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Crystal Bases for Quantum Superalgebras

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§1. Introduction

Associated with each integrable module M for the quantized enveloping algebra $U_q(\mathfrak{g})$ of a symmetrizable Kac-Moody Lie algebra $\mathfrak{g}$, there is a remarkable basis at $q = 0$, the *crystal base*, which was introduced by Kashiwara [Ka1]. If $\mathbf{A}$ denotes the local ring of all rational functions $f/g \in \mathbf{Q}(q)$ with $g(0) \neq 0$, then M contains an $\mathbf{A}$-lattice L, called the *crystal lattice.* The crystal base is a certain basis B for the $\mathbf{Q}$-vector space L/qL, which possesses many noteworthy features. It is well-behaved with respect to tensor products; it is preserved under the action of the modified root vector operators $\tilde{e}_i$ and $\tilde{f}_i$ (what are often called *Kashiwara operators*); and it has important connections with combinatorial bases of tableaux (see [MM], [KN], [KM], and [L]). Crystal bases play a prominent role in two-dimensional solvable lattice models, where the parameter q corresponds to the temperature in the lattice model. Since $q = 0$ corresponds to absolute zero temperature, one expects special behavior at this particular value, and the crystal base reflects this exceptional behavior.

In this work we describe a crystal base theory for quantum superalgebras. Basic definitions and general results on crystal bases for Kac-Moody superalgebras are presented in Sections 2, 3, and 4. Section 5 describes crystal bases for the orthosymplectic Lie superalgebra $\mathfrak{osp}(1, 2n)$, and Section 6, for affine Kac-Moody superalgebras. Sections 7, 8, and

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9 discuss crystal bases for the general linear Lie superalgebra $\mathfrak{gl}(m,n)$. More details on crystal bases for $\mathfrak{gl}(m,n)$ can be found in [BKK] where the theory is developed, in [MZ] and [C] for $\mathfrak{osp}(1,2n)$, and in [Je] for Kac-Moody Lie superalgebras.

§2. Quantized Universal Enveloping Algebras for Kac-Moody Superalgebras

We first recall the definition of the *quantized universal enveloping algebra* for a Kac-Moody superalgebra (cf. [BKM]). Let I be a finite index set, and assume $A = (a_{i,j})_{i,j\in I}$ is a generalized Cartan matrix. Thus A satisfies: (i) $a_{i,i} = 2$ for all $i \in I$, (ii) $a_{i,j} \in \mathbb{Z}_{\leq 0}$ for $i \neq j$, (iii) $a_{i,j} = 0$ if and only if $a_{j,i} = 0$ $(i, j \in I)$. In this paper, we assume that A is symmetrizable; i.e., there exists an invertible diagonal matrix $D = \mathrm{diag}(\ell_i |\ i \in I)$ with $\ell_i \in \mathbb{Z}_{>0}$ such that DA is symmetric.

Let I^{odd} be a subset of I and set $I^{\mathrm{even}} = I \setminus I^{\mathrm{odd}}$. The elements of I^{odd} (resp. I^{even}) are called the *odd* (resp. *even*) indices. The *parity function* p is defined by $p(i) = 1$ if $i \in I^{\mathrm{odd}}$ and $p(i) = 0$ if $i \in I^{\mathrm{even}}$. We say that the generalized Cartan matrix A is *colored by* I^{odd} if $a_{i,j} \in 2\mathbb{Z}$ for all $i \in I^{\mathrm{odd}}$, $j \in I$.

Let $\mathfrak{h}$ be a vector space of dimension $2|I| - \mathrm{rank}A$. Let $\Pi = \{\alpha_i |\ i \in I\} \subset \mathfrak{h}^*$ and $\Pi^\vee = \{h_i |\ i \in I\} \subset \mathfrak{h}$ be linearly independent sets such that $\alpha_j(h_i) = a_{i,j}$ for all $i, j \in I$. Then the triple $(\mathfrak{h}, \Pi, \Pi^\vee)$ forms a realization of A in the sense of [K4, Chap. 1].

Definition 2.1. *Assume $A = (a_{i,j})_{i,j\in I}$ is a generalized Cartan matrix colored by $I^{\mathrm{odd}} \subset I$, and let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realization of A. The* **Kac-Moody superalgebra** $\mathfrak{g} = \mathfrak{g}(A, I^{\mathrm{odd}})$ *is the Lie superalgebra generated by e_i, f_i $(i \in I)$ and $\mathfrak{h}$ with defining relations*

$$\begin{aligned}
&[h, h'] = 0 \quad \textit{for all} \quad h, h' \in \mathfrak{h},\\
&[h, e_i] = \alpha_i(h)e_i, \quad [h, f_i] = -\alpha_i(h) f_i,\\
&[e_i, f_j] = \delta_{i,j} h_i,\\
&(ad\, e_i)^{1-a_{i,j}}(e_j) = (ad\, f_i)^{1-a_{i,j}}(f_j) = 0 \quad (i \neq j).
\end{aligned} \tag{2.2}$$

