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Special Polynomials and Generalized Painlevé Equations

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Abstract.

We will review recent developments on the special polynomials arising in Painlevé equations and their generalizations.

§1. Introduction

The following six equations are called Painlevé equations;

(P_I) $\quad y'' = 6y^2 + t,$

(P_{II})
$$y'' = 2y^3 + ty + \alpha,$$

(P_{III})
$$y'' = \frac{1}{y}y'^2 - \frac{1}{t}y' + \frac{1}{t}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y},$$

(P_{IV})
$$y'' = \frac{1}{2y}y'^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y},$$

(P_V)
$$\begin{aligned} y'' = &\left(\frac{1}{2y} + \frac{1}{y-1}\right)y'^2 - \frac{y'}{t} \\ &+ \frac{(y-1)^2}{t^2}\left(\alpha y + \frac{\beta}{y}\right) + \frac{\gamma y}{t} + \frac{\delta y(y+1)}{y-1}, \end{aligned}$$

(P_{VI})
$$\begin{aligned} y'' = &\frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)y'^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)y' \\ &+ \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left[\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right], \end{aligned}$$

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where $y = y(t)$ is the unknown function, $' = d/dt$ and $\alpha, \beta, \gamma, \delta$ are complex parameters. These equations have the Painlevé property i.e. any movable singularity (depending on initial data) is a pole. This property is known as a practical method to test the integrability of differential equations.

By the work of K. Okamoto, the following facts are known for the Painlevé equations P_J J =II,III,IV,V or VI (see [1,2] for example).

(1) P_J has affine Weyl group symmetry of type (II) $A_1^{(1)}$, (III) $C_2^{(1)}$, (IV) $A_2^{(1)}$, (V) $A_3^{(1)}$ or (VI) $D_4^{(1)}$.
(2) For special values of the parameters, P_J is solved by hypergeometric functions such as (II) Airy, (III) Bessel, (IV) Hermite, (V) Laguerre or (VI) Gauss.
(3) There are also other rational (or algebraic) solutions such as (II) Yablonskii-Vorob'ev, (IV) Okamoto or (III,V,VI) Umemura.

We will study the polynomials in (3) from the points of view of combinatorial structure and determinant formula.

These polynomials arise as the τ-functions for the special solutions of the Painlevé equations, and defined by some recurrence relations. The origin of such recurrence relations (Toda equations) is the Bäcklund symmetry of the Painlevé equations. We shall explain these in the simplest example of P_{II}.

The second Painlevé equation P_{II} is a hamiltonian system

$$q' = \frac{\partial H}{\partial p}, \quad p' = -\frac{\partial H}{\partial q}.$$

where

$$H = \frac{1}{2}p^2 - (q^2 + \frac{t}{2})p - bq, \quad (b = \alpha + \frac{1}{2}).$$

The τ-function is defined as

$$\tau = \exp(\int H dt), \quad H = (\log \tau)'.$$

The P_{II} equation has the symmetry given by the Bäcklund transformation such as

$$s_1(q) = q + \frac{b}{p}, \quad s_1(p) = p, \quad s_1(b) = -b.$$
$$r(q) = -q, \quad r(p) = -p + 2q^2 - t, \quad r(b) = 1 - b.$$

These transformations s_1, r and $s_0 = rs_1r$ generate the affine Weyl group $W = \langle s_1, s_0, r \mid s_1^2 = s_0^2 = r^2 = 1, \ s_0 r = r s_1 \rangle$ of type $A_1^{(1)}$. There is an translation $T = rs_1$ such as

$$T^{-1}(q) = -q - \frac{b}{p}, \quad T(p) = -p + 2q^2 + t, \quad T(b) = b - 1.$$

These transformations commute with the derivation. Given a solution (p, q, b) for P_{II}, one obtains a sequence of solutions

$$(p_n, q_n, b_n) = T^n(p, q, b), \quad n \in Z.$$

K. Okamoto proved that the corresponding τ-functions τ_n satisfy the Toda equation

$$(\log \tau_n)'' = c_n \frac{\tau_{n-1}\tau_{n+1}}{\tau_n^2},$$

where c_n is a nonzero constant.

