Advanced Studies in Pure Mathematics 29, 2000
Singularities - Sapporo 1998
pp. 163–180

Geometry of complex surface singularities

Lê Dũng Tráng

§ Introduction.

In the local study of complex analytic spaces, it is natural to investigate the behaviour of the tangent spaces near a singular point. In the general case of equidimensional singularities, after choosing a local embedding of the singular space into a complex affine space, B. Teissier and the author have given the structure of the limit of tangent hyperplanes, i.e. hyperplanes containing a tangent space at a non-singular point, in terms of a family of cones contained in the tangent cone of the singularity and called the Auréole of the singularity (see [LT2]).

In the case of surface singularities, the Auréole is given by the tangent cone and a finite number of generatrices of the tangent cone called the exceptional tangents. Recent works of J. Snoussi showed that these exceptional tangents coincide with the special generatrices of Gonzalez and Lejeune ([GL]). His result is based on the fact that, after choosing a local embedding of the surface into a complex affine space, a hyperplane is not a limit of tangent hyperplanes if and only if its intersection with the normal surface singularity is a curve with a Milnor number (in the sense of Buchweitz and Greuel [BG]) which is minimum. This work enhances the interest in the local geometry of complex surface singularities that we began in [L3] and [LT1].

This paper is essentially a survey of results about the limits of tangent hyperplanes of a normal surface singularity. It gives a geometrical approach in the study of a normal surface singularity and suggests new research interests in effective resolutions of normal surface singularities. In particular, it should lead to effective bounds for the number of normalized blowing-up needed to solve the singularity.

Received March 29, 1999
Revised May 17, 1999

§1. An example.

1.1. Let $f\colon U \to \mathbf{C}$ be a complex analytic function defined on an open neighbourhood U of 0 in $\mathbf{C}^3$. We assume that $f(0) = 0$ and the function f has an isolated critical point at 0. The function f defines a complex analytic surface X closed in U. The analytic local ring $\mathcal{O}_{X,0}$ of X at 0 is

$$\mathcal{O}_{X,0} = \mathbf{C}\{X, Y, Z\}/(f)$$

quotient of the local ring of convergent series $\mathbf{C}\{X, Y, Z\}$ at 0 by the principal ideal generated by f in $\mathbf{C}\{X, Y, Z\}$.

Since we have assumed that 0 is an isolated critical point of f, the element f is irreducible in $\mathbf{C}\{X, Y, Z\}$, i.e. the principal ideal (f) generated by f in the ring $\mathbf{C}\{X, Y, Z\}$ is prime. Therefore, the ring $\mathcal{O}_{X,0}$ is an integral domain, i.e. it has no zero divisors. Furthermore, since it is the local ring of a hypersurface whose singularities are in codimension 2, a criterion of J.P. Serre (see [S] (IV D) §4) implies that the ring $\mathcal{O}_{X,0}$ is normal, i.e. it is integrally closed in its field of fractions.

1.2. As B. Teissier did in [T] (Chap. 1), we can associate to the germ $(X, 0)$ of the surface X at 0 the following invariants.

First, to any hypersurface V with an isolated singularity at the point 0, J. Milnor has associated an integer ([M] §7) called the Milnor number $\mu(V, 0)$ of V at 0. In our case, the Milnor number $\mu(X, 0)$ of the surface X at 0 is given by the complex dimension of the vector space

$$\mathcal{M}_{X,0} := \mathbf{C}\{X, Y, Z\}/(\partial f/\partial X, \partial f/\partial Y, \partial f/\partial Z)$$

quotient of $\mathbf{C}\{X, Y, Z\}$ by the ideal generated by the partial derivatives of f. Hilbert-Rückert Nullstellensatz (see [N] Chap. III §2 Theorem 2) implies that the $\mathbf{C}$-vector space $\mathcal{M}_{X,0}$ is finite dimensional over the field $\mathbf{C}$ if and only if f has an isolated critical point at 0. We have:

Lemma 1.2.1. *The Milnor number is a topological invariant of the hypersurface X at 0, in the sense that, for any hypersurface Y of $\mathbf{C}^3$ which has a singularity at the point y and for which there is a germ of homeomorphism of $(\mathbf{C}^3, 0)$ onto $(\mathbf{C}^3, y)$ which sends $(X, 0)$ onto (Y, y), we have that Y has an isolated singularity at 0 and $\mu(X, 0) = \mu(Y, y)$.*

Proof. Actually, the result is true in any dimension, but we give a proof for hypersurfaces in $\mathbf{C}^3$.

We need a topological interpretation of the Milnor number. In the case of isolated singularities, following J. Milnor [M] (Corollary 2.9), one can prove that there is $\epsilon_0 > 0$, such that, for any ϵ, $\epsilon_0 > \epsilon > 0$, the real sphere $S_\epsilon(0)$ (boundary of the open ball $B_\epsilon(0)$) of $\mathbf{C}^3$ centered at 0 with radius ϵ is transverse to the hypersurface $X := \{f = 0\}$. Let us fix ϵ, $\epsilon_0 > \epsilon > 0$. By the openness of the transversality, there is $\eta(\epsilon) > 0$, such that for any $t \in \mathbf{C}, 0 < |t| < \eta(\epsilon)$, the hypersurface $\{f = t\}$ intersects $S_\epsilon(0)$ transversally. So, the space $\{f = t\} \cap B_{\epsilon_0}(0)$ is a smooth manifold of real dimension 4. In [M] (Theorem 5.11 and Theorem 6.5), it is proven that the homotopy type of $\{f = t\} \cap B_{\epsilon_0}(0)$ is the one of a bouquet of $\mu(X, 0)$ 2-spheres, i.e. a space union of $\mu(X, 0)$ 2-spheres with one point in common. The space $\{f = t\} \cap B_{\epsilon_0}(0)$ is called a **Milnor fiber** of X at 0. On the other hand, Ehresmann Lemma (see e.g. [D] §20.8 Problème 4) implies that, for any η, $0 < \eta < \eta(\epsilon)$, the function f induces a locally trivial smooth fibration of $f^{-1}(\delta D_\eta) \cap B_\epsilon(0)$ onto δD_η, where δD_η is the circle of $\mathbf{C}$ centered at 0 with radius η. In [L2], we show that the homotopy class of this fibration is a topological invariant of the hypersurface X at 0. We call this fibration the **Milnor fibration** of X at 0.

