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Approximation Results for Kazhdan-Lusztig Polynomials

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§1. Introduction

In their fundamental paper [7] Kazhdan and Lusztig defined, for every Coxeter group W, a family of polynomials, indexed by pairs of elements of W, which have become known as the Kazhdan-Lusztig polynomials of W (see, e.g., [6], Chap. 7). These polynomials are intimately related to the Bruhat order of W and to the geometry of Schubert varieties, and have proven to be of fundamental importance in representation theory. In order to prove the existence of these polynomials Kazhdan and Lusztig used another family of polynomials (see [7], §2) which are intimately related to the multiplicative structure of the Hecke algebra associated to W. These polynomials are known as the R-polynomials of W (see, e.g., [6], §7.5) and their importance stems mainly from the fact that their knowledge is equivalent to that of the Kazhdan-Lusztig polynomials.

The main idea of this work is to use the theory of P-kernels developed by Stanley in [10] to approximate the Kazhdan-Lusztig polynomials with other "KLS-functions" (see §2 for definitions) that are easier to compute. In particular, we characterize the pairs $u, v \in W$ such that the Kazhdan-Lusztig polynomials of the subintervals of $[u, v]$ satisfy certain vanishing properties or, more generally, coincide with some given function in the incidence algebra of W, up to a given order. Two of our results generalize and refine previous ones that have appeared in [7] and [3].

The theory of P-kernels also naturally leads to define and study certain polynomials, indexed by pairs of elements of W, that are "dual"

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to the R-polynomials of W in a very precise sense. To the best of our knowledge, although their definition is quite natural, these polynomials have never been considered before in the literature. Similarly, we are led to the study of the "dual" of the zeta function of a locally Eulerian poset, which also seems to be a new object.

The organization of the paper is as follows. In the next section we collect notation, definitions, and results, that are used in the sequel. In §3 we prove our main results (Theorems 3.1 and 3.2). These are purely combinatorial results that "compare" two KLS-functions in terms of their kernels. In section 4 we define a natural involution on kernels and KLS-functions and study in some detail the dual of the zeta function of a locally Eulerian poset, and of the R-polynomials of a Coxeter group. To the best of our knowledge, these objects have never been considered before. We also study how a local change in a KLS-function affects the corresponding kernel. In section 5 we apply the results obtained in the two previous ones to the Kazhdan-Lusztig polynomials. In particular, we characterize the intervals of W such that the Kazhdan-Lusztig polynomials of its subintervals (respectively, lower subintervals) are equal to 1 up to a given order. Finally, in section 6, we discuss some conjectures and open problems arising from the present work.

§2. Notation, Definitions, and preliminaries

In this section we collect some definitions, notation and results that will be used in the rest of this work. We let $\mathbf{P} \stackrel{\text{def}}{=} \{1, 2, 3, \dots\}$, $\mathbf{N} \stackrel{\text{def}}{=} \mathbf{P} \cup \{0\}$, $\mathbf{Z}$ be the ring of integers, $\mathbf{Q}$ be the field of rational numbers, and $\mathbf{R}$ be the field of real numbers; for $a \in \mathbf{N}$ we let $[a] \stackrel{\text{def}}{=} \{1, 2, \dots, a\}$ (where $[0] \stackrel{\text{def}}{=} \emptyset$). The cardinality of a set A will be denoted by $|A|$. We write $A \subset B$ to mean that $A \subseteq B$ and $A \neq B$. Given a polynomial $P(q)$, and $i \in \mathbf{Z}$, we denote by $[q^i](P(q))$ the coefficient of q^i in $P(q)$. For $a \in \mathbf{Q}$ we let $\lfloor a \rfloor$ (respectively, $\lceil a \rceil$) denote the largest integer $\leq a$ (respectively, smallest integer $\geq a$). Given $A(q) \in \mathbf{R}[q]$ and $d \in \mathbf{P}$ we say that $A(q)$ is *symmetric* (respectively, *antisymmetric*) with respect to d if $q^d A\left(\frac{1}{q}\right) = A(q)$ (respectively, $q^d A\left(\frac{1}{q}\right) = -A(q)$).

For $j \in \mathbf{Q}$ we define operators U_j, $D_j : \mathbf{R}[q] \to \mathbf{R}[q]$ by letting

$$U_j\left(\sum_{i\geq 0} a_i q^i\right) \stackrel{\text{def}}{=} \sum_{i \geq j} a_i q^i,$$

and

$$D_j\left(\sum_{i\geq 0} a_i q^i\right) \stackrel{\text{def}}{=} \sum_{i=0}^{\lfloor j\rfloor} a_i q^i.$$

Note that U_j and D_j are linear and idempotent, and that $D_j = D_{\lfloor j\rfloor}$ and $U_j = U_{\lceil j\rceil}$, for all $j \in \mathbf{Q}$. The following lemma will be used repeatedly in this paper and its simple verification is omitted.

Lemma 2.1. *Let $A(q), B(q) \in \mathbf{R}[q]$, and $k \in \mathbf{Z}$. Then*

$$D_k(AB) = D_k(D_k(A)D_k(B)).$$

We follow [9], Chap. 3, for notation and terminology concerning partially ordered sets. In particular, given a partially ordered set (or, poset, for short) P we let $\mathrm{Int}(P) \stackrel{\text{def}}{=} \{(x,y) \in P^2 : x \leq y\}$, and given $u, v \in P$ we let $[u,v] \stackrel{\text{def}}{=} \{x \in P : u \leq x \leq v\}$, and define $[u,v)$ and $(u,v]$ similarly. We consider $[u,v]$ as a poset with the partial ordering induced by P. We say that a poset P is *locally finite* if $|[x,y]| < +\infty$ for all $(x,y) \in \mathrm{Int}(P)$, and in this case we denote by ζ_P (respectively, μ_P, δ_P) the *zeta* (respectively, *Möbius, delta*) function of P. We will usually omit the index P if there is no danger of confusion.

Given a finite graded poset P and $S \subseteq \mathbf{N}$ we let $P_S \stackrel{\text{def}}{=} \{x \in P : l(x) \in S\}$, where $l : P \to \mathbf{N}$ is the rank function of P, and $\alpha(P;S)$ be the number of maximal chains of P_S. We also let $P_i \stackrel{\text{def}}{=} P_{\{i\}}$ if $i \in \mathbf{N}$. We call $G(P) \stackrel{\text{def}}{=} \sum_{i\geq 0} |P_i| q^i$ the *rank generating function* of P, $d \stackrel{\text{def}}{=} \deg(G(P))$ the *rank* of P, and the collection of numbers $\{\alpha(P;S)\}_{S\subseteq[d]}$ the *flag f-vector* of P. We say that a finite graded poset P as above is *rank symmetric* if $G(P)$ is symmetric with respect to d, and is *Eulerian* if P has a $\hat{0}$ and $\hat{1}$ and $\mu(x,y) = (-1)^{l(y)-l(x)}$ for all $x, y \in P$, $x \leq y$. Following [10, §7, p. 835] (respectively, [11]) we say that a locally finite poset P is *locally Eulerian* (respectively, *locally rank symmetric*) if $[x,y]$ is Eulerian (respectively, *rank symmetric*) for all $(x,y) \in \mathrm{Int}(P)$.

Recall (see, e.g., [9], §3.6) that given a locally finite poset P and a commutative ring R the *incidence algebra* of P with coefficients in R, denoted $I(P;R)$, is the set of all functions $f : \mathrm{Int}(P) \to R$ with sum and product defined by

$$(f+g)(x,y) \stackrel{\text{def}}{=} f(x,y) + g(x,y)$$

and

$$(fg)(x,y) \stackrel{\text{def}}{=} \sum_{x \le z \le y} f(x,z)\, g(z,y),$$

for all $f, g \in I(P;R)$ and $(x,y) \in \mathrm{Int}(P)$. It is well known (see, e.g., [9], §3.6, and Proposition 3.6.2) that $I(P;R)$ is an associative algebra having δ as identity element, and that an element $f \in I(P;R)$ is invertible if and only if $f(x,x) \in R$ is invertible for all $x \in P$. If f is invertible then we denote by f^{-1} its (two-sided) inverse. Given $f \in I(P;R)$ we define $f^* \in I(P^*;R)$ (where P^* denotes the order dual of P) by letting

$$f^*(v,u) \stackrel{\text{def}}{=} f(u,v)$$

for all $(v,u) \in \mathrm{Int}(P^*)$. Note that $\zeta_{P^*} = \zeta_P^*$, $\delta_{P^*} = \delta_P^*$, and $\mu_{P^*} = \mu_P^*$. We adopt the convention that $f(u,v) \stackrel{\text{def}}{=} 0$ if $f \in I(P;R)$ and $u, v \in P$, $u \not\leq v$.