§2. Special polynomials

2.1. Yablonskii-Vorob'ev polynomials for P_{II}

Recurrence relation : $(' = \frac{d}{dt})$

$$T_{m+1}T_{m-1} = tT_m^2 - 4(T''_m T_m - T'^2_m), \tag{2.1}$$

Initial condition : $T_0 = T_1 = 1$.

Examples.

$T_2 = t,$

$T_3 = 4 + t^3,$

$T_4 = -80 + 20t^3 + t^6,$

$T_5 = 11200t + 60t^4 + t^{10},$

$T_6 = -6272000 - 3136000t^3 + 78400t^6 + 2800t^9 + 140t^{12} + t^{15}.$

T_m is a monic polynomial of $\deg(T_m) = m(m-1)/2$. The rational function

$$y(t) = \frac{d}{dt}\left(\log \frac{T_{m+1}}{T_m}\right)$$

solves the second Painlevé equation P_{II} with parameter $\alpha = -m$.

We denote by $s_\lambda(t)$ the Schur function in terms of power sum variables $t_m = 1/m \sum x_i^m$,

$$s_\lambda(t) = \sum_{m_i \geq 0} \pi_\lambda(1^{m_1}2^{m_2}\cdots)\frac{t_1^{m_1}}{m_1!}\frac{t_2^{m_2}}{m_2!}\cdots.$$

It is known that the polynomials T_m are certain specialization of the Schur function [3],

Theorem 1. *We have*

$$T_n = c_n s_{(n-1,\ldots,2,1)}(t_1, t_2, t_3, \ldots),$$

where c_n is a constant and $t_1 = t, t_2 = 0, t_3 = -4/3, t_4 = \cdots = 0$.

Example. $s_{(3,2,1)} = \frac{t_1^6}{45} - \frac{t_1^3 t_3}{3} + t_5 - t_3^2 \to \frac{1}{45}(t^6 + 20t^3 - 80)$.

T_m is also a similarity reduction of a special solution of the KdV equation $4u_t = u_{xxx} + 6uu_x$, $u = 2(\log\tau)_{xx}$ such that $\tau(zx, z^3 t) = z^{m(m-1)/2}\tau(x,t)$ and $T_m(x) = c\tau(x, t = -\frac{4}{3})$.

2.2. Okamoto polynomials for P_{IV}

Recurrence relation: $(' = \frac{d}{dx})$

$$Q_{n+1}Q_{n-1} = (x^2 + 2n - 1)Q_n^2 + Q_n''Q_n - Q_n'^2. \tag{2.2}$$

Initial condition: $Q_0 = Q_1 = 1$.

Example.

$$Q_2 = x^2 + 1,$$
$$Q_3 = x^6 + 5x^4 + 5x^2 + 5,$$
$$Q_4 = x^{12} + 14x^{10} + 65x^8 + 140x^6 + 175x^4 + 350x^2 + 175.$$

These are also specialization of the Schur functions. [4,5]

Theorem 2. *We have*

$$Q_n(x) = c_n s_{(2n-2,\ldots,6,4,2)}(t_1, t_2, \ldots)$$

where c_n is a constant and $t_1 = x,\ t_2 = \frac{1}{2},\ t_3 = t_4 = \cdots = 0$.

Remark. The constant term of Q_m can be obtained by the formula

$$\pi_\lambda(r^k) = \pm \frac{r^k k!}{\prod_{h\equiv 0 (\mathrm{mod}\ r)} h} \quad \text{(or 0)}.$$

2.3. Umemura polynomials for P_V

Recurrence relation: $(' = \frac{d}{dt})$

$$T_{n+1}T_{n-1} = \left(\frac{t}{8} - v + \frac{3}{4}n\right)T_n^2 + T_n'T_n + t\left(T_n''T_n - T_n'^2\right), \tag{2.3}$$

Initial conditions: $T_0 = T_1 = 1$.