Now let Y be a complex analytic surface closed in an open neighborhood V of $y \in Y$ in $\mathbf{C}^3$. Assume that we have a homeomorphism φ of a neighborhood U_1 of 0 in U onto V_1 of y in V, such that $\varphi(X \cap U_1) = Y \cap V_1$ and $\varphi(0) = y$. First, we prove that Y has an isolated singularity at y. Let x be a non-singular point of $X \cap U_1$. The homeomorphism φ induces a germ of homeomorphism of the germ (X, x) onto $(Y, \varphi(x))$. To prove that Y has an isolated singularity at y, it is enough to show that the point $\varphi(x)$ is non-singular on Y. This fact is a consequence of a Theorem of A'Campo ([AC] Théorème 3) which states that the Lefschetz number of the monodromy of a Milnor fibration is not zero if and only if the hypersurface is non-singular. In fact, it is easy to see that the Milnor fibration of X at a non-singular point x is trivial and its fiber is contractible. Therefore the Milnor fibration of Y at $\varphi(x)$, which is homotopically isomorphic to the Milnor fibration of X at x, has contractible fibers and is trivial ([M] Lemma 2.13). By A'Campo's theorem this implies that Y is not singular at $\varphi(x)$. On the other hand, since the Milnor fibers of X at 0 and Y at y have the same homotopy type, their Milnor numbers are equal.

We put

$$\mu^3(X, 0) := \mu(X, 0).$$

Secondly, one can prove that there is an open Zariski dense subset Ω_2 of the space $\check{\mathbf{P}}^2$ of complex hyperplanes through 0 in $\mathbf{C}^3$ such that, for any $H \in \Omega_2$, the Milnor number $\mu(H \cap X, 0)$ does not depend on H. Then, for $H \in \Omega_2$,

$$\mu^{(2)}(X, 0) := \mu(X \cap H, 0).$$

Third, we consider the multiplicity $m(X,0)$ of X at 0:

$$\mu^{(1)}(X,0) := m(X \cap H, 0).$$

As it is easily seen for hypersurfaces, there is an open dense Zariski subset Ω_1 of the space $\mathbf{P}^2$ of complex lines through 0 in $\mathbf{C}^3$, such that, for any $\ell \in \Omega_1$, the Milnor number $\mu(X \cap \ell, 0)$ is finite and does not depend on ℓ. But in this case, $X \cap \ell$ is a zero dimensional hypersurface in ℓ, so that $\mu(X \cap \ell, 0) + 1$ is nothing but the multiplicity of X at 0.

Let

$$f = f_m + f_{m+1} + \dots$$

be the Taylor expansion of f at 0, where f_k is a homogeneous polynomial of degree k and m is the multiplicity of f at 0. It is known that, for a hypersurface X, the multiplicity of the function f defining X equals the multiplicity at 0 of X, i.e. the multiplicity of the local ring $\mathcal{O}_{X,0}$ (see [S] V A) §2).

Then, it is easy to show that one can choose

$$\Omega_1 = \mathbf{P}^2 - Proj|C_{X,0}|$$

where $Proj|C_{X,0}|$ is the projective curve associated to the reduced tangent cone of X at 0.

B. Teissier showed in [T] (Chap. 1 §2), that the 3-uple

$$\mu^*(X,0) = (\mu^{(3)}(X,0), \mu^{(2)}(X,0), \mu^{(1)}(X,0))$$

is an analytic invariant of the germ of hypersurface $(X,0)$, i.e. if the local rings $\mathcal{O}_{X,0}$ and $\mathcal{O}_{X',0}$ of two 2-dimensional hypersurfaces X and X' at 0 are isomorphic, then, we have

$$\mu^*(X,0) = \mu^*(X',0).$$

Notice that, since $\mu^{(3)}(X,0)$ is a topological invariant of $(X,0)$, it is obviously an analytic invariant of the germ of hypersurface $(X,0)$.

Recall that a hyperplane H is a limit of tangent hyperplanes of the hypersurface X at 0, if there is a sequence x_n of non singular points of X which converges to 0 such that the sequence of tangent hyperplanes T_{X,x_n} converges to H. Now, in [T] (Consequence of Proposition 2.9 of Chap. 1, see also [HL] Théorème 2.2), B. Teissier proves:

Theorem 1.2.2. *Let $X, 0$ be a germ of complex hypersurface in $\mathbf{C}^{n+1}, 0$ with an isolated singularity at 0. Let H be a complex hyperplane through 0. Then, the hyperplane is not a limit of tangent hyperplanes to X at 0 if and only if $X \cap H$ has an isolated singularity and $\mu(X \cap H, 0)$ is minimal.*

The preceding theorem allows us to define the open Zariski set Ω_2 considered above as the complement in $\check{\mathbf{P}}^2$ of the set of limits of tangent hyperplanes to X at 0. In the case of complex surfaces X in $\mathbf{C}^3$ having an isolated singularity at 0, this shows that a complex plane H of $\mathbf{C}^3$ through 0 is not a limit of tangent hyperplanes to X at 0 if and only if $X \cap H$ has an isolated singularity at 0 and $\mu(X \cap H, 0) = \mu^{(2)}(X, 0)$.

Proposition 1.2.3. *Let H be a complex plane of $\mathbf{C}^3$ through 0 so that $\mu(X \cap H, 0) = \mu^{(2)}(X, 0)$. The multiplicity of $X \cap H$ at 0 equals the multiplicity $\mu^{(1)}(X, 0)$ of X at 0.*

Proof. To prove this fact, it is enough to apply a result of the author in [L1] (see also [LR] §3), showing that, in an analytic family of plane curves having an isolated singularity at 0 with their Milnor numbers at 0 constant, the topology of these plane curves at 0 and, hence, their multiplicity at 0, do not vary. We obtain the assertion of our proposition by considering the analytic family of plane sections parametrized by the set Ω_2 of general planes of $\mathbf{C}^3$ through 0. Q.E.D.