The free abelian group $Q = \bigoplus_{i\in I} \mathbb{Z}\alpha_i$ is the *root lattice* associated with the data $(A, \Pi, \Pi^\vee)$. For an element $\alpha = \sum_i k_i\alpha_i \in Q$, the parity of α is defined to be $p(\alpha) = \sum_i k_i p(i) \in \mathbb{Z}_2$. We say that α is *even* (resp. odd) if $p(\alpha) = 0$ (resp. $p(\alpha) = 1$). Let $Q_+ = \bigoplus_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i$ and $Q_- = -Q_+$. There is a partial ordering $\geq$ on $\mathfrak{h}^*$ defined by $\lambda \geq \mu$ if and only if $\lambda - \mu \in Q_+$ ($\lambda, \mu \in \mathfrak{h}^*$). Since A is symmetrizable there is a nondegenerate symmetric bilinear form $(\ |\ )$ on $\mathfrak{h}^*$ satisfying $(\alpha_i|\alpha_j) = \ell_i a_{i,j}$ for $i, j \in I$. For each $i \in I$, let $r_i \in GL(\mathfrak{h}^*)$ be the simple reflection defined by $r_i(\lambda) = \lambda - \lambda(h_i)\alpha_i$ ($\lambda \in \mathfrak{h}^*$). The subgroup of $GL(\mathfrak{h}^*)$ generated by r_i's is the *Weyl group* of the data $(A, \Pi, \Pi^\vee)$.

Let q be an indeterminate and set $q_i = q^{\ell_i}$ ($i \in I$). For nonnegative integers n and N, we define the q-binomial coefficients as follows:

$$\begin{aligned} &[n]_i = \frac{(-1)^{np(i)}q_i^n - q_i^{-n}}{(-1)^{p(i)}q_i - q_i^{-1}}, \\ (2.3) \qquad &[n]_i! = [n]_i[n-1]_i \cdots [2]_i[1]_i, \qquad [0]_i! = 1, \\ &\begin{bmatrix} N \\ n \end{bmatrix}_i = \frac{[N]_i!}{[n]_i![N-n]_i!}. \end{aligned}$$

Let $P^\vee$ be a $\mathbb{Z}$-lattice of $\mathfrak{h}$ containing all h_i's and satisfying $\alpha_i(h) \in \mathbb{Z}$ for all $i \in I$ and $h \in P^\vee$. The lattice $P^\vee$ is referred to as the *dual weight lattice*, and $P = \{\lambda \in \mathfrak{h}^* |\ \lambda(P^\vee) \subset \mathbb{Z}\}$ is the *weight lattice.*

Definition 2.4. *Let $A = (a_{i,j})_{i,j\in I}$ be a generalized Cartan matrix colored by $I^{\text{odd}} \subset I$, and let $\mathfrak{g} = \mathfrak{g}(A, I^{\text{odd}})$ be the corresponding Kac-Moody superalgebra. The* **quantized universal enveloping algebra** *$U_q(\mathfrak{g})$ is the associative algebra over $\mathbf{Q}(q)$ with 1 generated by the elements q^h ($h \in P^\vee$), e_i, f_i ($i \in I$) with defining relations*

$$\begin{aligned} &q^0 = 1, \quad q^h q^{h'} = q^{h+h'} \text{ for all } h, h' \in P^\vee, \\ &q^h e_i q^{-h} = q^{\alpha_i(h)} e_i, \quad q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i, \\ (2.5) \qquad &e_i f_j - (-1)^{p(i)p(j)} f_j e_i = \delta_{i,j} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \\ &(ad_q e_i)^{1-a_{i,j}}(e_j) = (ad_q f_i)^{1-a_{i,j}}(f_j) = 0 \quad (i \neq j), \end{aligned}$$

where $K_i = q^{\ell_i h_i}$ for $i \in I$ and $ad_q\, x(y) = xy - q^{(\alpha|\beta)}(-1)^{p(\alpha)p(\beta)} yx$ for $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_\beta$.

Proposition 2.6. ([BKM]) *The algebra $U_q(\mathfrak{g})$ has a Hopf superalgebra structure with comultiplication Δ, counit ε, and antipode S*

defined by

$$\begin{aligned} &\Delta(q^h) = q^h \otimes q^h \quad \text{for} \;\; h \in P^\vee, \\ (2.7) \qquad &\Delta(e_i) = e_i \otimes K_i^{-1} + 1 \otimes e_i, \\ &\Delta(f_i) = f_i \otimes 1 + K_i \otimes f_i \quad \text{for} \;\; i \in I, \end{aligned}$$

$$\begin{aligned} (2.8) \qquad &\varepsilon(q^h) = 1 \quad \text{for } h \in P^\vee, \\ &\varepsilon(e_i) = \varepsilon(f_i) = 0 \quad \text{for } i \in I, \end{aligned}$$

$$\begin{aligned} (2.9) \qquad &S(q^h) = q^{-h} \quad \text{for } h \in P^\vee, \\ &S(e_i) = -e_i K_i, \quad S(f_i) = -K_i^{-1} f_i \quad \text{for } i \in I. \end{aligned}$$

The $\mathbf{Q}(q)$-subalgebra U_q^0 of $U_q(\mathfrak{g})$ generated by the elements q^h ($h \in P^\vee$) is isomorphic to the group algebra $\mathbf{Q}(q)[P^\vee]$. Let U_q^+ (resp. U_q^-) be the $\mathbf{Q}(q)$-subalgebra of $U_q(\mathfrak{g})$ with 1 generated by the elements e_i (resp. f_i) for $i \in I$. Then we have the following *triangular decomposition* which can be regarded as a quantum analogue of Poincaré-Birkhoff-Witt Theorem.

