ and $3A_5 + 4A_1$. These two cases are closely related to each other. Let C be a nodal cubic curve and let l_1, l_2, and l_3 be three tangent lines at three inflection points of C (C has exactly three inflection points). A sextic curve for No. 16 is given by $C + l_1 + l_2 + l_3$. Next, consider a Cremona transformation given by these three tangent lines. Then the image of C gives a sextic curve for No. 2.

Remark 3.3. By Proposition 5.6 in [34] and [23], one can see that sextic curves for No. 1 - 10 are irreducible torus curves of type $(2,3)$ (see [23] for torus curves). It might be interesting to study them systematically as in [23].

Remark 3.4. The author does not know any single example of B with $\deg B \geq 8$ satisfying the inequality in Theorem 0.8. The condition may be too strong for curves of higher degree. In fact, in [26], Sakai proved:

Theorem 3.5 (Sakai). *Let b be the number of cusps. Then*

$$b \leq \frac{5}{16}d^2 - \frac{3}{8}d.$$

Suppose that B has only cusps. Then Sakai's inequality implies that there is no B with $3b > d^2 - 3d + 3$ if $d \geq 29$. Hence our inequality is too strong for curves of higher degree. This is something one can expect, since Theorem 0.8 comes from Theorem 2.1, which gives very rough information on $\pi_1(\mathbf{P}^2 \setminus B)$. Nevertheless, as we see in next section, the inequality in Theorem 0.8 is very nice estimate for sextic curves when $p = 3$.

§4. 4 Some sextic curves with $l_3 + \mu_B = 21$

We look into what happens for sextic curves when the equality $l_3 + \mu_B = 21$ holds. For such cases, as we have already seen Theorem 0.9, we are not able to determine whether $\pi_1(\mathbf{P}^2 \setminus B)$ is abelian or not. In this section, we give other examples of Zariski pairs with equality $l_3 + \mu_B = 21$. More precisely, we give two kinds of sextic curves, B_1 and B_2, such that (i) both of them have the same configuration of singularities, (ii) B_1 is the branch locus for some $\mathcal{D}_6$ covering, while B_2 can never be. This means that $\mathcal{D}_6$ is a homomorphic image of $\pi_1(\mathbf{P}^2 \setminus B_1)$, while there is no homomorphism from $\pi_1(\mathbf{P}^2 \setminus B_2)$ to $\mathcal{D}_6$. Now we give a list for the configurations of singularities.

Theorem 4.1. *For each case in the following table, there exists a pair of irreducible sextic curves (B_1, B_2) with the properties* (i) *and* (ii) *as above.*

	Configuration of singularities of B
1	$E_6 + A_8 + A_2 + 2A_1$
2	$E_6 + A_5 + 2A_2 + 2A_1$
3	$E_6 + 4A_2 + 2A_1$
4	$2E_6 + A_5 + A_1$
5	$2E_6 + 2A_2 + A_1$

Remark 4.2. (i) Note that No 2 is not contained in the examples in [31] and [32]. We show that the example does exist in §5.

(ii) For all cases, one of geometric differences between B_1 and B_2 is the existence of a conic, C, as in Example 0.3. Namely, for B_1 there exists a conic, C, with properties (i) $C \cap B_1 \subset \mathrm{Sing}(B_1)$; and the type of singularities in $C \cap B_1$ are either A_{3k-1} or E_6, and (ii) the intersection multiplicity at A_{3k-1} (resp. E_6) is $2k$ (resp. 4), while there exists no such conic for B_2. In [9], Degtyarev conjectured that there exist exact one rigid isotopy class for a sextic curve having the configuration of singularities No 1, 2 and 4 in Theorem 4.1. Our examples show that his conjecture is false for these cases.

§5. Existence of sextic curves

The main purpose of this section is to explain how one gets sextic curves with the prescribed properties as in §3 and §4. The method that we explain here is the one in [31] and [32].