Remark 1.2.4. In fact, a remarkable result of B. Teissier shows that, for any germ of complex hypersurface $X, 0$ in $\mathbf{C}^{n+1}$ with an isolated singularity, if the Milnor number $\mu(X \cap H, 0)$ is minimal, for any general flag

$$\{0\} \subset H_1 \subset \ldots \subset H_n \subset \mathbf{C}^{n+1} = H_{n+1}$$

in which the Milnor number $\mu(X \cap H_i)$ is minimal among i-dimensional sections of X at 0, we have

$$\mu^*(X \cap H, 0) = (\mu(X \cap H_n, 0), \ldots, \mu(X \cap H_1, 0)).$$

1.3. In the case of complex analytic surfaces, we can summarize the results of the preceding section by:

Proposition 1.3.1. *Let $X, 0$ be a germ of complex analytic surface in $\mathbf{C}^3$ with an isolated singularity at 0. A plane H of $\mathbf{C}^3$ through 0 is not a limit of tangent planes to X at 0 if and only if we have*

$$\mu(X \cap H, 0) = \mu^{(2)}(X, 0),$$

in which case, the plane H is not contain in the tangent cone $C_{X,0}$ of X at 0.

Using this result, J.P.G. Henry and the author prove in [HL] (Théorème 3.8):

Theorem 1.3.2. *Let $X, 0$ be a germ of complex analytic surface in $\mathbf{C}^3$. There are a finite number of complex generatrices of the tangent cone $C_{X,0}$ of X at 0, such that the set of limits of tangent planes $\mathcal{T}_{X,0}$ to X at 0 is the union of the set of limits of tangent planes to $C_{X,0}$ at 0 and the pencils of planes $\mathcal{L}_i$ $(1 \leq i \leq k)$ through these generatrices:*

$$\mathcal{T}_{X,0} = Proj|C_{X,0}|\check{\ } \cup \mathcal{L}_1 \cup \ldots \cup \mathcal{L}_k.$$

We call these generatrices the **exceptional tangents** of X at 0.

There are several ways to find the exceptional tangents of a complex analytic surface X at 0. One of the most useful ways is:

Proposition 1.3.3. *Let Ω be the set of finite projections of $X, 0$ into $\mathbf{C}^2, 0$ induced by linear projections of $\mathbf{C}^3$ onto $\mathbf{C}^2$ with a local degree equal to the multiplicity $m(X, 0)$ at 0. Let $p \in \Omega$ and $\Gamma(p)$ be the critical curve of p. The set of exceptional tangents of X at 0 is the set of tangents of $\Gamma(p)$ which do not depend on $p \in \Omega$.*

We shall give below generalizations of these Propositions in the case of normal surfaces.

§2. Tangents on Normal surfaces.

In all this paragraph, we shall consider a normal surface singularity (X, x) (this means that the local ring $\mathcal{O}_{X,x}$ of the germ is an integral domain and is integrally closed in its field of fractions). The criterion of Serre already used above shows that a surface singularity is normal if and only if it is isolated and its local ring is Cohen-Macaulay. We choose a representative X of of the germ (X, x) such that $X - \{x\}$ is non-singular and X is closed in an open neighbourhood of x in $\mathbf{C}^N$.

2.1. In [GL] (Définition 2.1) G. Gonzalez and M. Lejeune-Jalabert gave a definition of a **general hyperplane section** of X at x.

Definition 2.1.1. Let $\sigma: \overline{X}_1 \to X$ be the normalized blowing-up of the maximal ideal M which defines x on X. Let H be a hyperplane of $\mathbf{C}^N$ through x. We say that H is a general hyperplane for X at x (or $X \cap H$ is a general hyperplane section), if the strict transform of $X \cap H$ by σ does not go through the singular points of $\overline{X}_1$, intersects transversally in $\overline{X}_1$ the reduced exceptional divisor $\overline{E}$ of $\overline{X}_1$ and does not contain any non-singular point of $\overline{E}$ where the restriction of the normalisation of the blowing-up of X at x is critical.

Remark 2.1.2. Let $e: X_1 \to X$ be the blowing-up of X at x, call E the exceptional divisor of X_1. Call n the normalisation

$$n: \overline{X}_1 \to X_1$$

of X_1. Then, $X \cap H$ is a general hyperplane section if and only if strict transform of $X \cap H$ by e does not contain the images by n of the singular points of $\overline{X}_1$ and the ramification points of the map from $\overline{E}$ to E induced by n, and the hyperplane $Proj(H)$ intersects $Proj|C_{X,x}|$ at non-singular points transversally in $Proj(\mathbf{C}^N) = \mathbf{P}^{N-1}$.

In [GL] (§2) G. Gonzalez and M. Lejeune-Jalabert called **special generatrices** the generatrices of the tangent cone $C_{X,x}$ which correspond to the images by n of the singular points of $\overline{X}_1$ and the ramification points of the map from $\overline{E}$ to E induced by n.

Now, recall that a hyperplane H is a tangent hyperplane at a non-singular point y of X, if it contains the tangent plane $T_{X,y}$. Then the hyperplane H is a limit of tangent hyperplanes of the surface X at x, if there is a sequence x_n of non singular points of X which converges to x and a sequence of complex hyperplanes H_n tangent to X at x_n such that H_n converges to H.

Of course, the set of limits of tangent hyperplanes of X at x is algebraic. In fact, one considers the closure $C(X)$ in $X \times \check{\mathbf{P}}^{N-1}$ of the set of points (y, H), where y is a non-singular point of X and H is a hyperplane tangent to X at y. Using a classical result of Remmert (see [RS] Satz 13), one can prove that $C(X)$ is a complex analytic space. The projection onto X induces a morphism

$$\kappa \colon C(X) \to X$$

which is analytic and proper. A result of Chow ([C], see [GR] Chapter 9 §5) implies that the fiber of κ over x which is analytic and closed in $\check{\mathbf{P}}^{N-1}$ is actually algebraic. The space $C(X)$ is called the conormal space of X in $\mathbf{C}^N$.