Proposition 2.10. ([BKM]) *There is a* $\mathbf{Q}(q)$*-linear isomorphism*

$$(2.11) \qquad U_q(\mathfrak{g}) \cong U_q^- \otimes U_q^0 \otimes U_q^+.$$

Example 2.12. The simplest example of a quantized universal enveloping algebra associated with a Kac-Moody superalgebra is the *orthosymplectic quantum superalgebra* $U_q(\mathfrak{osp}(1,2))$, which is generated by the elements e, f, and $K^{\pm 1}$ and has defining relations

$$(2.13) \qquad KeK^{-1} = q^2 e, \quad KfK^{-1} = q^{-2} f, \quad ef + fe = \frac{K - K^{-1}}{q - q^{-1}}.$$

§3. Integrable Representations

In this section, we introduce the notion of integrable $U_q(\mathfrak{g})$-modules in category $\mathcal{O}$. Let $\mathfrak{g} = \mathfrak{g}(A, I^{\mathrm{odd}})$ be a Kac-Moody superalgebra associated with a generalized Cartan matrix $A = (a_{i,j})_{i,j \in I}$ colored by the set of odd indices I^{odd}. Let $U_q(\mathfrak{g})$ be the corresponding quantized universal enveloping algebra.

A $U_q(\mathfrak{g})$-module M is a *weight module* if it admits a *weight space decomposition*

$$M = \bigoplus_{\lambda \in P} M_\lambda, \quad \text{where } M_\lambda = \{v \in M \mid q^h v = q^{\lambda(h)} v \text{ for all } h \in P^\vee\}.$$

The category $\mathcal{O}$ consists of the weight modules with finite dimensional weight spaces such that there exist $\mu_1, \cdots, \mu_s$ in P satisfying

$$\mathrm{wt}(M) \subset D(\mu_1) \cup \cdots \cup D(\mu_s),$$

where $\mathrm{wt}(M) = \{\lambda \in P \mid M_\lambda \neq 0\}$ and $D(\mu) = \{\tau \in P \mid \tau \leq \mu\}$. The morphisms are $U_q(\mathfrak{g})$-module homomorphisms.

Among the $U_q(\mathfrak{g})$-modules in category $\mathcal{O}$, the most interesting ones are highest weight modules. A $U_q(\mathfrak{g})$-module M is a *highest weight module with highest weight* $\lambda \in P$ if there exists a nonzero vector $v_\lambda \in M$ such that (i) $q^h v_\lambda = q^{\lambda(h)} v_\lambda$ for all $h \in P^\vee$, (ii) $e_i v_\lambda = 0$ for all $i \in I$, (iii) $M = U_q(\mathfrak{g}) v_\lambda$. The vector v_λ, which is called a *highest weight vector* of M, is unique up to a constant multiple.

Let $\lambda \in P$ and let $J(\lambda)$ be the left ideal of $U_q(\mathfrak{g})$ generated by e_i ($i \in I$) and $q^h - q^{\lambda(h)} 1$ ($h \in P^\vee$). Then the quotient $M(\lambda) = U_q(\mathfrak{g})/J(\lambda)$ is given a $U_q(\mathfrak{g})$-module structure by left multiplication. It is easy to see that $M(\lambda)$ is a highest weight module with highest weight λ and highest weight vector $v_\lambda = 1 + J(\lambda)$. The $U_q(\mathfrak{g})$-module $M(\lambda)$ is the *Verma module* with highest weight λ. As a U_q^--module, $M(\lambda)$ is free of rank 1 generated by the highest weight vector $v_\lambda = 1 + J(\lambda)$, and every highest weight $U_q(\mathfrak{g})$-module with highest weight λ is a homomorphic image of $M(\lambda)$. The module $M(\lambda)$ has a unique maximal submodule $N(\lambda)$, and its unique irreducible quotient $V(\lambda) = M(\lambda)/N(\lambda)$ is again a highest weight $U_q(\mathfrak{g})$-module with highest weight λ.