Let P be a locally finite poset. We say that a function $\rho : \mathrm{Int}(P) \to \mathbf{N}$ is a *weak rank function* for P if it has the following two properties:

i): if $u < v$ then $\rho(u,v) > 0$;

ii): if $u \le a \le v$ then $\rho(u,v) = \rho(u,a) + \rho(a,v)$.

Note that a weak rank function always exists and that if ρ is a weak rank function for P then ρ^* is a weak rank function for P^*. The concept of a weak rank function enables us to extend the main definitions of §6 of [10] from the locally graded case (i.e., posets P such that $[x,y]$ is a finite graded poset for all $(x,y) \in \mathrm{Int}(P)$) to the locally finite case.

Let P and ρ be as above and $I(P) \stackrel{\text{def}}{=} I(P; \mathbf{R}[q])$. Following Stanley (see [10],p. 830, and Proposition 6.11, p. 835) we let

$$\tilde{I}(P) \stackrel{\text{def}}{=} \{f \in I(P) : \ \deg(f(x,y)) \le \rho(x,y), \quad \text{for all } (x,y) \in \mathrm{Int}(P)\},$$

and

$$\begin{aligned} I_{\frac{1}{2}}(P) \ \stackrel{\text{def}}{=}\ & \{f \in \tilde{I}(P) : \ \deg(f(u,v)) \le \frac{1}{2}(\rho(u,v) - 1) \text{ for } u < v, \\ & \text{and } \ f(u,u) = 1 \ \text{ for all } \ u \in P\}. \end{aligned}$$

Note that $\tilde{I}(P)$ is a subalgebra of $I(P)$ and that, if $f \in I(P)$ is invertible, then $f \in \tilde{I}(P)$ if and only if $f^{-1} \in \tilde{I}(P)$. Given $f \in \tilde{I}(P)$ and $k \in \mathbf{Q}$ we let

$$\overline{f}(u,v) \stackrel{\text{def}}{=} q^{\rho(u,v)} f(u,v) \left(\frac{1}{q}\right),$$

and

$$D_k(f)(u,v) \stackrel{\text{def}}{=} D_k(f(u,v)),$$

for all $u, v \in P$, $u \leq v$. Notice that $\tilde{I}(P)$, $I_{\frac{1}{2}}(P)$, and the involution $^{-}$ all depend also on ρ. However, throughout this work ρ will always be fixed, so no confusion should arise. Recall (see [10], Definition 6.2, p. 830) that an element $K \in I(P)$ is called a *P-kernel* (or, more simply, a *kernel*) if K is *unitary* (i.e., $K(u,u) = 1$ for all $u \in P$) and there exists an element $f \in I(P)$ such that:

i): f is invertible in $I(P)$;

ii): $fK = \overline{f}$.

An element $f \in I(P)$ satisfying ii) above is called *K-totally acceptable* (see [10], Definition 6.2, p.830). The next result was first proved by Stanley in the locally graded case (see [10], Corollary 6.7), and by the author in the locally finite one (see [5, Theorem 6.2]).

Theorem 2.2. *Let P be a locally finite poset and $K \in I(P)$ a P-kernel. Then there exists a unique K-totally acceptable element $\gamma \in I_{\frac{1}{2}}(P)$.*

We call the element γ whose existence and uniqueness is guaranteed by the preceding theorem the *Kazhdan-Lusztig-Stanley function* (or *KLS-function*, for short) of K. As noted in [10], §§6 and 7, the function γ specializes to many interesting objects depending on the particular choice of the poset P and kernel K.

There is a simple way to decide if a given element $K \in I(P)$ is a P-kernel or not. The following result was first proved by Stanley in [10] (see Theorem 6.5, p. 831) in the case that P is locally graded. However, his proof carries over unchanged to the present more general setting.

Theorem 2.3. *Let P be a locally finite poset and $K \in I(P)$ be such that $K(u,u) = 1$ for all $u \in P$. Then K is a P-kernel if and only if $K\overline{K} = \delta$.*

Note that Theorem 2.2 defines a map from the set of P-kernels to $I_{\frac{1}{2}}(P)$ and that, by Theorem 2.3, the map $f \mapsto f^{-1}\overline{f}$ is its inverse. Thus the correspondence $K \mapsto \gamma$ of Theorem 2.2 is a bijection. We call this bijection the *KLS-correspondence* of P and the elements of $I_{\frac{1}{2}}(P)$ the *KLS-functions* of P.

For a locally finite poset P define an element $\chi_P \in I(P)$ by letting

$$\chi_P(u,v) \stackrel{\text{def}}{=} \sum_{a \in [u,v]} \mu(u,a) q^{\rho(a,v)} \tag{1}$$

for all $(u,v) \in \mathrm{Int}(P)$ ($\chi_P(u,v)$ is often called the *characteristic polynomial* of $[u,v]$, see, e.g., [9, §3.10, p.128]). It is then clear from the definitions (see also [10, Example 6.8, p. 833]) that χ_P is a P-kernel, and that ζ is its KLS-function. We call χ_P the *characteristic kernel* of P. Note that, in general, $\chi_P^* \neq \chi_{P^*}$ even if P^* is weakly graded by ρ^*.

We follow [6] for general Coxeter groups notation and terminology. Given a Coxeter system (W,S) and $\sigma \in W$ we denote by $l(\sigma)$ the length of σ in W, with respect to S, and we let $D(\sigma) \stackrel{\mathrm{def}}{=} \{s \in S : \ l(s\,\sigma) < l(\sigma)\}$, and $\varepsilon_\sigma \stackrel{\mathrm{def}}{=} (-1)^{l(\sigma)}$. We denote by e the identity of W, and we let $T \stackrel{\mathrm{def}}{=} \{\sigma s \sigma^{-1} : \sigma \in W, \ s \in S\}$ be the set of reflections of W. We will always assume that W is partially ordered by (strong) *Bruhat order.* Recall (see, e.g., [6], §5.9) that this means that $x \leq y$ if and only if there exist $r \in \mathbf{N}$ and $t_1, \dots, t_r \in T$ such that $t_r \dots t_1\, x = y$ and $l(t_i \dots t_1\, x) > l(t_{i-1} \dots t_1 x)$ for $i = 1, \dots, r$. It is well known (see, e.g., [6, §8.5], Proposition 1, iv)) that intervals of W are finite Eulerian posets, and it is clear that $\rho(x,y) \stackrel{\mathrm{def}}{=} l(y) - l(x)$ for $(x,y) \in \mathrm{Int}(W)$ is a weak rank function for W. The following two results are well known and we refer the reader to [6, §7.5] and to [6, §§7.9-11] for their proofs.

Theorem 2.4. *Let (W,S) be a Coxeter system. Then there is a unique family of polynomials $\{R_{u,v}(q)\}_{u,v \in W} \subseteq \mathbf{Z}[q]$ such that, for all $u,v \in W$:*

i): $R_{u,v}(q) = 0$ *if* $u \not\leq v$*;*
ii): $R_{u,u}(q) = 1$*;*
iii): *if* $u < v$ *and* $s \in D(v)$ *then*

$$R_{u,v}(q) = \begin{cases} R_{su,sv}(q), & \text{if } s \in D(u), \\ (q-1)R_{u,sv}(q) + qR_{su,sv}(q), & \text{if } s \notin D(u). \end{cases}$$

Theorem 2.5. *Let (W,S) be a Coxeter system. Then there is a unique family of polynomials $\{P_{u,v}(q)\}_{u,v \in W} \subseteq \mathbf{Z}[q]$, such that, for all $u,v \in W$:*

i): $P_{u,v}(q) = 0$ *if* $u \not\leq v$*;*
ii): $P_{u,u}(q) = 1$*;*
iii): $deg(P_{u,v}(q)) \leq \lfloor \frac{1}{2}(l(v) - l(u) - 1) \rfloor$, *if* $u < v$*;*
iv):

$$q^{l(v)-l(u)}\, P_{u,v}\left(\frac{1}{q}\right) = \sum_{u \leq z \leq v} R_{u,z}(q)\, P_{z,v}(q)\,,$$

if $u \leq v$.