In his thesis, Jawad Snoussi proved:

Theorem 2.1.3. *A hyperplane H of $\mathbf{C}^N$ is general for X at x if and only if it is not a limit of tangent hyperplanes.*

In view of Teissier's theorem 1.2.2, J. Snoussi proves:

Theorem 2.1.4. *A hyperplane H of $\mathbf{C}^N$ is general for X at x if and only if the number of points in $H \cap Proj|C_{X,x}|$ equals the degree of $Proj|C_{X,x}|$ and the generalized Milnor number of Buchweitz and Greuel of the curve $H \cap X$ at x is minimal.*

In [BG] R. Buchweitz and G.-M. Greuel have defined a generalized Milnor number for any reduced curve. Namely let C be a reduced curve and O be a point of C. Denote the local ring of C at O by $\mathcal{O}_{C,O}$ and let

$\overline{\mathcal{O}}_{C,O}$ be its normalisation. Define $\delta(C,O)$ to be the dimension of the complex vector space $\overline{\mathcal{O}}_{C,O}/\mathcal{O}_{C,O}$

$$\delta(C,O) := \dim_{\mathbf{C}} \overline{\mathcal{O}}_{C,O}/\mathcal{O}_{C,O}.$$

Then, the generalized Milnor number of C at O is

$$\mu(C,O) := 2\delta(C,O) - r(C,O) + 1$$

where $r(C,O)$ is the number of analytic branches of C at O.

In [M] (Theorem 10.5), J. Milnor proved this relation between the Milnor number and $\delta(C,O)$, when C,O is the germ of a reduced plane curve.

A topological interpretation of the Milnor number for a curve (see [BG]) on a normal surface singularity defined by one equation is the following. We may assume that the singularity is locally embedded in some non-singular space $\mathbf{C}^N$ and that the curve is given by $\varphi = 0$, where φ is a holomorphic function defined in a neighbourhood of the singularity in $\mathbf{C}^N$. Then, there is $\epsilon_0 > 0$, such that, for any ϵ, $\epsilon_0 > \epsilon > 0$, there is $\eta_\epsilon > 0$, such that, for any complex number t, $\eta_\epsilon > |t| > 0$, the Milnor number of the curve singularity is equal to the first Betti number of the Riemann surface $B_\epsilon(0) \cap \{\varphi = t\}$, where $B_\epsilon(0)$ is the open ball of $\mathbf{C}^N$ centered at 0 with radius ϵ.

The key points to prove 2.1.3 and 2.1.4 are results of R. Buchweitz and G.-M. Greuel who show the semi-continuity of their generalized Milnor number in analytic families of curves and the equiresolution of analytic families with generalized Milnor number constant when these families are non-singular outside a section (see [BG]).

2.2. An important consequence of Snoussi's result (compare with Theorem 1.3.2) is:

Theorem 2.2.1. *Let (X,x) be a germ of normal complex analytic surface in $\mathbf{C}^N$. The set of limits of tangent hyperplanes $\mathcal{H}_{X,x}$ to X at x is the union of the set of limits of tangent hyperplanes to the tangent cone $C_{X,x}$ of X at x and the linear systems $\mathcal{L}_i$ $(1 \le i \le k)$ of hyperplanes through the special generatrices of $C_{X,x}$:*

$$\mathcal{H}_{X,x} = Proj|C_{X,x}|^{\check{}} \cup \mathcal{L}_1 \cup \ldots \cup \mathcal{L}_k.$$

As a consequence, following the terminology already used in [LT1] (1.3.2), it will be more convenient to call the special generatrices of $C_{X,x}$ **the exceptional tangents** of (X,x).

In fact, J. Snoussi also obtains the description of the set of limits of tangent spaces to X at x.

Theorem 2.2.2. *Let (X,x) be a germ of normal complex analytic surface in $\mathbf{C}^N$. The set of limits of tangent spaces to $\mathcal{T}_{X,x}$ to X at x is the union of the set of limits of tangent spaces to the tangent cone $C_{X,x}$ at its vertex and of 1-dimensional subspaces $\mathcal{G}_i$ $(1 \le i \le k)$ of the grassmanian space $G(2,N)$ of 2-planes in $\mathbf{C}^N$ through x which contain the exceptional tangents ℓ_i $(1 \le i \le k)$.*

Note that the set of limits of tangent spaces to X at x is algebraic. This was predictable since it is the fiber of the Nash modification

$$\nu\colon \tilde{X} \to X$$

where $\tilde{X}$ is the closure in $X \times G(2,N)$ of the set of points $(y, T_{X,y})$, where y is a non-singular point of X and $T_{X,y}$ is the tangent space to X at y, and ν is induced by the projection onto X. A theorem of Remmert (see [RS] Satz 13) implies that $\tilde{X}$ is an analytic space.

2.3. From the results of [LT2], we can generalize the result of 1.3.3. Namely, we have:

Theorem 2.3.1. *Let Ω be the set of finite projections of $(X,0)$ into $(\mathbf{C}^2,O)$ induced by linear projections of $\mathbf{C}^N$ onto $\mathbf{C}^2$ and which have a local degree equal to $m(X,x)$ at x. Let $p \in \Omega$ and $\Gamma(p)$ be the critical curve of p. The set of exceptional tangents of X at x is the set of tangent lines of $\Gamma(p)$ which do not depend on $p \in \Omega$.*

Proof. We shall adapt the proof of [LT2] (Théorème 2.1.1) to the case of dimension 2.

Consider the blowing-up

$$e'\colon EC(X) \to C(X)$$

of the analytic subspace $\kappa^{-1}(x)$ in the conormal space $C(X)$ of X. Of course, it factors through the blowing-up

$$e\colon X_1 \to X$$

of the point x in X. We have the following commutative diagram

$$\begin{array}{ccccccc} X \times \check{\mathbf{P}}^{N-1} \times \mathbf{P}^{N-1} & \supset & EC(X) & \xrightarrow{e'} & C(X) & \subset & X \times \check{\mathbf{P}}^{N-1} \\ & & \downarrow \kappa' & & \downarrow \kappa & & \\ X \times \mathbf{P}^{N-1} & \supset & X_1 & \xrightarrow{e} & X & & \end{array}$$