A $U_q(\mathfrak{g})$-module M is said to be *integrable* if all e_i and f_i ($i \in I$) are locally nilpotent on M. We denote by $\mathcal{O}_{int}$ the subcategory of $\mathcal{O}$ consisting of integrable $U_q(\mathfrak{g})$-modules in category $\mathcal{O}$. For each $i \in I$, let $U_q(\mathfrak{g})_i$ be the subalgebra of $U_q(\mathfrak{g})$ generated by e_i, f_i, $K_i^{\pm 1}$. Then it is easy to verify that

$$U_q(\mathfrak{g})_i \cong \begin{cases} U_q(\mathfrak{sl}_2) & \text{if } i \in I^{\mathrm{even}}, \\ U_q(\mathfrak{osp}(1,2)) & \text{if } i \in I^{\mathrm{odd}}. \end{cases} \tag{3.1}$$

A $U_q(\mathfrak{g})$-module M is in category $\mathcal{O}_{int}$ if and only if it has a weight space decomposition with finite dimensional weight spaces, and for each $i \in I$, M is locally $U_q(\mathfrak{g})_i$-finite, i.e., for each $v \in M$, $\dim U_q(\mathfrak{g})_i v < \infty$. In particular, a $U_q(\mathfrak{g})$-module in category $\mathcal{O}_{int}$ is a direct sum of finite dimensional irreducible $U_q(\mathfrak{g})_i$-modules for all $i \in I$.

The category $\mathcal{O}_{int}$ is semisimple, and its irreducible objects can be characterized as follows.

Proposition 3.2. ([Je]) *The irreducible highest weight $U_q(\mathfrak{g})$-module $V(\lambda)$ is integrable if and only if $\lambda \in P^+$, where*

$$P^+ = \{\lambda \in P \mid \lambda(h_i) \geq 0 \ \ \text{for all } i \in I, \ \ \lambda(h_i) \in 2\mathbf{Z} \ \ \text{for all } i \in I^{\text{odd}}\}.$$

The irreducible integrable highest weight modules over $U_q(\mathfrak{g})$ are quantum deformations of those over the Kac-Moody superalgebra $\mathfrak{g}$.

Proposition 3.3. ([BKM]) *If $\lambda \in P^+$, then the irreducible highest weight module $V(\lambda)$ over the quantized universal enveloping algebra $U_q(\mathfrak{g})$ is a quantum deformation of the irreducible highest weight module $\overline{V}(\lambda)$ over the Kac-Moody superalgebra $\mathfrak{g}$. The character of $V(\lambda)$ is given by the Weyl-Kac character formula:*

$$(3.4) \qquad ch V(\lambda) = \frac{\sum_{w \in W} (-1)^{l(w)} e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha \in Q_+} (1 - (-1)^{p(\alpha)} e^{-\alpha})^{1-2p(\alpha) \dim \mathfrak{g}_\alpha}},$$

where $\rho \in P$ is a linear functional satisfying $\rho(h_i) = 1$ for all $i \in I$, the parity function p is as in Section 2, and W is the Weyl group.

§4. Crystal Bases for Kac-Moody Superalgebras

Let $M = \bigoplus_{\lambda \in P} M_\lambda$ be a $U_q(\mathfrak{g})$-module in category $\mathcal{O}_{int}$. Fix $i \in I$ and for any $k \in \mathbf{Z}_{\geq 0}$ define

$$e_i^{(k)} = \frac{1}{[k]_i!} e_i^k, \quad f_i^{(k)} = \frac{1}{[k]_i!} f_i^k.$$

Then every element $u \in M_\lambda$ can be uniquely expressed as

$$(4.1) \quad u = \sum_{k=0}^{N} f_i^{(k)} u_k, \quad \text{where } N \in \mathbf{Z}_{\geq 0} \text{ and } u_k \in M_{\lambda+k\alpha_i} \cap \ker e_i.$$

We define the *Kashiwara operators* $\tilde{e}_i$ and $\tilde{f}_i$ by

$$\tilde{e}_i u = \sum_{k=1}^{N} f_i^{(k-1)} u_k, \quad \tilde{f}_i u = \sum_{k=0}^{N} f_i^{(k+1)} u_k. \tag{4.2}$$

Let $\mathbf{A} = \{f(q)/g(q) \in \mathbf{Q}(q) \mid f, g \in \mathbf{Q}[q],\ g(0) \neq 0\}$ be the localization of $\mathbf{Q}[q]$ at $q = 0$. Thus $\mathbf{A}$ consists of all rational functions that are regular at $q = 0$.

Definition 4.3. *Let M be a $U_q(\mathfrak{g})$-module in the category $\mathcal{O}_{\mathrm{int}}$. A free $\mathbf{A}$-submodule L of M is a* **crystal lattice** *if*

(i) *L generates M as a vector space over $\mathbf{Q}(q)$.*
(ii) *L has a weight decomposition $L = \bigoplus_{\lambda \in P} L_\lambda$, where $L_\lambda = L \cap M_\lambda$.*
(iii) *$\tilde{e}_i L \subset L$ and $\tilde{f}_i L \subset L$ for any $i \in I$.*

Definition 4.4. *Let M be a $U_q(\mathfrak{g})$-module in the category $\mathcal{O}_{\mathrm{int}}$. A* **crystal base** *of M is a pair (L, B) such that*