The polynomials $R_{u,v}(q)$ and $P_{u,v}(q)$, whose existence is guaranteed by the two previous theorems, are called the *R-polynomials* and

Kazhdan-Lusztig polynomials of W. There is one more property of the polynomials $R_{u,v}(q)$ that we will use, and that we recall here for the reader's convenience. A proof of it can be found in [6, §7.8].

Proposition 2.6. *Let (W, S) be a Coxeter system. Then*

$$q^{l(v)-l(u)} R_{u,v}\left(\frac{1}{q}\right) = (-1)^{l(v)-l(u)} R_{u,v}(q)$$

for all $u, v \in W$.

We define two elements $\Re, \wp \in \tilde{I}(W)$ by letting

$$\Re(u,v) \stackrel{\text{def}}{=} (-1)^{l(v)-l(u)} R_{u,v}(q), \tag{2}$$

and

$$\wp(u,v) \stackrel{\text{def}}{=} P_{u,v}(q), \tag{3}$$

for all $u, v \in W$, $u \leq v$. It then follows immediately from Proposition 2.6 and Theorem 2.5 that $\overline{\Re}\wp = \overline{\wp}$ and that $\wp \in I_{\frac{1}{2}}(W)$. Therefore $(\overline{\Re})^*$ is a W^*-kernel and $\wp^*$ is its KLS-function.

Given $u, v \in W$, $u \leq v$, we define a polynomial $Q_{u,v}(q) \in \mathbf{Z}[q]$ by letting

$$Q_{u,v}(q) \stackrel{\text{def}}{=} (-1)^{l(v)-l(u)} \wp^{-1}(u,v). \tag{4}$$

(note that $\wp$ is invertible in $I(W)$ by part ii) of Theorem 2.5). It then follows immediately from well known results (see, e.g., [8], p. 190) that $Q_{u,v}(q)$ is the inverse Kazhdan-Lusztig polynomial of u, v.

The R-polynomials are much better understood than the Kazhdan-Lusztig polynomials (see, e.g., [6, p. 159]). For example, it is well known (see, e.g., [6, §§7.4-5]) and easy to see, that $R_{x,y}(q)$ is always a monic polynomial of degree $l(y) - l(x)$, while neither the degree nor the leading term of $P_{x,y}(q)$ can be easily predicted. Therefore, some of the recent research on Kazhdan-Lusztig polynomials (see, e.g., [3]) has focused on using the R-polynomials to gain information on the Kazhdan-Lusztig polynomials. This is the case for the present work also.

Throughout this paper, unless otherwise explicitly stated, (W, S) denotes a Coxeter system, P a locally finite poset, and $\rho : \text{Int}(P) \to \mathbf{N}$ a weak rank function for P.

§3. Comparison Results

In this section we derive the main results on which our applications to Kazhdan-Lusztig polynomials are based. These are purely combinatorial results which "compare" two KLS-functions in terms of their kernels. They can also be seen as giving some fundamental properties of the KLS-correspondence of a locally finite poset.

Theorem 3.1. *Let $k \in \mathbf{Z}$, $u, v \in P$, K_1, K_2 be P-kernels, and γ_1, γ_2 be their KLS-functions. Then the following are equivalent:*

i): $D_k(\gamma_1(x,y)) = D_k(\gamma_2(x,y))$ *for all* $x, y \in [u,v]$;
ii): $D_k(K_1(x,y)) = D_k(K_2(x,y))$ *for all* $x, y \in [u,v]$.

Proof. Assume that ii) holds. We proceed by induction on $\rho(u,v)$. If $\rho(u,v) = 0$ then $u = v$ and i) coincides with ii). So assume $\rho(u,v) > 0$, and let $x, y \in [u,v]$. Then from ii) and our induction hypothesis we conclude that

$$\begin{aligned}
D_k(\overline{\gamma_1}(x,y) - \gamma_1(x,y)) &= D_k\left(\sum_{x \le a < y} \gamma_1(x,a)K_1(a,y)\right) \\
&= D_k\left(\sum_{x \le a < y} D_k(\gamma_1(x,a))D_k(K_1(a,y))\right) \\
&= D_k\left(\sum_{x \le a < y} D_k(\gamma_2(x,a))D_k(K_2(a,y))\right) \\
&= D_k\left(\sum_{x \le a < y} \gamma_2(x,a)K_2(a,y)\right) \\
&= D_k(\overline{\gamma_2}(x,y) - \gamma_2(x,y)).
\end{aligned}$$

Since $\gamma_1, \gamma_2 \in I_{\frac{1}{2}}(P)$ this implies that $D_k(\gamma_1(x,y)) = D_k(\gamma_2(x,y))$ and this proves i).

Assume now that i) holds. We proceed again by induction on $\rho(u,v)$, ii) being clearly true if $\rho(u,v) = 0$. So assume that $\rho(u,v) > 0$ and let $x, y \in [u,v]$, $x < y$. Then from i) and the induction hypothesis we conclude that

$$\begin{aligned}
D_k(\overline{\gamma_1}(x,y) - K_1(x,y)) &= D_k\left(\sum_{x<a\leq y} \gamma_1(x,a)K_1(a,y)\right) \\
&= D_k\left(\sum_{x<a\leq y} D_k(\gamma_1(x,a))D_k(K_1(a,y))\right) \\
&= D_k\left(\sum_{x<a\leq y} D_k(\gamma_2(x,a))D_k(K_2(a,y))\right) \\
&= D_k\left(\sum_{x<a\leq y} \gamma_2(x,a)K_2(a,y)\right) \\
&= D_k(\overline{\gamma_2}(x,y) - K_2(x,y)). \qquad (5)
\end{aligned}$$

Now if $k < \left\lceil \frac{\rho(x,y)+1}{2} \right\rceil$ then since $\gamma_1, \gamma_2 \in I_{\frac{1}{2}}(P)$ we have from (5) that

$$D_k(K_1(x,y)) = D_k(K_2(x,y)),$$

as desired. If $k \geq \frac{\rho(x,y)+1}{2}$ then we conclude from our hypothesis i) that

$$\gamma_1(x,y) = D_k(\gamma_1(x,y)) = D_k(\gamma_2(x,y)) = \gamma_2(x,y).$$

Hence

$$\begin{aligned}
D_k(K_1(x,y)) &= D_k(K_1(x,y) - \overline{\gamma_1}(x,y)) + D_k(\overline{\gamma_1}(x,y)) \\
&= D_k(K_2(x,y) - \overline{\gamma_2}(x,y)) + D_k(\overline{\gamma_2}(x,y)) \\
&= D_k(K_2(x,y))
\end{aligned}$$

and ii) holds also in this case.

Note that it is not true, in general, that if K is a P-kernel and $k \in \mathbf{N}$, then $D_k(K)$ is also a P-kernel. For example, if P is the Boolean algebra of rank 2 and $K = \chi_P$ (the characteristic kernel of P) then $D_1(K)$ is not a P-kernel since

$$\sum_{\hat{0}\leq a\leq \hat{1}} D_1(\chi_P)(\hat{0},a)\,\overline{D_1(\chi_P)}(a,\hat{1}) = (q^2-2q) - 2(q-1)^2 + (1-2q) \neq 0,$$

which would contradict Theorem 2.3. Thus Theorem 3.1 is not a special case of Theorem 2.2.

The next result is also a "comparison result" except that it does not require any knowledge of the P-kernel corresponding to one of the KLS-functions involved.

Theorem 3.2. *Let $k \in \mathbf{Z}$, $u, v \in P$, $f \in I_{\frac{1}{2}}(P)$, K be a P-kernel, and γ be its KLS-function. Then the following are equivalent:*

i): $D_k(\gamma(u,x)) = D_k(f(u,x))$ *for all* $x \in [u,v]$;

ii): $D_k\left(\sum_{a\in[u,x)} f(u,a)K(a,x)\right) = D_k(\overline{f}(u,x) - f(u,x))$ *for all* $x \in [u,v]$.

Proof. Assume that i) holds. Then we have from our hypotheses that

$$\begin{aligned}
D_k\left(\sum_{a\in[u,x)} f(u,a)K(a,x)\right) &= D_k\left(\sum_{a\in[u,x)} D_k(f(u,a))D_k(K(a,x))\right) \\
&= D_k\left(\sum_{a\in[u,x)} D_k(\gamma(u,a))D_k(K(a,x))\right) \\
&= D_k\left(\sum_{a\in[u,x)} \gamma(u,a)K(a,x)\right) \\
&= D_k(\overline{\gamma}(u,x) - \gamma(u,x)) \\
&= D_k(\overline{f}(u,x) - f(u,x))
\end{aligned}$$

for all $x \in [u,v]$, as desired.