The rational function

$$y(t) = -\frac{T_n(t, v+\frac{1}{2})T_{n+1}(t, v+\frac{1}{4})}{T_n(t,v)T_{n+1}(t, v+\frac{3}{4})}$$

solves the equation P_V with parameters $\alpha = 2v^2$, $\beta = -2(v - \frac{n}{2})^2$, $\gamma = n$ and $\delta = -\frac{1}{2}$. The Umemura polynomial T_m is also a specialization of the Schur function as follows, [6]

Theorem 3. *We have*

$$T_{n+1}(t,v) = c_n s_{(n,n-1,\ldots,2,1)}(t_1, t_2, t_3, \ldots)$$

where c_n is a constant and $t_m = t/2 + (-4v + n + 2)/m$.

Example.

$$\begin{aligned} 2^2T_2 &= x + l_3, \\ 2^6T_3 &= x^3 + 3l_4x^2 + 3l_3l_5x + l_3l_4l_5, \\ 2^{12}T_4 &= x^6 + 6l_5x^5 + 15l_4l_6x^4 + (10l_4l_5l_6 + 10l_3l_5l_7)\,x^3 + \\ &\quad 15l_3l_5^2l_7x^2 + 6l_3l_4l_5l_6l_7x + l_3l_4l_5^2l_6l_7, \end{aligned}$$

where $l_k = k - 4v$, $x = t/2$.

The polynomial T_m can be represented as a Wronskian determinant of Laguerre polynomials. It is interesting to note that the polynomial T_m is also a τ-function for discrete P_{II}, if the parameter v is regarded as discrete time.[7]

Theorem 4. *Put $T_n^m = T_{n+1}(t, v = (n+1-m)/4)$, then the rational function*

$$y_m = \frac{T_n^{m-1}T_{n+1}^{m-1}}{T_n^mT_{n+1}^m} - 1,$$

solves the second discrete Painlevé equation

$$(dP_{II}) \qquad y_{m+1} + y_{m-1} = \frac{4}{t}\,\frac{(m+1)y_m + (n+1)}{y_m^2 - 1}.$$

2.4. Umemura Polynomials for P_{VI}

Recurrence relation: $(' = \frac{d}{dv})$

(2.4)

$$U_{n-1}U_{n+1} = \{\frac{1}{4}(-2b_1^2 - 2b_2^2 + (b_1^2 - b_2^2)v) + (n - \frac{1}{2})\}U_n^2 + \frac{1}{4}(v^2 - 4)^2(U_n U_n'' - U_n'^2) + \frac{1}{4}(v^2 - 4)vU_n U_n'.$$

Initial condition: $U_0 = U_1 = 1$.

The following is conjectured in [2] and proved in [8].

Theorem 5.

$$2^{-n(n-1)}U_n = \sum_{I \subset [n-1]} \dim(V_{\lambda_I}^{GL_n}) \prod_{i \in I} c_i \prod_{j \in [n-1]-I} d_j.$$

where $[n-1] = \{1, 2, \ldots, n-1\}$,

$$c_i = \prod_{k=1}^{i} \left((-4b_1^2 + (2k-1)^2) \frac{2-v}{4} \right),$$

$$d_i = \prod_{k=1}^{i} \left((-4b_2^2 + (2k-1)^2) \frac{2+v}{4} \right),$$

and $\lambda_I = (I|I)$ *(Frobenius symbol).*

Example.

$$2^2 T_2 = c_1 + d_1,$$

$$2^6 T_3 = c_1c_2 + 3c_1d_2 + 3d_1c_2 + d_1d_2,$$

$$2^{12} T_4 = c_1c_2c_3 + 6d_1c_2c_3 + 15c_1d_2c_3 + (10d_1d_2c_3 + 10c_1c_2d_3) + 15d_1c_2d_3 + 6c_1d_2d_3 + d_1d_2d_3.$$