Let $\varphi : \mathcal{E} \to \mathbf{P}^1$ be an elliptic $K3$ surface with a section s_0, i.e., a Jacobian elliptic $K3$ surface. It is well-known that such surfaces are always obtained in the following way(cf. [17]):

Let $\mathbf{F}_4$ be the Hirzebruch surface of degree 4, i.e., $\mathbf{F}_4 = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(4))$. Let Δ_0 and Δ_∞ be the negative and positive section, respectively. Let T be a reduced divisor on $\mathbf{F}_4$ such that (i) $T \sim 3\Delta_\infty$ and (ii) T has at most simple singularities. As $\Delta_0 + T \sim 3\Delta_\infty \sim 4\Delta_0 + 12f$, where f denotes the class of a fiber $\mathbf{F}_4 \to \mathbf{P}^1$, there exists a double covering, $\mathcal{E}'$, of $\mathbf{F}_4$ branched along $\Delta_0 + T$. Let $\mu : \mathcal{E} \to \mathcal{E}'$ be the canonical resolution, which satisfies the following diagram:

$$\begin{array}{ccc} \mathcal{E}' & \overset{\mu}{\leftarrow} & \mathcal{E} \\ \downarrow & & \downarrow \\ \mathbf{F}_4 & \leftarrow & \Sigma, \end{array}$$

where $\Sigma \to \mathbf{F}_4$ is a composition of blowing-ups so that the branch locus of $\mathcal{E} \to \Sigma$ is smooth. Then $\mathcal{E}$ is a $K3$ surface with a Jacobian elliptic fibration induced by the ruling $\mathbf{F}_4 \to \mathbf{P}^1$; and its section s_0 comes from Δ_0.

We can also explain the above construction in another way as follows:

Since $\varphi : \mathcal{E} \to \mathbf{P}^1$ is a Jacobian elliptic fibration, the generic fiber of φ is an elliptic curve, $\mathcal{E}_{\mathbf{C}(\mathbf{P}^1)}$, over $\mathbf{C}(\mathbf{P}^1)$ (s_0 gives a reference point). Considering s_0 as the zero, we can equip $\mathcal{E}_{\mathbf{C}(\mathbf{P}^1)}$ with additive group structure. Let σ denote the inverse morphism with respect to the group law on $\mathcal{E}_{\mathbf{C}(\mathbf{P}^1)}$. It induces a fiber preserving involution on $\mathcal{E}$, which we also denote by σ. Consider the quotient surface $\mathcal{E}/\langle\sigma\rangle$. $\mathcal{E}/\langle\sigma\rangle$ is nothing but Σ in the above diagram, and it is not minimal in general. Blowing down (-1) curves contained in fibers not meeting Δ_0 in an appropriate order, we have $\mathbf{F}_4$. Let $MW(\mathcal{E})$ be the Mordell-Weil group of $\mathcal{E}$, i.e., the group of sections of φ. Now we can easily see:

Lemma 5.1. (i) *$\Delta_0 + T$ is the image of the locus of 2-torsions, $T_2(\mathcal{E})$, with respect to the group law.*

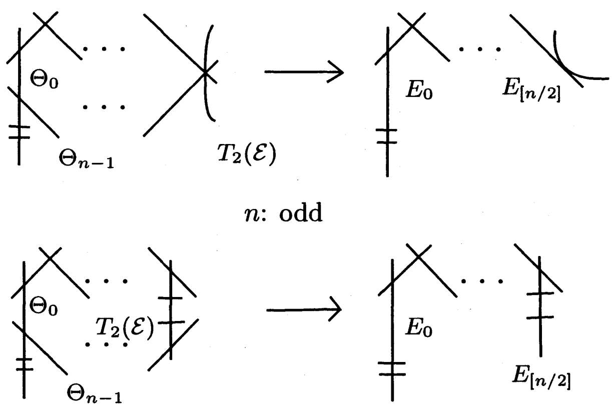
(ii) *T is irreducible if and only if the Mordell-Weil group, $MW(\mathcal{E})$, has no 2-torsion point.*

We now consider when one can blow down $\Sigma(=\mathcal{E}/\langle\sigma\rangle)$ to $\mathbf{P}^2$, not to $\mathbf{F}_4$, in such a way that the image of $T_2(\mathcal{E})$ is a sextic curve. There are several ways to do it ([24]), and we here explain one of them.