Conversely, assume that ii) holds. We proceed by induction on $\rho(u,v)$, i) being clear if $u = v$. So assume that $\rho(u,v) \geq 1$. Then by our induction hypothesis we have that $D_k(\gamma(u,x)) = D_k(f(u,x))$ for all $x \in [u,v)$. Hence we have from our hypothesis ii) that

$$\begin{aligned}
D_k(\overline{f}(u,v) - f(u,v)) &= D_k\left(\sum_{a\in[u,v)} f(u,a)K(a,v)\right) \\
&= D_k\left(\sum_{a\in[u,v)} D_k(f(u,a))D_k(K(a,v))\right) \\
&= D_k\left(\sum_{a\in[u,v)} D_k(\gamma(u,a))D_k(K(a,v))\right) \\
&= D_k\left(\sum_{a\in[u,v)} \gamma(u,a)K(a,v)\right) \\
&= D_k(\overline{\gamma}(u,v) - \gamma(u,v)).
\end{aligned}$$

Since $\gamma, f \in I_{\frac{1}{2}}(P)$ this implies that $D_k(f(u,v)) = D_k(\gamma(u,v))$, and i) follows.

§4. New kernels from old

The applicability of the results obtained in the previous section depends to some extent on the explicit knowledge of P-kernels and their corresponding KLS-functions. Although on almost all posets there are infinitely many P-kernels it is difficult to find pairs of a P-kernel and its KLS-function that can *both* be described explicitly. For example, it follows easily from Theorem 2.3 that if K is a P-kernel then $\overline{K}$ is also a P-kernel and K^* is a P^*-kernel. Thus, to each kernel K there are naturally associated three other kernels, namely $\overline{K}$, K^*, and $\overline{K^*}$ (note that this process does not go on indefinitely, since $(\overline{K})^* = \overline{K^*}$ if P^* is weakly graded by ρ^*, as is usually the case). However, while it is known (see [10, Proposition 8.1]) that the KLS-function of $\overline{K^*}$ is $(\gamma^{-1})^*$ if γ is the KLS-function of K, no simple expression is known for the KLS-functions of K^* or of $\overline{K}$ in terms of the KLS-function of K. Similarly, it is obvious that if $\gamma \in I_{\frac{1}{2}}(P)$ then $\gamma^* \in I_{\frac{1}{2}}(P^*)$, and $\gamma^{-1}, D_k(\gamma) \in I_{\frac{1}{2}}(P)$, but no simple expression is known for the corresponding kernels in terms of the kernel of γ.

In this section we examine in some detail two particularly interesting such pairs, and we introduce a process that, given a pair (K, γ) of a P-kernel and its corresponding KLS-function, produces explicitly another such pair. The results in this section are applied in the next one to the study of Kazhdan-Lusztig polynomials.

We begin by studying a process that could be called "deformation" of a KLS-function. For $g \in \mathbf{R}[q]$ and $x, y \in P$, $x < y$, we define an element $g_{x,y} \in I(P)$ by letting

$$g_{x,y}(u,v) \stackrel{\text{def}}{=} \begin{cases} 0, & \text{if } (u,v) \neq (x,y), \\ g(q), & \text{if } (u,v) = (x,y), \end{cases}$$

for all $(u,v) \in \text{Int}(P)$.

Proposition 4.1. *Let $f \in I(P)$ be unitary, $g \in \mathbf{R}[q]$, and $x, y \in P$, $x < y$. Then*

$$(f + g_{x,y})^{-1}(u,v) = f^{-1}(u,v) - g(q) f^{-1}(u,x) f^{-1}(y,v)$$

for all $(u,v) \in Int(P)$.

Proof. We proceed by induction on $\rho(u,v)$, the result being clear if $\rho(u,v) = 0$. So let $\rho(u,v) \geq 1$. We may clearly assume that $[x,y] \subseteq$

$[u, v]$. Then we have that, if $u < x$,

$$\begin{aligned}
(f + g_{x,y})^{-1}(u, v) &= -\sum_{u<a\leq v} (f + g_{x,y})(u, a)(f^{-1}(a, v) - g(q)f^{-1}(a, x)f^{-1}(y, v \\
&= -\sum_{u<a\leq v} f(u, a)(f^{-1}(a, v) - g(q)f^{-1}(a, x)f^{-1}(y, v)) \\
&= f^{-1}(u, v) + g(q) \sum_{u<a\leq v} f(u, a)f^{-1}(a, x)f^{-1}(y, v) \\
&= f^{-1}(u, v) + g(q)f^{-1}(y, v) \sum_{u<a\leq x} f(u, a)f^{-1}(a, x) \\
&= f^{-1}(u, v) - g(q)f^{-1}(y, v)f^{-1}(u, x),
\end{aligned}$$

as desired. On the other hand, if $u = x$ then

$$\begin{aligned}
(f + g_{x,y})^{-1}(x, v) &= -\sum_{x<a\leq v} (f + g_{x,y})(x, a)(f^{-1}(a, v) - g(q)f^{-1}(a, x)f^{-1}(y, v \\
&= -\sum_{x<a\leq v} (f + g_{x,y})(x, a)f^{-1}(a, v) \\
&= -\sum_{x<a\leq v} f(x, a)f^{-1}(a, v) - g(q)f^{-1}(y, v) \\
&= f^{-1}(x, v) - g(q)f^{-1}(y, v),
\end{aligned}$$

and the result again follows.

Suppose now that $f \in I_{\frac{1}{2}}(P)$, K is the P-kernel of f, and $g \in \mathbf{R}[q]$, $x, y \in P$, $x < y$, are such that $\deg(g) \leq \frac{1}{2}(\rho(x, y) - 1)$. Then $f + g_{x,y} \in I_{\frac{1}{2}}(P)$, and we denote by $K_{x,y}(g)$ the P-kernel corresponding to $f + g_{x,y}$. The next result gives an explicit expression for $K_{x,y}(g)$ in terms of K, g, f, and x, y.

Theorem 4.2. *Let $f \in I_{\frac{1}{2}}(P)$, $g \in \mathbf{R}[q]$, and $x, y \in P$, $x < y$, be such that $f + g_{x,y} \in I_{\frac{1}{2}}(P)$. Then*

$$K_{x,y}(g)(u, v) = \begin{cases} K(u, v) + f^{-1}(u, x)\left(q^{\rho(x,y)}g\left(\frac{1}{q}\right) - g(q)\right), & \text{if } y = v, \\ K(u, v) - g(q)f^{-1}(u, x)K(y, v), & \text{otherwise,} \end{cases}$$

for all $(u, v) \in Int(P)$.

Proof. Since $K_{x,y}(g)$ is the P-kernel corresponding to $f + g_{x,y}$ and K is the P-kernel of f there follows from the definitions that $K_{x,y}(g) =$

$(f+g_{x,y})^{-1}\overline{(f+g_{x,y})}$, and $K = f^{-1}\overline{f}$. We may clearly assume that $[x,y] \subseteq [u,v]$. Then we have from Proposition 4.1 that, if $y < v$,

$$\begin{aligned} K_{x,y}(g)(u,v) &= \sum_{u \le a \le v} (f+g_{x,y})^{-1}(u,a)\overline{(f+g_{x,y})}(a,v) \\ &= \sum_{u \le a \le v} (f+g_{x,y})^{-1}(u,a)\overline{f}(a,v) \\ &= \sum_{u \le a \le v} (f^{-1}(u,a) - g(q)f^{-1}(u,x)f^{-1}(y,a))\overline{f}(a,v) \\ &= K(u,v) - g(q)f^{-1}(u,x)\sum_{y \le a \le v} f^{-1}(y,a)\overline{f}(a,v) \\ &= K(u,v) - g(q)f^{-1}(u,x)K(y,v), \end{aligned}$$

as desired. On the other hand, if $y = v$ then

$$\begin{aligned} K_{x,y}(g)(u,y) &= \sum_{u \le a \le y} (f+g_{x,y})^{-1}(u,a)\overline{(f+g_{x,y})}(a,y) \\ &= \sum_{u \le a \le y} (f^{-1}(u,a) - g(q)f^{-1}(u,x)f^{-1}(y,a))\overline{(f+g_{x,y})}(a,y) \\ &= \sum_{u \le a \le y} f^{-1}(u,a)\overline{(f+g_{x,y})}(a,y) - g(q)f^{-1}(u,x) \\ &= \sum_{u \le a \le y} f^{-1}(u,a)\overline{f}(a,y) + f^{-1}(u,x)q^{\rho(x,y)}g\left(\frac{1}{q}\right) - g(q)f^{-1}(u,x) \\ &= K(u,y) + f^{-1}(u,x)\left(q^{\rho(x,y)}g\left(\frac{1}{q}\right) - g(q)\right), \end{aligned}$$

and the result again follows.