Remark. Note that the same coefficients appear in the Umemura polynomials U_m for P_{VI} and T_m for P_V. Such a relation was also observed for polynomials arising in the third Painlevé equation P_{III}.[2,14]

§3. Generalization for the root systems

We shall generalize the story above for other Painlevé type equations with affine Weyl group symmetry besides $A_1^{(1)}$, $A_2^{(1)}$, $A_3^{(1)}$, $C_2^{(1)}$ or $D_4^{(1)}$. To do this, first we generalize the representations of the affine Weyl groups in terms of root system data (Cartan matrix).

3.1. A representation of Weyl groups

Let $A = (a_{ij})_{i,j \in I}$ be a generalized Cartan Matrix such as

$$a_{ii} = 2, \quad a_{ij} \in Z_{\leq 0}, \quad a_{ij} = 0 \leftrightarrow a_{ji} = 0 \ (i \neq j).$$

The corresponding Weyl group $W = W(A)$ is defined as

$$W = \langle s_i (i \in I) \mid s_i^2 = 1, \ (s_i s_j)^{m_{ij}} = 1 \rangle,$$

where $m_{ij} = 2, 3, 4, 6$ or ∞ when $a_{ij} a_{ji} = 0, 1, 2, 3$ or ≥ 4. We introduce additional data $U = (u_{ij})_{i,j \in I}$ such that $u_{ii} = 0$, $u_{ij} = 0 \leftrightarrow a_{ij} = 0$ and $u_{ij} a_{ij} = -u_{ji} a_{ji}$, $(i \neq j)$. Let $R = C(\alpha_i \ ; f_i \ ; \ i \in I)$ be the field of rational functions. Then we have [9]

Theorem 6. *There is a representation of W on R such that*

$$s_i(\alpha_j) = \alpha_j - \alpha_i a_{ij}, \quad s_i(f_j) = f_j + \frac{\alpha_i}{f_i} u_{ij}.$$

This representation has the following applications

(1) Bäcklund transformations of known Painlevé equations.
(2) Discrete integrable dynamical systems
(3) Generalized Painlevé equations for root systems.

We shall explain these in the next subsections.

3.2. Symmetric Form

For the Painlevé equations P_{IV}, P_V and P_{VI}, the Bäcklund transformations take the universal form as above by suitable choice of dependent variables.

Example. Symmetric form of P_{IV} [4]

$$\begin{aligned} f_0' &= f_0(f_1 - f_2) + \alpha_0, \\ f_1' &= f_1(f_2 - f_0) + \alpha_1, \\ f_2' &= f_2(f_0 - f_1) + \alpha_2. \end{aligned} \tag{3.1}$$

3.3. Discrete integrable systems

(Extended) affine Weyl group is a semi-direct product of Lattice M and finite Weyl group W_0. Let $\{T_i\}$ be the generators of M. In the above representation, these are non trivially commuting bi-rational mapping.[9]

Example. $A = A_3^{(1)}$. Put $T_1 = \pi s_3 s_2 s_1$, then

$$\begin{aligned} T_1(f_2) &= \pi s_3 s_2 s_1(f_2) = \pi s_3 s_2(f_2 + \frac{\alpha_1}{f_1}) \\ &= \pi s_3(f_2 + \frac{\alpha_1+\alpha_2}{f_1 - \frac{\alpha_2}{f_2}}) = \pi(f_2 - \frac{\alpha_3}{f_3} + \frac{\alpha_1+\alpha_2+\alpha_3}{f_1 - \frac{\alpha_2+\alpha_3}{f_2 - \frac{\alpha_3}{f_3}}}) \\ &= f_3 - \frac{\alpha_0}{f_0} + \frac{\alpha_2+\alpha_3+\alpha_0}{f_2 - \frac{\alpha_3+\alpha_0}{f_3 - \frac{\alpha_0}{f_0}}}. \end{aligned}$$

The commuting family of rational mappings are considered as discrete analogue of the Painlevé equations (see section 4).