Lemma 5.2. *If $\varphi : \mathcal{E} \to \mathbf{P}^1$ has a singular fiber of type I_n $(n \geq 6)$, then one can blow down Σ to $\mathbf{P}^2$; and the image of $T_2(\mathcal{E})$ is a sextic curve with an E_6 singularity.*

Proof. The action of σ on an I_n fiber is as follows (cf. [5], [20]): Label irreducible components of an I_n fiber in such a way that

$$\Theta_0\Theta_1 = \cdots = \Theta_{n-1}\Theta_0 = 1, \quad \Theta_0 s_0 = 1.$$



(Figure 1)

Then $\sigma^*\Theta_i = \Theta_{n-i}$ and $\sigma^*\Theta_0 = \Theta_0$. Hence the image of an I_n fiber in Σ is a tree of $([n/2]+1)$ $\mathbf{P}^1$'s, E_i $(i = 0, ..., [n/2])$, such that

$$E_iE_{i+1} = 1, \quad \left(0 \leq i \leq \left[\frac{n}{2}\right] - 1\right), \quad E_0\bar{s}_0 = 1,$$

where $\bar{s}_0$ is the image of s_0, and

$$E_0^2 = E_{[n/2]}^2 = -1, \quad E_i^2 = -2, \quad 1 \leq i \leq \left[\frac{n}{2}\right] - 1.$$

In blowing down Σ to $\mathbf{F}_4$, we first blow down $E_{[n/2]}$, then $E_{[n/2]-1}$, $E_{[n/2]-1}$ and so on. In order to blow down Σ to $\mathbf{P}^2$, we do it in a different way. Namely, we first blow down E_0, then E_1 and E_2 in this order. Then $\bar{s}_0$ becomes a (-1) curve; and one can blow down it to a point, x. Then we blow down $E_{[n/2]}$, $E_{[n/2]-1}, \ldots$, E_4 in this order. Blowing down (-1) curves in the other fibers in the same way as we do in blowing down Σ to $\mathbf{F}_4$, we have $\mathbf{P}^2$. Since (i) the image of $T_2(\mathcal{E})$ has an E_6 singularity at x, (ii) the image of a general fiber for elliptic fibration is a line through x, we infer that the image of $T_2(\mathcal{E})$ is a sextic curve, $B_{\mathcal{E}}$, with an E_6 singularity. Q.E.D.

Remark 5.3. In a similar manner, one can also blow down Σ to $\mathbf{P}^2$ if φ has $3I_2$ (resp. I_4 and I_2) singular fibers. In this case, the corresponding triple point is D_4 (resp. D_5).

Corollary 5.4. *$B_{\mathcal{E}}$ is irreducible if and only if $MW(\mathcal{E})$ has no 2-torsion point.*

Definition 5.5. We call the singular fibers as in Lemma 5.2 and Corollary 5.3 the *preferred fibers.*

As one can easily see from its construction $B_{\mathcal{E}}$, the type of a singularity of $B_{\mathcal{E}}$ other than E_6, D_4 and D_5 as in Lemma 5.2 and Remark 5.3 has something to do with that of the corresponding singular fiber of φ. We give a table for its correspondence (cf. [17]):

Lemma 5.6. *The relation between the type of a non-preferred singular fiber of φ and that of the corresponding singularity of $B_{\mathcal{E}}$ is as follows;*

Type of a singular fiber	I_n $(n \geq 2)$	I_1	I_n^*
Type of a singular point	A_{n-1}	*a smooth point*	D_{n+4}

II	II^*	III	III^*	IV	IV^*
a smooth point	E_8	A_1	E_7	A_2	E_6

With the argument so far, the existence of the sextic curves as in Example 3-15 is reduce to that of Jacobian elliptic $K3$ surfaces with the prescribed configuration of singular fibers. Here we give a table for that.