As noted at the beginning of this section, given a P-kernel K and its KLS-function γ, no simple formula is known for the KLS-function of $\overline{K}$ nor for the P-kernel of γ^{-1}. We believe that these objects are interesting and worthy of investigation. For this reason, and for convenience, we introduce here a notation for them. Namely, given $\gamma \in I_{\frac{1}{2}}(P)$ we let γ' be the KLS-function of $\overline{K}$ (where K is the P-kernel of γ). Similarly, given a P-kernel K we let K' be the P-kernel of γ^{-1} (where γ is the KLS-function of K). Note that these definitions don't overlap since no element of $I_{\frac{1}{2}}(P) \setminus \{\delta\}$ can be a P-kernel, by Theorem 2.3. Also, note that $(\gamma')' = \gamma$ and $(K')' = K$ for any $\gamma \in I_{\frac{1}{2}}(P)$ and P-kernel K. In the rest of this section we look in some detail at two particularly interesting cases of this operation. Namely, we look at ζ' and $\Re'$.

Let $\chi \in I(P)$ be the characteristic kernel of P. Then from (1) we have that

$$\overline{\chi}(u,v) = \sum_{a\in[u,v]} \mu(u,a) q^{\rho(u,a)}, \tag{6}$$

for all $(u,v) \in \mathrm{Int}(P)$. Since the KLS-function of χ is the zeta function of P, we expect ζ' to be a fundamental enumerative invariant of P. For simplicity, and because of the applications that we are interested in, we limit ourselves to the case that P is locally Eulerian. As a weak rank function for P we take, for $(x,y) \in \mathrm{Int}(P)$, $\rho(x,y)$ to be the common length of all the maximal chains in $[x,y]$ (see also [10, p. 829]). Note that in this case we have from (6) that

$$\overline{\chi}(u,v) = \sum_{a\in[u,v]} (-q)^{\rho(u,a)} \tag{7}$$

for all $(u,v) \in \mathrm{Int}(P)$.

We begin by showing that there is one case in which ζ' is extremely easy to compute.

Proposition 4.3. *Let P be a locally Eulerian poset and $u,v \in P$, $u < v$. Then $[u,v]$ is locally rank symmetric if and only if $\zeta'(x,y) = (-1)^{\rho(x,y)}$ for all $(x,y) \in Int([u,v])$.*

Proof. It is clear from our definition (1) and (7) that $[u,v]$ is locally rank symmetric if and only if $\overline{\chi}(x,y) = (-1)^{\rho(x,y)}\chi(x,y)$ for all $(x,y) \in \mathrm{Int}([u,v])$. But it is easy to see that $(-1)^{\rho(x,y)}\chi(x,y)$ is a P-kernel and $(-1)^{\rho(x,y)}$ is its KLS-function, so the result follows from Theorem 3.1.

If Proposition 4.3 does not apply, however, things are considerably more subtle. Given three integers s,k,d with $0 \le k \le d$ and $0 \le s$ we let $\mathbf{S}_{s,k}(d)$ be the set of all sequences $(a_1,\dots,a_{2s+1}) \in [d]^{2s+1}$ such that:

i): $a_1 \le a_2 \le \dots \le a_{2s+1}$;
ii): $\sum_{j=1}^{i}(-1)^{j+i}a_{2j-1} > \frac{1}{2}a_{2i}$ for $i = 1,\dots,s$;
iii): $\sum_{j=1}^{s+1}(-1)^{s+1-j}a_{2j-1} = d-k$.

We then let

$$\mathbf{S}_k(d) = \bigcup_{s\ge 0} \mathbf{S}_{s,k}(d).$$

For example, $\mathbf{S}_{0,k}(d) = \{(d-k)\}$, $\mathbf{S}_{1,1}(d) = \{(1,1,d)\}$, $\mathbf{S}_{1,2}(d) = \{(1,1,d-1), (2,2,d), (2,3,d)\}$, $\mathbf{S}_{2,2}(d) = \{(1,1,3,3,d)\}$, and $\mathbf{S}_{2,3}(6) = \{(1,1,3,3,5), (1,1,4,4,6), (1,1,4,5,6), (2,2,5,5,6), (2,3,5,5,6)\}$.

Notice that if $(a_1, \ldots, a_{2s+1}) \in \mathbf{S}_{s,k}(d)$, with $s \geq 1$, then from ii) (for $i = s$) and iii) we conclude that $a_{2s+1} - d + k > \frac{a_{2s}}{2}$. In particular, this shows that if $(a_1, \ldots, a_{2s+1}) \in \mathbf{S}_k(d) \setminus \mathbf{S}_{0,k}(d)$ and $k < \frac{d}{2}$ then $a_{2s} \leq d - 1$.

It is not apparent from our definitions that $\mathbf{S}_{s,k}(d) = \emptyset$ for $s > k$, but this is indeed the case.

Lemma 4.4. *Let $s, k, d \in \mathbf{N}$, $k \leq d$, and $(a_1, \ldots, a_{2s+1}) \in \mathbf{S}_{s,k}(d)$. Then*

$$a_1 < a_3 - a_1 < a_5 - a_3 + a_1 < \ldots < a_{2s-1} - a_{2s-3} + \ldots \tag{8}$$

In particular, $s \leq k$.

Proof. From part ii) of the definition of $\mathbf{S}_{s,k}(d)$ we deduce that

$$\sum_{j=1}^{i} (-1)^{j+i} a_{2j-1} > \frac{a_{2i-1}}{2}$$

which can be written as

$$\sum_{j=1}^{i-1} (-1)^{j+i-1} a_{2j-1} < \sum_{j=1}^{i} (-1)^{j+i} a_{2j-1}$$

for $i = 1, \ldots, s$, and this proves (8). In particular, (8) implies that $a_{2s-1} - a_{2s-3} + \ldots \geq s$. Therefore, using iii),

$$d \geq a_{2s+1} = \sum_{j=1}^{s+1} (-1)^{s+1-j} a_{2j-1} + \sum_{j=1}^{s} (-1)^{s-j} a_{2j-1} \geq d - k + s,$$

and the second statement also follows.

We are now ready to state and prove the second main result of this section. This gives an explicit formula for ζ' in terms of the flag f-vector of the intervals of P. For a sequence $A \stackrel{\text{def}}{=} (a_1, \ldots, a_r) \in \mathbf{N}^r$ with $a_1 \leq \ldots \leq a_r$ and $u, v \in P$ we let $\alpha([u,v]; (a_1, \ldots, a_r)) \stackrel{\text{def}}{=} \alpha([u,v]; \{x \in \mathbf{N} : x = a_i \text{ for some } i \in [r]\})$, and $\sum_{a \in A} a \stackrel{\text{def}}{=} a_1 + \ldots + a_r$. So, for example, $\alpha([u,v]; (1,1,3,3,5)) = \alpha([u,v]; \{1,3,5\})$.

Theorem 4.5. *Let P be a locally Eulerian poset, $u, v \in P$, $u < v$, and $k \leq \frac{1}{2}(\rho(u,v) - 1)$. Then*

$$[q^k](\zeta'(u,v)) = \sum_{A \in \mathbf{S}_k(d)} (-1)^{\sum_{a \in A} a} \alpha([u,v]; A),$$

where $d \stackrel{\text{def}}{=} \rho(u,v)$.