We may consider that $E_1 := (e' \circ \kappa)^{-1}(x) = (\kappa' \circ e)^{-1}(x)$ is embedded in $\check{\mathbf{P}}^{N-1} \times \mathbf{P}^{N-1}$. The spaces $D := e^{-1}(x)$ and $D_1 := \kappa^{-1}(x)$ are contained in $\check{\mathbf{P}}^{N-1}$ and $\mathbf{P}^{N-1}$ respectively. Let $E_1(\alpha)$, $\alpha \in A$, be the irreducible components of E_1. Let $D_1(\alpha)$ and $D(\alpha)$ be the images of $E_1(\alpha)$ by κ' and e'. We have (see [LT2] Théorème 2.1.1)

Lemma 2.3.2. *For each $\alpha \in A$, the variety $D(\alpha)$ is the dual of $D_1(\alpha)$ and the correspondence is given by $E_1(\alpha)$.*

Proof. In fact, this is a consequence of a lemma of Whitney (see [L3]) which states that, for any $(H,\ell) \in E_1$, we have

$$\ell \subset H.$$

The dimension of the components of E_1 is $N-2$. If $D_1(\alpha)$ has dimension 0, $E_1(\alpha)$ is isomorphic to $D(\alpha)$ and consists of all the hyperplanes which contain the point $\{D_1(\alpha)\}$. If $D_1(\alpha)$ has dimension 1, it is a component of the Projective set associated to the tangent cone and at a general point l_1 of $D_1(\alpha)$, the points (H,l_1) in $\kappa^{-1}(l_1)$ consists of hyperplanes containing the tangent plane to the tangent cone $|C_{X,x}|$ along the line l_1 (see [L3] (Théorème 1.2.1)). Then, the image of $E_1(\alpha)$ by e' is the closure of the set of hyperplanes which contain the tangent planes at non-singular points of the component $D_1(\alpha)$ of $Proj(|C_{X,x}|)$ which contains l_1. By definition $E_1(\alpha)$ is the correspondence variety of $D_1(\alpha)$ and its dual variety $e'(E_1(\alpha)$.

Now, we can prove Theorem 2.3.1. Let $\mathcal{D}(A)$ be the projective subvariety of $\check{\mathbf{P}}^{N-1}$ which consists of the hyperplanes through x which contain a codimension 2 space A through x. Then, $\mathcal{D}(A)$ is isomorphic to a projective space of dimension 1. Let p be the projection on $\mathbf{C}^2$ induced by the linear projection p_A of $\mathbf{C}^N$ onto $\mathbf{C}^2$ with Kernel A. When A is sufficiently general, say if A belongs to an open Zariski subset Ω' of the Grassmannian manifold of codimension 2 projective subspaces in $\mathbf{P}^{N-1}$, the projection p_A restricted to (X,x) has local degree equal to the multiplicity $m(X,x)$, so $p_A \in \Omega$. With p_2 being the map from $EC(X)$ into $\check{\mathbf{P}}^{N-1}$, we observe that the subspace $p_2^{-1}(\mathcal{D}(A))$ of $EC(X)$ is non-empty, if $m(X,x) \geq 2$, so it is a curve. It is easy to show that the curve $\Gamma(p)$ is

$$\Gamma(p) = e \circ \kappa'(p_2^{-1}(\mathcal{D}(A)).$$

Therefore the curve $\kappa'(p_2^{-1}(\mathcal{D}(A))$ is the strict transform of $\Gamma(p)$ by e and it intersects the exceptional divisor at points which correspond to the tangents of $\Gamma(p)$ at x. Since $\mathcal{D}(A)$ has dimension 1 it meets the set of hyperplanes $E(\alpha) := e'(E_1(\alpha))$ which contain $D_1(\alpha)$ when $D_1(\alpha)$ has dimension 0. Therefore $\kappa'(p_2^{-1}(\mathcal{D}(A))$ contain the sets $D_1(\alpha)$ of dimension 0. The other points of $\kappa'(p_2^{-1}(\mathcal{D}(A))$ contained in the exceptional divisor of e are in the dual of the intersections of $\mathcal{D}(A)$ and the sets $D(\alpha)$ dual to the components of $Proj(|C_{X,x}|)$. Since $|C_{X,x}|$ is a cone, these points are the generatrices of $|C_{X,x}|$ which are the closure of the components of the critical locus of the restriction to the non-singular part of $|C_{X,x}|$ of the linear projection p_A. These latter generatrices depend on the projection p_A.

Hence, this shows that the tangent lines in the tangent cone of $\Gamma(p)$ consist of lines of $|C_{X,x}|$ which depend on $p \in \Omega$ and of the exceptional tangents which do not depend on $p \in \Omega$.

§3. Resolutions of Normal surfaces.

3.1. Let (X,x) be a normal complex surface singularity. We choose a representative X of (X,x) such that $X - \{x\}$ is non-singular.

Definition 3.1.1. We say that a complex analytic map $\pi\colon Z \to X$ is a resolution of singularity of (X,x), if

i) the space Z is non-singular;
ii) the map π is proper;
iii) the map π induces an isomorphism of $Z - \pi^{-1}(x)$ onto $X - \{x\}$ and $Z - \pi^{-1}(x)$ is dense in Z.

An important result by R. Walker and O. Zariski ([W] and [Z] VI §21) is:

Theorem 3.1.2. *Any normal surface singularity has a resolution obtained by composing a finite number of compositions of a point blowing-up and a normalisation.*

In [BPV] (III §6), one can find a more geometrical construction of a resolution of a normal complex surface singularity (due to Jung [J], see [Hi]) by using the embedded resolution of the discriminant of a finite projection of the singularity onto a 2-dimensional complex plane.

Remark 3.1.3. Notice that there are many resolutions of the singularity (X,x). For instance, the identity is a resolution of the non-singular germ $(\mathbf{C}^2,0)$ and the blowing-up of the point 0 in $(\mathbf{C}^2,0)$ is also a resolution.

We shall see below that all resolutions are obtained from one of them.

3.2. Given a resolution π of the complex analytic normal surface singularity (X,x), most of the topological information of (X,x) is obtained from the geometry of the space $\pi^{-1}(x)$. In fact, we have first:

Theorem 3.2.1 (Main theorem of Zariski). *Let π be a resolution of a complex analytic normal surface singularity. The space $\pi^{-1}(x)$ is connected.*

One may find proof of this result in [H] (Chap. III, Corollary 11.4). The main argument comes from the fact that, since (X, x) is a normal singularity, there are regular neighbourhoods U of x in X whose boundary ∂U (called the **local link** of X at x) is a connected 3-manifold.