Lemma 5.7. *A sextic curve with singularities as in one of the left column exists if a Jacobian elliptic $K3$ surface with the configuration of singular fibers in the same row of the right column exists.*

3	$2E_6 + 2A_2 + 2A_1$	I_6, IV^*, $2I_3$, $2I_2$
4	$E_6 + 4A_2 + 3A_1$	I_6, $4I_3$, $3I_2$
5	$E_6 + A_5 + 4A_2$	$2I_6$, $4I_3$
6	$E_6 + A_{11} + A_2$	I_6, I_{12}, I_3, $3I_1$
7	$E_6 + A_8 + A_3 + A_2$	I_6, I_9, I_4, I_3, $2I_1$
8	$E_6 + A_8 + 2A_2 + A_1$	I_6, I_9, $2I_3$, I_2, I_1
9	$E_6 + A_5 + A_4 + 2A_2$	$2I_6$, I_5, $2I_3$, I_1
10	$D_5 + A_8 + 3A_2$	I_4, I_2, I_9, $3I_3$
11	$E_6 + A_5 + A_3 + 2A_2 + A_1$	$2I_6$, I_4, $2I_3$, I_2
12	$E_6 + 2A_5 + A_3$	$3I_6$, I_4, $2I_1$
13	$D_5 + 2A_5 + 2A_2$	I_2, I_4, $2I_6$, $2I_3$
14	$D_4 + 3A_5$	$3I_2$, $3I_6$
15	$D_4 + A_{11} + 2A_2$	$3I_2$, I_{12}, $2I_3$

For No. 4 - 15, such elliptic K3 surfaces exist by [18]. For No. 3, one obtains it in the same way as in Lemma 4.2, [32]. Hence, by Lemma 5.7, there exist sextic curves for No. 3 - 15 in Example 3.2.

Now we go on to Theorem 4.1. An easy but key lemma to obtain a pair of sextic curves having the same configuration of singularities is as follows:

Lemma 5.8. *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be Jacobian elliptic $K3$ surfaces such that*

(i) *the configurations of non semi-stable singular fibers of $\mathcal{E}_1$ and $\mathcal{E}_2$ are the same, and*

(ii) *the configurations of semi-stable singular fibers of $\mathcal{E}_1$ is I_6, $I_{n_1}, \ldots, I_{n_s}$ $(n_i \geq 2)$, $r\,I_1$, while that of $\mathcal{E}_2$ is I_7, $I_{n_1}, \ldots, I_{n_s}$ $(n_i \geq 2)$, $(r-1)\,I_1$.*

Then the configuration of singularities of $B_{\mathcal{E}_1}$ is the same as that of $B_{\mathcal{E}_2}$.

Proof. From Lemmas 5.2 and 5.6, the statement follows. Q.E.D.

Corollary 5.9. *Let $\varphi_1 : \mathcal{E}_1 \to \mathbf{P}^1$ and $\varphi_2 : \mathcal{E}_2 \to \mathbf{P}^1$ be elliptic $K3$ surfaces having the configurations of singular fibers as in the table below. Then the configurations of singularities of $B_{\mathcal{E}_1}$ and $B_{\mathcal{E}_2}$ are the right column in the table.*