Proof. We proceed by induction on $\rho(u,v)$, the result being clear if $\rho(u,v)=1$. From the definition of ζ', (7), and the fact that $k \leq \frac{1}{2}(d-1)$ we conclude that

$$[q^{d-k}](\overline{\zeta'}(u,v) - \overline{\chi}(u,v)) = [q^{d-k}]\left(\sum_{u<a\leq v} \zeta'(u,a)\overline{\chi}(a,v)\right)$$

$$= \sum_{u<a<v} \sum_{i=0}^{\lfloor \frac{\rho(u,a)-1}{2} \rfloor} [q^i](\zeta'(u,a))[q^{d-k-i}](\overline{\chi}(a,v))$$

$$= \sum_{u<a<v} \sum_{i=0}^{\lfloor \frac{\rho(u,a)-1}{2} \rfloor} \sum_{A \in \mathbf{S}_i(\rho(u,a))} (-1)^{\sum_{x\in A} x} \alpha([u,a];A)(-1)^{d-k-i}|[a,v]_{d-k-i}|$$

$$= \sum_{j=1}^{d-1} \sum_{i=0}^{\lfloor \frac{j-1}{2} \rfloor} \sum_{A \in \mathbf{S}_i(j)} (-1)^{\sum_{x\in A} x+d-k-i} \sum_{a\in[u,v]_j} \alpha([u,a];A)|[a,v]_{d-k-i}|$$

(9)

$$= \sum_{j=1}^{d-1} \sum_{i=max(0,j-k)}^{\lfloor \frac{j-1}{2} \rfloor} \sum_{A \in \mathbf{S}_i(j)} (-1)^{\sum_{x\in A} x+d-k-i} \alpha([u,v];A,j,j+d-k-i).$$

Now notice that if $A \in \mathbf{S}_i(j)$, $0 \leq i \leq \lfloor \frac{j-1}{2} \rfloor$ and $j \in [d-1]$ (with $i \geq j-k$) then $(A,j,j+d-k-i) \in \mathbf{S}_k(d) \setminus \mathbf{S}_{0,k}(d)$. Conversely, if $B=(b_1,\ldots,b_{2s+3}) \in \mathbf{S}_k(d) \setminus \mathbf{S}_{0,k}(d)$ then $(b_1,\ldots,b_{2s+1}) \in \mathbf{S}_{b_{2s+2}+d-k-b_{2s+3}}(b_{2s+2})$, $0 \leq b_{2s+2}+d-k-b_{2s+3} < \frac{b_{2s+2}}{2}$, (by Lemma 4.4), $b_{2s+2} \in [d-1]$, by the remarks preceding Lemma 4.4, and $b_{2s+2}+d-k-b_{2s+3} \geq b_{2s+2}-k$. Therefore we conclude from (9) that

$$[q^{d-k}](\overline{\zeta'}(u,v) - \overline{\chi}(u,v)) = \sum_{B \in \mathbf{S}_k(d)\setminus \mathbf{S}_{0,k}(d)} (-1)^{\sum_{x\in B} x} \alpha([u,v];B)$$

and the result follows since

$$[q^{d-k}](\overline{\chi}(u,v)) = (-1)^{d-k}|[u,v]_{d-k}| = (-1)^{d-k}\alpha([u,v];d-k)$$

and $\mathbf{S}_{0,k}(d) = \{(d-k)\}$. This concludes the induction step and hence the proof.

Using the Bayer-Billera relations for flag f-vectors of Eulerian posets (see [1, Theorem 2.1]) it is possible to simplify somewhat the expression given in Theorem 4.5, especially for small values of k.

Corollary 4.6. *Let P be a locally Eulerian poset, and $u, v \in P$, $u < v$. Then*

$$[q^0](\zeta'(u,v)) = (-1)^{\rho(u,v)},$$

$$[q](\zeta'(u,v)) = (-1)^{\rho(u,v)}(|[u,v]_1| - |[u,v]_1^*|),$$

$$[q^2](\zeta'(u,v)) = (-1)^{\rho(u,v)}(|[u,v]_2| - \alpha([u,v];\{1,\rho(u,v)-1\}) + |[u,v]_2^*|).$$

Proof. The first two formulas follow immediately from Theorem 4.5 and the definition of $\mathbf{S}_k(d)$, keeping in mind that $\mathbf{S}_{s,k}(d) = \emptyset$ if $s > k$ by Lemma 4.4. For $k = 2$ we obtain in the same way that

$$\begin{aligned}[q^2](\zeta'(u,v)) &= (-1)^{\rho(u,v)}(|[u,v]_2^*| - \alpha([u,v];\{1,\rho(u,v)-1\}) + |[u,v]_2| \\ &\quad -\alpha([u,v];\{2,3\}) + \alpha([u,v];\{1,3\})).\end{aligned}$$

But since $[u,v]$ is Eulerian we have that

$$2\alpha([u,v];\{2,3\}) = \alpha([u,v];\{1,2,3\}) = 2\alpha([u,v];\{1,3\}),$$

and the result follows.

We conclude this section by looking at the W-kernel $\mathfrak{R}'$. Recall from section 2 that $\overline{\mathfrak{R}}^*$ is a W^*-kernel and $\wp^*$ is its KLS-function. It therefore follows from Proposition 8.1 of [10] that $\mathfrak{R}$ is a W-kernel and $\wp^{-1}$ is its KLS-function. Therefore, by our definition, $\mathfrak{R}'$ is the W-kernel of $\wp$. Note that, from this point of view, $\mathfrak{R}'$ is an even more natural object to consider than $\mathfrak{R}$ itself. We let

$$S_{x,y}(q) \stackrel{\text{def}}{=} \mathfrak{R}'(x,y)$$

for all $(x,y) \in \mathrm{Int}(W)$.

Our aim is to obtain some information about the polynomials $S_{x,y}(q)$. Despite the naturality of their definition these polynomials seem to have never been considered before. As the following results show, they have properties that are very similar to those of the R-polynomials.

Proposition 4.7. *Let $x, y \in W$, $x \leq y$. Then*

$$S_{x,y}(q) = \sum_{x \leq a \leq y} \varepsilon_x \varepsilon_a Q_{x,a}(q)\, q^{l(y)-l(a)} P_{a,y}\left(\frac{1}{q}\right). \tag{10}$$

In particular, $S_{x,y}(q)$ is a monic polynomial of degree $l(y)-l(x)$, and $S_{x,y}(0)=\varepsilon_x\varepsilon_y$.

Proof. The first assertion is essentially just a restatement of our definitions. In fact, it follows from them that $\wp\Re' = \overline{\wp}$ in $I(W)$ and hence that $\Re' = \wp^{-1}\overline{\wp}$, which, by (3) and (4), implies (10). The second statement follows from the first one and the facts that $\wp$, $\wp^{-1} \in I_{\frac{1}{2}}(W)$ and $Q_{x,y}(0)=P_{x,y}(0)=1$ for $(x,y)\in \mathrm{Int}(W)$.

Note the similarity of (10) with the formula for the R-polynomials

$$R_{x,y}(q) = \sum_{x\le a\le y} \varepsilon_x\varepsilon_a P_{x,a}(q) q^{l(y)-l(a)} Q_{a,y}\left(\frac{1}{q}\right) \tag{11}$$

for all $(x,y)\in \mathrm{Int}(W)$. Because of (10) and (11), many other formulas for the R-polynomials have analogues for the polynomials $S_{x,y}(q)$. We give below two as an example (cf. Corollaries 5.3 and 7.7 in [3]).

Corollary 4.8. *Let $x,y\in W$, $x\le y$. Then*

$$[q](S_{x,y}) = \varepsilon_x\varepsilon_y([q](Q_{x,y}) - |[x,y]_1^*|)$$

and

$$[q^{l(y)-l(x)-1}](S_{x,y}) = [q](P_{x,y}) - |[x,y]_1|.$$

It is of course possible to obtain from (10) similar formulas for all the coefficients of $S_{x,y}(q)$, but we see no reason to do this explicitly here.

Proposition 4.9. *Let (W,S) be a finite Coxeter system, and $x,y\in W$, $x\le y$. Then*

$$q^{l(y)-l(x)} S_{x,y}\left(\frac{1}{q}\right) = \varepsilon_x\varepsilon_y S_{w_0y,w_0x}(q),$$

where w_0 denotes the longest element of W.

Proof. It is well known (see, e.g., [6, Proposition 7.13]) that if W is finite then $Q_{x,y}(q) = P_{w_0y,w_0x}(q)$ for all $(x,y)\in \mathrm{Int}(W)$. Hence we conclude from (10) that

$$\begin{aligned} q^{l(y)-l(x)} S_{x,y}\left(\frac{1}{q}\right) &= \sum_{x\le a\le y} \varepsilon_x\varepsilon_a q^{l(a)-l(x)} P_{w_0a,w_0x}\left(\frac{1}{q}\right) P_{a,y}(q) \\ &= \sum_{w_0y\le b\le w_0x} \varepsilon_x\varepsilon_{w_0b} q^{l(w_0x)-l(b)} P_{b,w_0x}\left(\frac{1}{q}\right) Q_{w_0y,b}(q) \\ &= \varepsilon_x\varepsilon_y S_{w_0y,w_0x}(q), \end{aligned}$$

as desired.