Theorem 3.2.1 shows that, if π is a resolution of a complex analytic normal surface singularity the fiber is either a point or a connected curve. It is a point only if the surface is non-singular and π is the identity, as a consequence of the following theorem of D. Mumford (see [Mu])

Theorem 3.2.2. *The local link of a normal surface singularity (X, x) is simply connected if and only if X is non-singular at the point x.*

Therefore, if the normal surface singularity (X, x) is really singular, for any resolution π of (X, x), the fiber $\pi^{-1}(x)$ is a connected curve.
There is another important theorem of D. Mumford ([Mu], see §1) which characterizes the fiber $\pi^{-1}(x)$ (the **exceptional fiber** of π) of a resolution $\pi\colon Z \to X$ of (X, x), when it is a curve. Let $E_1, \ldots, E_k$ be the irreducible components of $\pi^{-1}(x)$.

Theorem 3.2.3. *The intersection matrix $(E_i.E_j)_{1\leq i,j\leq k}$ is definite negative.*

This fact allows us to associate some important combinatorial invariants to a resolution π of (X, x). For instance, there is a theorem of Zariski (see [A] Proposition 2) which states

Theorem 3.2.4. *Let I be negative definite bilinear form on a free abelian group G generated by $e_1, \ldots, e_k$, there are elements $z \neq 0$*

$$z = \sum_1^k m_i e_i$$

of this group such that

$$I(z, e_i) \leq 0$$

for any i, $1 \leq i \leq k$. Furthermore, these elements make a semi-group $E^+(I)$ which has a smallest element $z_0 = \sum_1^k a_i e_i$, such that $a_i \geq 1$, for any i, $1 \leq i \leq k$. We shall call z_0 the fundamental element of I.

Definition 3.2.5. The fundamental element of the intersection form I on the free abelian group generated by the components of a resolution π of a complex normal surface is called the **fundamental cycle** of this resolution. The semi-group $E^+(I)$ is called the **Lipman semi-group** of the resolution π and is also denoted by $E^+(\pi)$.

An important remark is that, given a resolution $\pi\colon Z \to X$ of (X, x), any function $\varphi \in \mathcal{O}_{X,x}$ defines a divisor $(\varphi \circ \pi)$ on Z and the compact part of this divisor is an element of $E^+(\pi)$. For example, in a resolution π for which the inverse image $\pi^* M_{X,x}$ of the maximal ideal $M_{X,x}$ of $\mathcal{O}_{X,x}$ is invertible, the maximal cycle of the resolution is given by the compact part of the divisor given by a general element of $M_{X,x}$ (see [Y]).

In [Li] (§18), J. Lipman proved that the semi-group $E^+(\pi)$ of a resolution π of a rational singularity is given by the general elements of ideals I of $\mathcal{O}_{X,x}$ whose inverse images $\pi^* I$ are invertible.

3.3. It is useful to recall the notion of minimal resolution.

Definition 3.3.1. A resolution of a surface singularity (X, x) is called **minimal** if its exceptional divisor does not contained a non-singular rational curve of self-intersection -1. Such a curve is called an **exceptional curve of the first kind**.

The basic theorem about surface resolutions is (see [La] Chapter 5):

Theorem 3.3.2. *Minimal resolutions of a surface singularity (X, x) are isomorphic, i.e. if X is a representative of (X, x) such that $X - \{x\}$ is non-singular, $\pi_1\colon Z_1 \to X$ and $\pi_2\colon Z_2 \to X$ are two minimal resolutions of (X, x), there is an isomorphism $\varphi\colon Z_1 \to Z_2$, such that $\pi_2 \circ \varphi = \pi_1$.*

A consequence is the factorization theorem:

Corollary 3.3.3. *Let $\pi\colon Z \to X$ be a resolution of the surface singularity (X, x) and $\pi_0\colon Z_0 \to X$ be a minimal resolution of the surface singularity (X, x). There is a unique holomorphic map $\psi\colon Z \to Z_0$, such that $\pi = \pi_0 \circ \psi$, and ψ is the composition of a finite sequence of point blowing-ups.*

§4. General sections and Tjurina-Spivakovsky components.

4.1. Given a resolution $\pi\colon Z \to X$ of a normal surface singularity (X, x), let $E_1, \ldots, E_k$ be the components of the exceptional divisor $\pi^{-1}(x)$ of π. We consider a cycle a in the Lipman semi-group E^+ of π, so, for $1 \leq i \leq k$, we have $a.E_i \leq 0$. The **Tjurina-Spivakovsky components** of a (compare with [Sp] Chap. III, Definition 3.1) are the maximal connected curves contained in the exceptional divisor $\pi^{-1}(x)$ whose components are components E_i such that $a.E_i = 0$. Therefore, the components of $\pi^{-1}(x)$ which are not contained in a Tjurina-Spivakovsky component of a are the components E_j of $\pi^{-1}(x)$ such that $a.E_j < 0$.

Consider an ideal I of the local ring $\mathcal{O}_{X,x}$ such that $I\mathcal{O}_Z$ is locally invertible. Following Lipman (see [Li] §18), the ideal $I\mathcal{O}_Z$ defines an element a_I in the semi-group E^+. One can check that a_I is the compact part of the divisor on Z defined by a general element of the ideal I. The following lemma shows the interest of Tjurina-Spivakovsky components:

Lemma 4.1.1. *Let $\pi\colon Z \to X$ be a resolution of a normal surface singularity (X, x) such that the maximal ideal M of $\mathcal{O}_{X,x}$ defines a locally invertible ideal $M\mathcal{O}_Z$. Let $q\colon Z \to \overline{X_1}$ be the factorisation of π through the normalized blowing-up $\overline{e}\colon \overline{X_1} \to X$ of the point $\{x\}$ in X. The connected curves of the exceptional divisor of π which are mapped by q to the singular points of $\overline{X}_1$ are Tjurina-Spivakovsky components of the cycle defined by $M\mathcal{O}_Z$.*