(i) *L is a crystal lattice of M,*
(ii) *B is a pseudo-basis of L/qL, that is, $B = B' \cup (-B')$ for some $\mathbf{Q}$-basis B' of L/qL,*
(iii) *B has a weight decomposition $B = \bigsqcup_{\lambda \in P} B_\lambda$, where $B_\lambda = B \cap (L_\lambda/qL_\lambda)$,*
(iv) *$\tilde{e}_i B \subset B \sqcup \{0\}$ and $\tilde{f}_i B \subset B \sqcup \{0\}$ for all $i \in I$,*
(v) *for any $b, b' \in B$ and $i \in I$, we have $b = \tilde{f}_i b'$ if and only if $b' = \tilde{e}_i b$.*

The set $B/\{\pm 1\}$ is given a colored oriented graph structure with the i-arrow defined by $b \xrightarrow{\mathrm{i}} b'$ if and only if $\tilde{f}_i b = b'$ for $b, b' \in B/\{\pm 1\}$. We call $B/\{\pm 1\}$ the *crystal graph* of M.

For $b \in B/\{\pm 1\}$ and $i \in I$, let

$$\begin{aligned} \varepsilon_i(b) &= \max\{n \in \mathbf{Z}_{\geq 0} \mid \tilde{e}_i^n b \neq 0\}, \\ \varphi_i(b) &= \max\{n \in \mathbf{Z}_{\geq 0} \mid \tilde{f}_i^n b \neq 0\}. \end{aligned} \tag{4.5}$$

Then it follows from the representation theory of $U_{q_i}(\mathfrak{sl}_2)$ and $U_{q_i}(\mathfrak{osp}(1,2))$ that

$$\langle h_i, \mathrm{wt}(b) \rangle = \varphi_i(b) - \varepsilon_i(b) \quad \text{for } i \in I,$$

where wt(b) is the weight of b.

As in the case of Kac-Moody algebras, the crystal graphs for Kac-Moody superalgebras exhibit nice behavior with respect to tensor products.

Theorem 4.6. *Let M_ν be a $U_q(\mathfrak{g})$-module in category $\mathcal{O}_{\text{int}}$, and let (L_ν, B_ν) be a crystal base of M_ν for $\nu = 1, 2$. Set $L = L_1 \otimes_{\mathbf{A}} L_2$ and $B = B_1 \otimes B_2 = B_1 \times B_2$. Then (L, B) is a crystal base of $M_1 \otimes M_2$, and the crystal graph structure of $B_1 \otimes B_2$ is given by*

$$(4.7)\qquad \begin{aligned} \tilde{e}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{e}_i(b_1) \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i(b_2) & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases} \\ \tilde{f}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{f}_i(b_1) \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2) \\ b_1 \otimes \tilde{f}_i(b_2) & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases} \end{aligned}$$

Hence the tensor product rule for crystal graphs of Kac-Moody superalgebras is the same as the one for Kac-Moody algebras (cf. [Ka2]).

For a dominant integral weight $\lambda \in P^+$, let $V(\lambda)$ be the irreducible highest weight $U_q(\mathfrak{g})$-module with highest weight λ, and let $L(\lambda)$ be the free $\mathbf{A}$-submodule of $V(\lambda)$ spanned by the vectors of the form $\tilde{f}_{i_1}\tilde{f}_{i_2}\cdots\tilde{f}_{i_r}v_\lambda$, where v_λ is the highest weight vector of $V(\lambda)$. Set

$$(4.8)\qquad B(\lambda) = \{\tilde{f}_{i_1}\tilde{f}_{i_2}\cdots\tilde{f}_{i_r}v_\lambda \in L(\lambda)/qL(\lambda) \mid i_k \in I\} \setminus \{0\}.$$

Then, just as in the case of Kac-Moody algebras, we have the existence theorem for the crystal base for Kac-Moody superalgebras.

Theorem 4.9. ([Je]) *The pair $(L(\lambda), B(\lambda))$ is a crystal base of $V(\lambda)$.*

Although the proof of Theorem 4.9 is rather long and complicated, it is still a straightforward generalization of Kashiwara's *grand loop argument* ([Ka2]) once we define the super-version of the q-analogue of bosons. See [Je] for more details. The uniqueness theorem for the crystal base for Kac-Moody superalgebras can be proved in the same manner as for Kac-Moody algebras.