Proposition 4.9 also holds for the R-polynomials (see, e.g., [6, Propositions 7.6 and 7.8]). After seeing all these similarities it is natural to suspect that the polynomials $S_{x,y}(q)$ might just be the R-polynomials in disguise. This, however, is not true even for finite Weyl groups. For example, if $W = S_4$ then one can compute that

$$S_{1234,3412}(q) = q^4 - 2q^3 + 4q^2 - 4q + 1,$$

and this is not an R-polynomial by Proposition 2.6.

§5. Applications to Kazhdan-Lusztig polynomials

In this section we apply the results obtained in the two previous ones to Kazhdan-Lusztig polynomials. In particular, we characterize the intervals $[u, v]$ in W such that the Kazhdan-Lusztig polynomials of its subintervals coincide with ζ, ζ', or $\wp^{-1}$ up to a given order, and we obtain refinements of two results that originally appeared in [3] and [7].

We begin by comparing a deformation of the characteristic kernel of W^* with the kernel $(\overline{\mathfrak{R}})^*$, where $\mathfrak{R}$ is defined by (2). For brevity, throughout this section, we write χ instead of χ_{W^*}. Note first that, since W^* is locally Eulerian, we obtain from (1) that

$$\chi(y, x) = \varepsilon_x \varepsilon_y \sum_{i=0}^{l(y)-l(x)} |[x, y]_i| (-q)^i \tag{12}$$

for all $(y, x) \in \mathrm{Int}(W^*)$.

Theorem 5.1. *Let $u, v \in W$, $u < v$, $k \in \mathbf{N}$, and $f \in \mathbf{R}[q]$, deg$(f) \leq \frac{1}{2}(l(v) - l(u) - 1)$. Then the following are equivalent:*

i): *$D_k(P_{y,x}) = 1$ for all $[y, x] \subset [u, v]$, and $D_k(P_{u,v}) = D_k(1 + f)$;*

ii): *$D_k(R_{y,x}) = D_k(\chi(x, y))$ for all $[y, x] \subset [u, v]$, and*

$$D_k\left(f(q) - q^{l(v)-l(u)} f\left(\frac{1}{q}\right)\right) = D_k(\chi(v, u) - R_{u,v}).$$

Proof. Let $\chi_{v,u}(f)$ be the W^*-kernel of $\zeta + f_{v,u} \in I_{\frac{1}{2}}(W^*)$. Then we have from Theorem 4.2 that, if $[y, x] \subseteq [u, v]$

$$\chi_{v,u}(f)(x, y) = \chi(x, y) + \zeta^{-1}(x, v)\left(q^{l(v)-l(u)} f\left(\frac{1}{q}\right) - f(q)\right)$$

if $u = y$, while $\chi_{v,u}(f)(x, y) = \chi(x, y)$ otherwise.

On the other hand, we know that $(\Re)^*$ is a W^*-kernel and $\wp^*$ is its KLS function. Furthermore, it follows from our definitions that

$$[q^j](\overline{\Re}^*(x,y)) = [q^j](\overline{\Re}(y,x)) = [q^j]\left(q^{l(x)-l(y)}\varepsilon_x\varepsilon_y R_{y,x}\left(\frac{1}{q}\right)\right) = [q^j](R_{y,x}(q)),$$

by Proposition 2.6, and

$$[q^j](\wp^*(x,y)) = [q^j](\wp(y,x)) = [q^j](P_{y,x})$$

for all $(x,y) \in \mathrm{Int}(W^*)$, so the result follows from Theorem 3.1.

If $P_{y,x}(q) = 1$ for all $[y,x] \subset [u,v]$ then much more precise information can be obtained, as the next result shows. Note that if one uses Theorem 2.5 as a recursion for computing Kazhdan-Lusztig polynomials these are the first "non-trivial" (i.e., $\neq 1$) Kazhdan-Lusztig polynomials that one generates.

Proposition 5.2. *Let $u, v \in W$, $u < v$, and $d \stackrel{\text{def}}{=} l(v) - l(u)$. Suppose that $P_{y,x} = 1$ for all $[y,x] \subset [u,v]$. Then:*

i): $P_{u,v} = 1 + D_{\frac{d-1}{2}}(\chi(v,u) - R_{u,v})$;

ii): $R_{u,v} = \chi(v,u) + 1 - P_{u,v}(q) - q^d + q^d P_{u,v}\left(\frac{1}{q}\right)$;

iii): $(1 + \varepsilon_u\varepsilon_v)R_{u,v}(q) = \chi(v,u)(q) + q^d\chi(v,u)\left(\frac{1}{q}\right)$;

iv): $P_{u,v}(q) = 1 + \frac{1}{2}\sum_{i=0}^{\lfloor\frac{d-1}{2}\rfloor}(-q)^i(|[u,v]_i| - |[u,v]_{d-i}|)$ *if d is even;*

v): *$[u,v]$ is rank-symmetric if d is odd.*

Proof. Taking $k = l(v) - l(u)$, and $f(q) \stackrel{\text{def}}{=} P_{u,v}(q) - 1$ in Theorem 5.1 yields ii), from which i) follows immediately. From ii) we conclude that

$$\begin{aligned} q^d\left(-1 + \left(\frac{1}{q}\right)^d - \chi(v,u)\left(\frac{1}{q}\right)\right) &= q^d\left(\left(\frac{1}{q}\right)^d P_{u,v}(q) - P_{u,v}\left(\frac{1}{q}\right) - R_{u,v}\left(\frac{1}{q}\right)\right) \\ &= P_{u,v}(q) - q^d P_{u,v}\left(\frac{1}{q}\right) - \varepsilon_u\varepsilon_v R_{u,v}(q) \\ &= 1 - q^d + \chi(v,u)(q) - (1 + \varepsilon_u\varepsilon_v)R_{u,v}(q)\,, \end{aligned}$$

where we have used ii) again and Proposition 2.6 and iii) follows.

Now if $\varepsilon_u\varepsilon_v = 1$ we conclude from iii) and i) that

$$P_{u,v}(q) = 1 + \frac{1}{2}D_{\frac{d-1}{2}}\left(\chi(v,u)(q) - q^d\chi(v,u)\left(\frac{1}{q}\right)\right),$$

and iv) follows from (12). If $\varepsilon_u \varepsilon_v = -1$ it follows from iii) that

$$\chi(v,u)(q) = -q^d \chi(v,u)\left(\frac{1}{q}\right),$$

and v) follows from (12).

It should be noted that the preceding result is yet another piece of evidence in favor of the "feeling" mentioned in [3, p. 384], that the Kazhdan-Lusztig polynomials somehow "measure" the difference between the R-polynomials and the rank generating functions.

We note the following interesting reformulation of part v) of Proposition 5.2.

Corollary 5.3. *Let $u, v \in W$, $u < v$, be such that $[u, v]$ is not rank-symmetric and has odd rank. Then there exists $[x, y] \subset [u, v]$ such that $P_{x,y}(q) \neq 1$.*

The following result characterizes the intervals of W for which the zeta function is a "good approximation" of the Kazhdan-Lusztig polynomials.

Proposition 5.4. *Let $u, v \in W$, $u < v$, and $k \in \mathbf{N}$. Then the following are equivalent:*

i): $D_k(P_{x,y}) = 1$ *for all* $[x, y] \subseteq [u, v]$*;*

ii): $\varepsilon_x \varepsilon_y [q^j](R_{x,y}) = (-1)^j |[x, y]_j|$ *for all* $x, y \in [u, v]$ *and* $j \in [k]$.

Proof. This follows immediately from (12) and Theorem 5.1.

Note that, when $k = l(v) - l(u)$, Proposition 5.4 reduces to Proposition 5.6 of [3].

Most of the results that we have derived so far in this section have analogues that are obtained by considering the W^*-kernel $\overline{\chi}$ instead of χ. We state here one of them as an example. It is a "dual" of Proposition 5.4, and characterizes the intervals of W having the property that the Kazhdan-Lusztig polynomials of its subintervals coincide, up to a given order, with the function ζ' studied in section 4. Note that we write ζ' for $(\zeta_{W^*})'$.