Proof. Since $M\mathcal{O}_Z$ is locally invertible, the resolution π factorizes through the blowing-up of M, i.e. the blowing-up of the point $\{x\}$ in X. Since Z is non-singular, it is also normal, so this factorisation lifts to the normalized blowing-up of the point $\{x\}$ in X.

$$\begin{tikzcd} Z \arrow[d, "q"'] \arrow[dr, "\pi"] & \\ \overline{X_1} \arrow[r, "\overline{e}"] & X \end{tikzcd}$$

Let l be a general element of the maximal ideal. As we have noticed above, the cycle defined by M on Z coincide with the compact part of the divisor defined by l on Z. One can see that the components E_i of the exceptional divisor of π which are not in a Tjurina-Spivakovsky component, i.e. such that $a_M.E_i < 0$, are the components of the exceptional divisor which are intersected by the strict transform of the general element l. These components are in fact the strict transforms by π of the components of the projective set associated to the tangent cone of X at x. So, the images by the map q of the components contained in a Tjurina-Spivakovsky component of a_M must be points. Therefore q is obtained by contracting the Tjurina-Spivakovsky components of a_M. Since q is a resolution of the singularities of $\overline{X_1}$, the images of the Tjurina-Spivakovsky components of a_M contain the singular points of $\overline{X_1}$. If Z is an arbitrary resolution of (X, x), it is possible that, by contracting a Tjurina-Spivakovsky component of a_M, one obtains a non-singular point. However, if π is the minimal resolution of (X, x) in which $\pi^* M$ is invertible, the Tjurina-Spivakovsky components of a_M all contract in a singular point of X_1. This fact is consequence of the observation that in such resolution none of the components in a Tjurina-Spivakovsky component of a_M is a curve of the first kind (see 3.3.3).

Remark 4.1.2. In [Tj], G. Tjurina found that, in the case of a rational singularity (X, x), the Tjurina-Spivakovsky components of the fundamental cycle of the minimal resolution of (X, x) contract into the singular points of the blowing-up of X at $\{x\}$. The reason is that, for any resolution π of a rational singularity, the inverse image π^*M of the maximal ideal of x in X is invertible and that the blowing-up X_1 of the point x in X is already normal.

M. Spivakovsky extended the notion of Tjurina-Spivakovsky components to any cycle in the Lipman semi-group of a resolution of a rational singularity (see [Sp] Chap. III, Definition 3.1).

We have naturally generalized the definition of Spivakovsky to resolutions of general normal surface singularities, but unlike the case of rational singularities, the Lipman semi-group of a resolution π might be different from the semi-group of ideals I of the local ring $\mathcal{O}_{X,x}$ whose lifting π^*I is invertible on Z.

Recall that J. Snoussi proved that the exceptional tangents of a normal surface singularity are the special generatrices of Gonzalez and Lejeune (see above in 2.1.2). So the result of Lemma 4.1.1 says that the images of the singular points of X_1 under normalisation are exceptional tangents. On the other hand, images of the singular points of the exceptional set of the normalized blowing-up under normalisation give also special generatrices, these images are also exceptional tangents. In particular, singular generatices of tangent cones of rational singularities are exceptional tangents, so that Snoussi gives a positive answer to a question of M. Spivakovsky in [Sp] (Chap. III, Remark 3.12).

An interesting corollary is the following:

Proposition 4.1.3. *If a normal surface singularity has no exceptional tangent, the normalized blowing-up of its singular point is non-singular.*

In [LT2], we proved that an isolated singularity of surface which has a reduced tangent cone and which has no exceptional tangent is equisingular to its tangent cone. In the case of germs of hypersurface in $\mathbf{C}^3$, we proved that if there are no exceptional tangents, the tangent cone is reduced. Of course, in these two cases, the normalized blowing-up of the singular point is non-singular, since the blowing-up of the singular point is already non-singular.

4.2. Examples. In general the inverse image of the maximal ideal by a resolution of a normal surface singularity might not be invertible (see [Y]). For example, consider the minimal resolution of the hypersurface $x^2 + y^3 + z^6 = 0$ (see [GL]). One can show that the exceptional divisor of the minimal resolution is a non-singular elliptic curve of self intersection -1. In order to obtain a resolution in which the inverse image of the maximal ideal is invertible, we need to blow-up a point in this elliptic curve. In this new resolution where the inverse of the maximal ideal is invertible the Tjurina-Spivakovsky component of the maximal cycle is the elliptic curve. This elliptic curve contracts to a singularity whose minimal resolution has an exceptional divisor whose unique component is this elliptic curve with self-intersection -2. It is easy to show that there is only one exceptional tangent in this example.

Another interesting example is the hypersurface $x^2 + y^4 + z^4 = 0$. For this case, in the minimal resolution the inverse image of the maximal ideal is an elliptic curve with self-intersection -2. In fact, the normalized blowing-up of the singular point is non-singular. However, we have 4 exceptional tangents which correspond to the ramification points of the projection of the elliptic curve on the non-singular rational curve which is the projective curve associated to the tangent cone of the singularity.

The two preceding examples are simple elliptic singularities in the sense of K. Saito (see [Sa]). The singularity of $x^2+y^3+z^6=0$ is simple elliptic of type $\tilde{E}_8$ and the singularity of $x^2+y^4+z^4=0$ is of type $\tilde{E}_7$. It is interesting to see that the blowing-up of $x^2+y^3+z^6=0$ is normal and contains one singularity which is equisingular to $x^2+y^4+z^4=0$, but not analytically isomorphic to it. In fact, using the deformation of $\tilde{E}_8$ type singularities given in Satz 1.9 of [Sa], one can find a deformation of $x^2+y^3+z^6=0$ which gives $x^2+y^4+z^4=0$ in its blowing-up.

4.3. A natural question is to decide if, in terms of limits of tangent hyperplanes or in terms of limits of tangent spaces, the singularities of the normalized blowing-up are simpler than the given one.

Another interesting problem related to the preceding question is to find an effective bound on the number of normalized blowing-up necessary to solve a normal surface singularity. The description given above is a first step in an attempt to understand the geometry of the normalized blowing-up.