3.4. Higher Painlevé equations

Next problem is to find differential equations for which the Weyl groups act as the Bäcklund transformation.

There exist a series of such equations for $A_l^{(1)}$. These series contain P_{IV} (for $l = 2$) and P_V (for $l = 3$) as simplest examples and expected to have the Painlevé property.

Case $l = 2n$:

$$\frac{df_j}{dt} = f_j \sum_{1 \le r \le n} (f_{j+2r-1} - f_{j+2r}) + \alpha_j. \tag{3.2}$$

Case $l = 2n+1$:

$$\begin{aligned} \frac{df_j}{dt} = f_j &\sum_{1 \le r \le s \le n} (f_{j+2r-1} f_{j+2s} - f_{j+2r} f_{j+2s+1}) \\ &+ (\frac{k}{2} - \sum_{1 \le r \le n} \alpha_{j+2r}) f_j + \alpha_j (\sum_{1 \le r \le n} f_{j+2r}). \end{aligned} \tag{3.3}$$

For each case, $f_j = f_j(t)$ are the unknown functions and α_j are constants such as $\alpha_0 + \cdots + \alpha_l = k$ $(0 \le j \le l)$.

Remark.

(1) These equations can be obtained as a continuum limit of the discrete system previously discussed.
(2) These equations admit a hamiltonian formulation.[10]
(3) These systems also have a Lax formalism and can be considered as a similarity reduction of (modified) KP equations.

§4. Special polynomials arising from the representation

In $A_l^{(1)}$ case, for any $w \in W$, $w(f_i)$ is always factorized into four polynomials as

$$w(f_i) = \frac{PQ}{RS}.$$

Example. For $A_3^{(1)}$, we see

$$T_1(f_2) = \frac{(f_3 f_0 - \alpha_0)(f_2 f_3 f_0 + \alpha_2 f_0 - \alpha_0 f_2)}{f_0(f_2 f_3 f_0 - \alpha_0 f_2 - (\alpha_3 + \alpha_0) f_0)}.$$

These polynomials can be interpreted as the τ-function of our discrete system. Their polynomiality is closely related with the singularity confinement property, which is a discrete analog of the Painlevé property.

Remark. The following property of the difference system is called singularity confinement: "Any singularity depending on initial data will disappear after finite iteration of mapping and the initial data can be recovered after such iteration". [11].

Theorem 7. *The representation of Weyl groups can be extended to $C(\alpha_i\ ; f_i\ ; \tau_i\ ; i \in I)$ in such a way that*

$$s_i(\tau_i) = \frac{f_i}{\tau_i} \prod_{k \neq i} \tau_k^{|a_{ki}|}, \quad s_i(\tau_j) = \tau_j \ (i \neq j).$$

For any $w \in W$, $w(\tau_i)$ is factorized as $w(\tau_i) = \phi_{i,w} \prod_{j \in I} {\tau_j}^{m_j}$, where $m_j = \langle \alpha\check{}_j, w(\Lambda_i) \rangle \in Z$ and $\phi_{i,w} \in C(\alpha_i\ ; f_i\ ; i \in I)$. We observe that [9]

Conjecture. *$\phi_{i,w}$ is a polynomial.*

We have a proof of this conjecture in $A_l^{(1)}$ case, by using explicit determinant formulas.[12]

Theorem 8. *For any $w \in W$, the polynomial $\phi_{0,w}$ is given by the following determinant of Jacobi-Trudi type*

$$\phi_{0,w} = \frac{1}{N_w} \det \left[\pi^{1-j}(h_{\lambda_i - i + j})\right]_{1 \leq i,j \leq l(\lambda)}.$$

Where $N_w \in Z[\alpha_i; i \in I]$, $h_j \in Z[\alpha_i, f_i; i \in I]$ *and* λ *is a partition determined by* $w \in W$.

Note that the determinant structure of polynomial ϕ is the same as the 9th variation of the Schur function.[13]

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