Theorem 4.10. (cf. [Je], [Ka2]) *Let M be a $U_q(\mathfrak{g})$-module in category $\mathcal{O}_{\text{int}}$ and let $M \cong \bigoplus_{\lambda \in P^+} V(\lambda)$ be the decomposition of M into a direct sum of irreducible highest weight modules. Then for any crystal base (L, B) of M, there exists an isomorphism $\Phi : M \to \bigoplus_{\lambda \in P^+} V(\lambda)$ such that $\Phi(L) \cong \bigoplus_{\lambda \in P^+} L(\lambda)$ and $\overline{\Phi}(B) \cong \bigsqcup_{\lambda \in P^+} B(\lambda)$, where $\overline{\Phi} : L \to \bigoplus_{\lambda \in P^+} L(\lambda)/qL(\lambda)$ is the $\mathbf{Q}$-linear isomorphism induced by Φ. In particular, if $\lambda \in P^+$, any crystal base (L, B) of $V(\lambda)$ is isomorphic to $(L(\lambda), B(\lambda))$.*

§5. Crystal Graphs for the Quantum Superalgebra $U_q(\mathfrak{osp}(1, 2n))$

In this section, we present an explicit description of the crystal graph of a finite dimensional irreducible module for the quantum superalgebra $U_q(\mathfrak{osp}(1, 2n))$. Among the finite dimensional simple Lie superalgebras, the superalgebras $\mathfrak{osp}(1, 2n)$ are distinguished as the only ones that are Kac-Moody superalgebras as defined in Section 2. We apply the crystal base theory for Kac-Moody superalgebras described in Section 4 to the superalgebras $\mathfrak{osp}(1, 2n)$ and obtain a realization of the crystal graphs in terms of certain semistandard Young tableaux. The basic technique used here originated in [KN], where the crystal graphs for finite dimensional irreducible modules over classical Lie algebras were realized in terms of certain semistandard Young tableaux. Our presentation follows that in [C], which was based on the approach of [KN]. Musson and Zou [MZ] have also developed a crystal base theory for $\mathfrak{osp}(1, 2n)$. Their methods derive from those in [J] and [Ka1], and they work only for $\mathfrak{osp}(1, 2n)$. Our method is based on the general crystal base theory for arbitrary Kac-Moody superalgebras, which includes some affine cases. Moreover, it yields an explicit tableau description of crystal graphs.

The representation theory of the Lie superalgebra $\mathfrak{osp}(1, 2n)$ is known to closely resemble the representation theory of the Lie algebra $\mathfrak{so}(2n+1)$ (see for example, [RS]). As a result, the crystal graphs of finite dimensional irreducible representations of $U_q(\mathfrak{osp}(1, 2n))$ have virtually the same description as the crystal graphs for $U_q(\mathfrak{so}(2n+1))$-modules. There exist finite dimensional irreducible representations of $U_q(\mathfrak{osp}(1, 2n))$ that are not quantum deformations of $\mathfrak{osp}(1, 2n)$-modules (these can be found in [Z1]). Since in this paper we focus only on the $U_q(\mathfrak{osp}(1, 2n))$-modules that are quantum deformations of $\mathfrak{osp}(1, 2n)$-modules, we don't have to consider ones that correspond to the so-called *spinor representations* of $\mathfrak{so}(2n + 1)$.

The index set for $\mathfrak{osp}(1, 2n)$ is $I = \{1, 2, \cdots, n\}$, and there is just one odd index, $I^{odd} = \{n\}$. The associated Cartan matrix is a generalized

Cartan matrix of type $B(0,n)$:

$$(5.1) \qquad A = (a_{ij})_{i,j\in I} = \begin{pmatrix} 2 & -1 & \cdots & & & \\ -1 & 2 & \cdots & & & \\ & & \ddots & & & \\ & & \cdots & 2 & -1 & \\ & & \cdots & -1 & 2 & -1 \\ & & \cdots & & -2 & 2 \end{pmatrix}$$

which corresponds to the Dynkin diagram

Definition 5.2. *The* **quantum superalgebra** $U_q(\mathfrak{osp}(1,2n))$ *is the quantized universal enveloping algebra of the Kac-Moody superalgebra associated with the data* (A, I^{odd})*, where* $A = (a_{ij})_{i,j\in I}$ *is given in (5.1) and* $I^{odd} = \{n\}$.

Thus, $U_q(\mathfrak{osp}(1,2n))$ is the associative algebra over $\mathbf{Q}(q)$ generated by the elements e_i, f_i, $K_i^{\pm 1}$ $(i = 1, \cdots, n)$ with defining relations given by (2.5). Let $(\mathfrak{h}, \Pi, \Pi^\vee)$ denote a realization of the generalized Cartan matrix of type $B(0,n)$, and assume ϵ_i $(i = 1, 2, \cdots, n)$ is an orthonormal basis of $\mathfrak{h}^*$. Then the simple roots of the Lie superalgebra $\mathfrak{osp}(1,2n)$ are given by

$$\begin{aligned} \alpha_i &= \epsilon_i - \epsilon_{i+1} \quad (1 \le i \le n-1), \\ \alpha_n &= \epsilon_n. \end{aligned}$$