It is interesting to notice that the geometry involved in a normalized blowing-up is similar to the one used by M. Spivakovsky in [Sp] to resolve a normal surface singularity by a finite composition of normalized Nash modifications. In some sense, these two processes of resolutions are dual. The problem of giving an effective bound on the number of normalized Nash modifications needed to solve a normal surface singularity is also not solved. In [Sp], M. Spivakovsky gives a detailed study of the complexity of the resolution graph of the minimal resolution of a minimal rational singularity and the behaviour of this complexity after a normalized Nash modification. This is the key step to obtain the resolution of normal surface singularities by composition of a finite number of normalized Nash modifications.

Since rational surface singularities are absolutely isolated (see [Tj]), the complexity of the resolution graph of the minimal resolution decreases strictly after each blowing-up. However, it is not trivial to get a bound of the number of point blowing-ups needed in order to reach a resolution from the local ring of the singularity without having to calculate the resolution graph of the minimal resolution.

References

[AC] N. A'Campo, *Le nombre de Lefschetz d'une monodromie*, Indag. Math. **35** (1973), 113–118.

[A] M. Artin, *Isolated rational singularities of surfaces*, Amer. J. Math. **88** (1966), 129–136.

[BPV] W. Barth, C. Peters, A. Van de Ven, Compact complex surfaces, Ergebnisse der Math. und ihrer Grenzgebiete **3**, Band 4, Springer Verlag (1984).

[BG] R. Buchweitz - G.M. Greuel, *The Milnor number and deformations of complex curve singularities*, Inv. Math. **58** (1980), 241–281.

[C] W.L. Chow, *On compact analytic varieties*, Amer. J. Math. **71** (1949), 893–914.

[D] J. Dieudonné, Eléments d'Analyse 4, Cahiers Scientifiques **XXXIV**, Gauthier-Villars, 1971.

[GL] G. Gonzalez-Sprinberg - M. Lejeune-Jalabert, *Courbes lisses, cycle maximal et points infiniment voisins des singularités de surfaces*, Pub. Mat. Urug. **7** (1997), 1–27.

[GR] H. Grauert - R. Remmert, Coherent analytic sheaves, Grundl. der math. Wiss. **265**, Springer-Verlag, 1984.

[H] R. Hartshorne, Algebraic Geometry, GTM **52**, Springer-Verlag, 1977.

[HL] J.P.G. Henry - Lê D.T., *Limites d'espaces tangents*, in Fonctions de Plusieurs variables II, SLN **482**, Springer-Verlag, 1975.

[Hi] F. Hirzebruch, *Über vierdimensionale Riemannsche Flächen mehrdeutiger analytischer Funktionen von zwei Veränderlichen*, Math. Ann. **126** (1953), 1–22.

[Ho] C. Houzel, *Géométrie analytique locale I*, in Sém. H. Cartan, 1960-1961, Notes Miméographiées, I.H.P. (1962), Paris.

[J] H.W.E. Jung, *Darstellung der Funktionen eines algebraischen Körpers zweier unabhängigen Veränlichen x, y in der Umgebung einer Stelle $x = a$, $y = b$*, J. Reine angw. Math. **133**, 1908.

[La] H. Laufer, Normal surface singularities, Ann. Math. Stud. **71**, Princeton University Press (1971), Princeton.

[Li] J. Lipman, *Rational singularities with applications to algebraic surfaces and unique factorization*, Pub. IHES **36** (1969), 195–279.

[L1] Lê D.T. *Sur un critère d'équisingularité*, C.R.Acad.Sc. Série A **272** (1970), 138–140.

[L2] Lê D.T. *Calcul des cycles évanouissants d'une hypersurface complexe*, Ann. Inst. Fourier **23** (1973), 261–270.

[L3] Lê D.T. *Limites d'espaces tangents sur les Surfaces*, Nova Acta Leopoldina **240**, Band **52**, Abh. Deutsche Akad. de Natur. Leopolodina (1981), 119–137.

[LR] Lê D.T. - C.P. Ramanujam, *The invariance of Milnor's number implies the invariance of the topological type*, Amer. J. Math. **98** (1976), 67–78.

[LT1] Lê D.T. - B. Teissier, *Sur la géométrie des surfaces complexes I. Tangentes exceptionnelles*, Amer. J. Math. **101** (1979), 420–452.

[LT2] Lê D.T. - B. Teissier, *Limites d'espaces tangents en Géométrie analytique*, Com. Helv. **63** (1988), 540–578.

[M] J. Milnor, Singular points of complex hypersurfaces, Ann. Math. Stud. **61**, Princeton U. P., 1968.

[Mu] D. Mumford, *The topology of normal singularities of an algebraic surface and a criterion for simplicity*, Pub. Math. I.H.E.S. **9** (1961), 229–246.

[N] R. Narashiman, Introduction to the theory of analytic spaces, SLN **25**, Springer-Verlag, 1966.

[RS] R. Remmert - K. Stein, *Über die wesentlichen Singularitäten analytischer Mengen*, Math. Ann. **126** (1953), 263–306.

[Sa] K. Saito, Einfach-elliptische Singularitäten, Inv. math. **23** (1974), 289–325.

[S] J.P. Serre, Algèbre Locale. Multiplicités, SLN **11**, Springer-Verlag, 1989.

[Sn] J. Snoussi, Limites d'espaces tangents à une surface normale, Thèse, Université de Provence, CMI, Marseille, 1998.

[Sp] M. Spivakovsky, *Sandwiched singularities and desingularization of surfaces by normalized Nash transformation*, Ann. Math. **131** (1990), 411–491.

[T] B. Teissier, *Cycles évanescents, sections planes et conditions de Whitney*, in Singularités à Cargèse, Astérisque **7-8**, S.M.F., 1974.

[Tj] G.N. Tjurina, *Absolute isolatedness of rational singularities and triple rational point*, Funk. An. i ievo pril. **2** (1968), 70–81.

[Y] S.T. Yau, *On maximally elliptic singularities*, Trans. Amer. Math. Soc. **257** (1980), 269–329.

[W] R.J. Walker, *Reduction of singularities of an algebraic surface*, Ann. Math. **36** (1935), 336–365.

[Z] O. Zariski, *The Reduction of the singularities of an algebraic surface*, Ann. Math. **40** (1939), 639–689.

CMI-Université de Provence
39 Rue Joliot Curie
F-13453 Marseille Cedex 13
France
ledt@gyptis.univ-mrs.fr