	Singular fibers of $\mathcal{E}_1$	*Singular fibers of $\mathcal{E}_2$*	*Singularities of $B_{\mathcal{E}_i}$ $(i = 1, 2)$*
1	I_6, I_9, I_3, $2I_2$, $2I_1$	I_7, I_9, I_3, $2I_2$, I_1	$E_6 + A_8 + A_2 + 2A_1$
2	$2I_6$, $2I_3$, $2I_2$, $2I_1$	I_7, I_6, $2I_3$, $2I_2$, I_1	$E_6 + A_5 + 2A_2 + 2A_1$
3	I_6, $4I_3$, $2I_2$, $2I_1$	I_7, $4I_3$, $2I_2$, I_1	$E_6 + 4A_2 + 2A_1$
4	$2I_6$, IV^*, I_2, $2I_1$	I_7, IV^*, I_2, I_1	$2E_6 + A_5 + A_1$
5	I_6, IV^*, $2I_3$, I_2, $2I_1$	I_7, IV^*, $2I_3$, I_2, I_1	$2E_6 + 2A_2 + A_1$

In order to prove Theorem 4.1, the following is crucial.

Proposition 5.10. *Let $B_{\mathcal{E}_1}$ and $B_{\mathcal{E}_2}$ as in Corollary* 5.9. *There exists a $\mathcal{D}_6$ covering branched along $B_{\mathcal{E}_i}$ if and only if $MW(\mathcal{E}_i)$ has a* 3*-torsion.*

We give here an idea for our proof. Let T_{φ_i} be the subgroup of $\mathrm{NS}(\mathcal{E}_i)$ generated by the zero section, a general fiber and irreducible components of singular fibers not meeting the zero section. Then $MW(\mathcal{E}_i) \cong \mathrm{NS}(\mathcal{E}_i)/T_\varphi$ by [27]. Hence this indicates our proof is done in a similar way to that of Theorem 2.1. For details, see [31], [32].

Now Theorem 4.1 easily follows from the below:

Proposition 5.11. *For each case in Corollary* 5.9, *there exist elliptic surfaces $\mathcal{E}_1$ and $\mathcal{E}_2$ satisfying the following properties:*

(i) *$MW(\mathcal{E}_i)$ has no* 2*-torsion. In particular, $B_{\mathcal{E}_i}$ is irreducible.*

(ii) *$MW(\mathcal{E}_2)$ has no* 3*-torsion.*

(iii) *$MW(\mathcal{E}_1)$ has a* 3*-torsion.*

Proof. For all the statements except those for No. 2, one can find their proof in [31] and [32]. Hence we give a proof for No. 2 only. By [18], there exists $\mathcal{E}_2$ and it satisfies (i) and (ii) by Lemma 1.7 in [32]. For $\mathcal{E}_1$, we construct it in the following way: Let $\psi : Y \to \mathbf{P}^1$ be a rational elliptic surface with singular fibers $3I_3$, I_2, I_1. Let v_1 and v_2 be points of $\mathbf{P}^1$ such that $\psi^{-1}(v_i)$ $(i = 1, 2)$ are I_3 fibers. Let g be a degree 2 map from $\mathbf{P}^1$ to $\mathbf{P}^1$ branched at v_1 and v_2. Consider an elliptic $K3$ surface, $\mathcal{E}_1$, obtained as the pull-back surface of Y by g, i.e., the relatively minimal smooth model of the fiber product $Y \times_g \mathbf{P}^1$. Then:

Claim. *$MW(\mathcal{E}_1)$ has* (i) *a* 3*-torsion, and* (ii) *no* 2*-torsion.*

Proof of Claim. Since $MW(Y)$ has a 3-torsion (see [25], for example), so does $MW(\mathcal{E}_1)$. Since the covering transformation of g commutes with the inverse morphism, σ, $T_2(\mathcal{E}_1)$ is a double covering of $T_2(Y)$, and it is branched at two points on $T_2(Y)$. Hence $T_2(\mathcal{E}_1)$ is irreducible. In particular, $MW(\mathcal{E}_1)$ has no 2-torsion by Corollary 5.4. Q.E.D.

Thus we have $\mathcal{E}_1$ with the desired properties.

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