The basic representation of $U_q(\mathfrak{osp}(1,2n))$ is its *vector representation*, which is the $(2n+1)$-dimensional space

$$\mathbf{V} = \left(\bigoplus_{i=1}^{n} \mathbf{Q}(q) v_i\right) \oplus \mathbf{Q}(q) v_0 \oplus \left(\bigoplus_{i=1}^{n} \mathbf{Q}(q) v_{\bar{i}}\right)$$

over $\mathbf{Q}(q)$ with basis $\{v_i, v_{\bar{i}} \mid i = 1, 2, \cdots, n\} \cup \{v_0\}$ and with $U_q(\mathfrak{osp}(1,2n))$-

action defined by

$$(5.3)\qquad \begin{aligned} K_i v_j &= \begin{cases} q v_j & \text{if } j = i, \overline{i+1}, \\ q^{-1} v_j & \text{if } j = i+1, \bar{i}, \\ v_j & \text{otherwise}, \end{cases} \\ e_i v_j &= \begin{cases} v_{j-1} & \text{if } j = i+1, \bar{i}, \\ 0 & \text{otherwise}, \end{cases} \\ f_i v_j &= \begin{cases} v_{j+1} & \text{if } j = i, \overline{i+1}, \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

for $i = 1, 2, \cdots, n-1$, and

$$(5.4)\qquad \begin{aligned} K_n v_j &= \begin{cases} q^2 v_j & \text{if } j = n, \\ q^{-2} v_j & \text{if } j = \bar{n}, \\ v_j & \text{otherwise}, \end{cases} \\ e_n v_j &= \begin{cases} v_0 & \text{if } j = \bar{n}, \\ [2]_n v_n & \text{if } j = 0, \\ 0 & \text{otherwise}, \end{cases} \\ f_n v_j &= \begin{cases} v_0 & \text{if } j = n, \\ [2]_n v_{\bar{n}} & \text{if } j = 0, \\ 0 & \text{otherwise}. \end{cases} \end{aligned}$$

In these expressions it is understood that $\bar{i} \pm 1 = \overline{i \mp 1}$ for $i = 1, 2, \cdots, n-1$.

Let

$$(5.5)\qquad \begin{aligned} \mathbf{L} &= \left(\bigoplus_{i=1}^{n} \mathbf{A} v_i \right) \oplus \mathbf{A} v_0 \oplus \left(\bigoplus_{i=1}^{n} \mathbf{A} v_{\bar{i}} \right) \\ \text{and} \quad \mathbf{B} &= \{ \overline{v_j}, \overline{v_{\bar{j}}} \mid j = 1, \cdots, n \} \cup \{ \overline{v_0} \}. \end{aligned}$$

Then $(\mathbf{L}, \mathbf{B})$ is a crystal base of $\mathbf{V}$ with crystal graph given by

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{n-1} \boxed{n} \xrightarrow{n} \boxed{0} \xrightarrow{n} \boxed{\bar{n}} \xrightarrow{n-1} \cdots \xrightarrow{2} \boxed{\bar{2}} \xrightarrow{1} \boxed{\bar{1}}$$

Here we identify $\overline{v_j} = \boxed{j}$ for $j = 1, \cdots, n, 0, \bar{n}, \overline{n-1}, \cdots, \bar{1}$.

The *fundamental weights* of $\mathfrak{osp}(1,2n)$ are defined by $\omega_i(h_j) = \delta_{i,j}$ $(i,j = 1,2,\cdots,n)$. Alternately, $\omega_i = \epsilon_1 + \cdots + \epsilon_i$, $(1 \le i \le n-1)$, and $\omega_n = \dfrac{1}{2}(\epsilon_1 + \cdots + \epsilon_n)$. The finite dimensional irreducible $\mathfrak{osp}(1,2n)$-modules are parametrized by their highest weights λ, which have the form

$$\begin{aligned}\lambda &= a_1\omega_1 + \cdots + a_{n-1}\omega_{n-1} + 2a_n\omega_n \quad \text{with} \quad a_i \in \mathbf{Z}_{\ge 0}, \\ &= \lambda_1\epsilon_1 + \cdots + \lambda_{n-1}\epsilon_{n-1} + \lambda_n\epsilon_n,\end{aligned}$$

where $\lambda_i = a_1 + \cdots + a_i$ $(i = 1,2,\cdots,n)$ ([K3]). Hence λ corresponds to a partition $(\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0)$ of $N = a_1 + 2a_2 + \cdots + na_n$ having at most n parts, and the finite dimensional irreducible $U_q(\mathfrak{osp}(1,2n))$-module $V(\lambda)$ can be embedded into $\mathbf{V}^{\otimes N}$. Therefore the crystal graph $B(\lambda)$ of $V(\lambda)$ is isomorphic to the connected component of $\mathbf{B}^{\otimes N}$ containing a highest weight vector u_λ of weight λ (i.e., $\mathrm{wt} u_\lambda = \lambda$ and $\tilde{e}_i u_\lambda = 0$ for all $i = 1,2,\cdots,n$).

The methods in [KN] allow us to identify the crystal graph $B(\lambda)$ with a certain set of Young tableaux which are semistandard relative to the ordering

$$1 < 2 < \cdots < n < 0 < \overline{n} < \cdots < \overline{1}$$

on the elements of $\mathbf{B}$ in the following way.