Proposition 5.5. *Let $u, v \in W$, $u < v$, and $k \in \mathbf{N}$. Then the following are equivalent:*

i): $D_k(P_{x,y}) = \varepsilon_x \varepsilon_y D_k(\zeta'(y, x))$ *for all* $x, y \in [u, v]$*;*

ii): $\varepsilon_x \varepsilon_y [q^j](R_{x,y}) = (-1)^j |[x, y]_j^*|$ *for all* $x, y \in [u, v]$ *and* $j \in [k]$.

The last two propositions have the following curious consequence.

Corollary 5.6. *Let $u, v \in W$, $u < v$. Then the following are equivalent:*

i): $P_{x,y} = 1$ *for all* $u \leq x \leq y \leq v$;

ii): $P_{x,y} = \varepsilon_x \varepsilon_y \zeta'(y, x)$ *for all* $u \leq x \leq y \leq v$.

Proof. This follows immediately from Propositions 5.4, 5.5 and 2.6.

As another application of Theorem 3.1 we obtain the following result which characterizes the intervals $[u, v]$ of W for which the Kazhdan-Lusztig and inverse Kazhdan-Lusztig polynomials coincide on the subintervals of $[u, v]$, up to a given order.

Proposition 5.7. *Let $u, v \in W$, and $k \in \mathbf{N}$. Then the following are equivalent:*

i): $D_k(P_{x,y}) = D_k(Q_{x,y})$ *for all* $x, y \in [u, v]$;

ii): $D_k(R_{x,y}) = D_k(S_{x,y})$ *for all* $x, y \in [u, v]$.

Proof. We know that $\Re$ and $\Re'$ are W-kernels and that $\wp^{-1}$ and $\wp$ are their respective KLS-functions. This easily implies that $(x, y) \mapsto R_{x,y}(q)$ is a W-kernel and that $(x, y) \mapsto Q_{x,y}(q)$ is its KLS-function, so the result follows from Theorem 3.1.

We conclude this section with an application of Theorem 3.2, which gives a refinement of Lemma 2.6 (ii) of [7] (the case $k = l(v) - l(u)$).

Corollary 5.8. *Let $k \in \mathbf{N}$, and $u, v \in W$, $u \leq v$. Then the following are equivalent:*

i): $D_k(P_{x,v}) = 1$ *for all* $x \in [u, v]$;

ii): $D_k\left(\sum_{a \in [x,v]} R_{x,a}\right) = D_k(q^{l(v)-l(x)})$ *for all* $x \in [u, v]$.

Proof. This follows immediately from Theorem 3.2 by taking $P = W^*$, $K = \overline{\Re}^*$, $\gamma = \wp^*$, and $f = \zeta_{W^*}$.

A dual result can be obtained by taking $P = W$, $K = \Re'$, $\gamma = \wp$, and $f = \zeta$, we leave its statement to the interested reader.

§6. Conjectures and open problems

In this section we discuss some conjectures and open problems arising from the present work.

The first one is naturally suggested by Corollary 5.6.

Conjecture 6.1. *Let $u, v \in W$, $u < v$, and $k \in \mathbf{N}$. Then the following are equivalent:*

i): $D_k(P_{x,y}) = 1$ *for all* $u \leq x \leq y \leq v$;

ii): $D_k(P_{x,y}) = \varepsilon_x \varepsilon_y D_k(\zeta'(y,x))$ *for all* $u \le x \le y \le v$.

By Propositions 5.4 and 5.5 Conjecture 6.1 is equivalent to the following one.

Conjecture 6.2. *Let* $u, v \in W$, $u < v$, *and* $k \in \mathbf{N}$. *Then the following are equivalent:*

i): $\varepsilon_x \varepsilon_y [q^j](R_{x,y}) = (-1)^j |[x,y]_j|$ *for all* $x, y \in [u,v]$ *and* $j \in [k]$;
ii): $\varepsilon_x \varepsilon_y [q^j](R_{x,y}) = (-1)^j |[x,y]_j^*|$ *for all* $x, y \in [u,v]$ *and* $j \in [k]$.

A consequence of Conjecture 6.2 is the following one.

Conjecture 6.3. *Let* $u, v \in W$, $u < v$, *and* $k \in \mathbf{N}$. *Suppose that* $D_k(P_{x,y}) = 1$ *for all* $(x,y) \in Int([u,v])$. *Then* $|[x,y]_j| = |[x,y]_j^*|$ *for all* $x, y \in [u,v]$ *and* $j \in [k]$.

Note that this conjecture holds for $k = 1$ by Proposition 5.4 and Corollary 5.3 of [3]. We now show that it also holds for $k = 2$.

Proposition 6.4. *Let* $u, v \in W$, $u < v$, *be such that* $[q](P_{x,y}) = [q^2](P_{x,y}) = 0$ *for all* $x, y \in [u,v]$. *Then* $|[x,y]_i| = |[x,y]_i^*|$ *for all* $x, y \in [u,v]$, $i = 1, 2$.

Proof. We already know that $|[x,y]_1| = |[x,y]_1^*|$ for all $x, y \in [u,v]$. Also, we know from [3, Corollary 5.4] that

$$\begin{aligned} [q^2](P_{x,y}) &= \varepsilon_x \varepsilon_y [q^2](R_{x,y}) + \sum_{a \in [x,y]_1^*} \varepsilon_x \varepsilon_a [q](R_{x,a}) + |[x,y]_2^*| \\ &\quad + \alpha([x,y]^*; \{1,3\}) - \sum_{a \in [x,y]_3^*} [q](R_{a,y}). \end{aligned} \tag{13}$$

On the other hand, from Proposition 5.4 and our hypotheses we deduce that

$$\varepsilon_x \varepsilon_y [q](R_{x,y}) = -|[x,y]_1| = -|[x,y]_1^*|$$

for all $x, y \in [u,v]$. Hence from (13) and our hypotheses we conclude that

$$\begin{aligned} 0 &= \varepsilon_x \varepsilon_y [q^2](R_{x,y}) - \sum_{a \in [x,y]_1^*} |[x,a]_1^*| + |[x,y]_2^*| \\ &\quad + \alpha([x,y]^*; \{1,3\}) - \sum_{a \in [x,y]_3^*} |[a,y]_1^*| \\ &= \varepsilon_x \varepsilon_y [q^2](R_{x,y}) - \alpha([x,y]^*; \{1,2\}) + |[x,y]_2^*| \\ &\quad + \alpha([x,y]^*; \{1,3\}) - \alpha([x,y]^*; \{1,3\}) \\ &= \varepsilon_x \varepsilon_y [q^2](R_{x,y}) - |[x,y]_2^*|, \end{aligned} \tag{14}$$

for all $x, y \in [u, v]$. On the other hand, from Proposition 5.4 we have that

$$\varepsilon_x \varepsilon_y [q^2](R_{x,y}) = |[x, y]_2| \tag{15}$$

for all $x, y \in [u, v]$, and the result follows from (14) and (15).

It is a well known conjecture (see, [6, p. 159]) that the coefficients of Kazhdan-Lusztig polynomials are always nonnegative. Using Theorem 5.1 we can derive from it the following (much weaker) conjecture, that should be more tractable.

Conjecture 6.5. *Let $k \in \mathbf{P}$ and $u, v \in W$, $u < v$, be such that $D_k(P_{x,y}) = 1$ for all $[x, y] \subset [u, v]$. Then*

$$\varepsilon_u \varepsilon_v (-1)^i |[u, v]_i| \geq [q^i](R_{u,v})$$

for $i = 0, \ldots, min(k, \lfloor \frac{d}{2} \rfloor)$, and

$$(-1)^i |[u, v]_i^*| \leq \varepsilon_u \varepsilon_v [q^i](R_{u,v}),$$

for $i = d - k, \ldots, \lfloor \frac{d}{2} \rfloor$.

There is another related conjecture which we wish to mention. It was observed in [3, p. 384] (see also [4], Problem 5.1), that the polynomial $\varepsilon_u \varepsilon_v (\chi(v, u)(-q) - R_{u,v}(-q))$ seems to have always nonnegative coefficients. If this is true, and the nonnegativity conjecture holds, then part ii) of Proposition 5.2, shows that the following must also hold.

Conjecture 6.6. *Let $u, v \in W$, $u < v$, be such that $P_{x,y} = 1$ for all $[x, y] \subset [u, v]$. Then:*

i): $P_{u,v}(q) = 1$ *if* $\varepsilon_u \varepsilon_v = 1$;
ii): $[q^{2i}](P_{u,v}) = 0$ *if* $\varepsilon_u \varepsilon_v = -1$, *and* $i \geq 1$.

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