Suppose first that $\lambda = \epsilon_1 + \cdots + \epsilon_k$ $(1 \le k \le n)$. Then $\lambda = \omega_k$ if $k = 1,\cdots,n-1$ and $\lambda = 2\omega_n$ if $k = n$. Let $u_\lambda = \boxed{1} \otimes \boxed{2} \otimes \cdots \otimes \boxed{k} \in \mathbf{B}^{\otimes k}$. Then it is easy to see that $\mathrm{wt} u_\lambda = \lambda = \epsilon_1 + \cdots + \epsilon_k$ and $\tilde{e}_i u_\lambda = 0$ for all $i = 1,\cdots,n$. The explicit description of the crystal graph $B(\epsilon_1 + \cdots + \epsilon_k)$ is given in the next result.

Proposition 5.6. ([KN]) *Let $B(Y_k)$ be the set of all vectors in $\mathbf{B}^{\otimes k}$ of the form* $\begin{array}{|c|}\hline j_1 \\ \hline \vdots \\ \hline j_k \\ \hline\end{array} = \boxed{j_1} \otimes \cdots \otimes \boxed{j_k} \in \mathbf{B}^{\otimes k}$ *satisfying the following conditions:*

(a) $1 \le j_1 \le \cdots \le j_k \le \overline{1}$, *but 0 is the only entry that can be repeated.*

(b) *if $j_r = p$ and $j_s = \overline{p}$ $(1 \le p \le n)$, then $r - s + k + 1 \le p$.*

Then $B(\epsilon_1 + \cdots + \epsilon_k) \cong B(Y_k)$.

Next suppose $\lambda = (\epsilon_1 + \cdots + \epsilon_k) + (\epsilon_1 + \cdots + \epsilon_l)$ with $1 \le k \le l \le n$. For

$$u_\lambda = \begin{array}{|c|c|}\hline 1 & 1 \\ \vdots & \vdots \\ \hline k & k \\ \hline \multicolumn{1}{|c|}{} \\ \cline{1-1} \multicolumn{1}{|c|}{l} \\ \cline{1-1}\end{array} = \begin{array}{|c|}\hline 1 \\ \hline \\ \hline k \\ \hline\end{array} \otimes \begin{array}{|c|}\hline 1 \\ \hline \\ \hline k \\ \hline \\ \hline l \\ \hline\end{array} = \boxed{1} \otimes \cdots \otimes \boxed{k} \otimes \boxed{1} \otimes \cdots \otimes \boxed{l} \in$$

$B(Y_k) \otimes B(Y_l) \subset \mathbf{B}^{\otimes(k+l)}$, it is easy to see that $\mathrm{wt} u_\lambda = \lambda = (\epsilon_1 + \cdots + \epsilon_k) + (\epsilon_1 + \cdots + \epsilon_l)$ and $\tilde{e}_i u_\lambda = 0$ for all $i = 1, 2, \cdots, n$.

To describe the connected component of $\mathbf{B}^{\otimes(k+l)}$ containing u_λ, we need to introduce some terminology.

Definition 5.7. *For*

$$w = u \otimes v = \begin{array}{|c|}\hline j_1 \\ \vdots \\ \hline j_k \\ \hline\end{array} \otimes \begin{array}{|c|}\hline i_1 \\ \\ \vdots \\ \\ \hline i_l \\ \hline\end{array} = \begin{array}{|c|c|}\hline i_1 & j_1 \\ \vdots & \vdots \\ \hline & j_k \\ \hline \multicolumn{1}{|c|}{} \\ \cline{1-1} \multicolumn{1}{|c|}{i_l} \\ \cline{1-1}\end{array} \in B(Y_k) \otimes B(Y_l) \subset \mathbf{B}^{\otimes(k+l)},$$

(a) w *is in the* (a,b)-**configuration** *for* $1 \le a \le b < n$, *if there exist positive integers* $1 \le p \le q < r \le s \le k$ *such that* $i_p = a$, $j_q = b$, $j_r = \overline{b}$, $j_s = \overline{a}$ *or* $i_p = a$, $i_q = b$, $i_r = \overline{b}$, $j_s = \overline{a}$.

(b) w *is in the* (a,n)-**configuration** *for* $1 \le a < n$ *if there exist positive integers* $1 \le p \le q < r = q+1 \le k$ *such that* $i_p = a$, $j_s = \overline{a}$, *and* $i_q, i_{q+1} \in \{n, 0, \overline{n}\}$ *or* $i_p = a$, $j_s = \overline{a}$, *and* $j_q, j_{q+1} \in \{n, 0, \overline{n}\}$.

(c) w *is in the* (n,n)-**configuration** *if there exist positive integers* $1 \le p < q \le r = s = k$ *such that* $i_p = n$ *or* 0, *and* $j_q = 0$ *or* $\overline{n}$.

Then the crystal graph $B(\lambda)$ with $\lambda = (\epsilon_1 + \cdots + \epsilon_k) + (\epsilon_1 + \cdots + \epsilon_l)$ for the finite dimensional irreducible $U_q(\mathfrak{osp}(1,2n))$-module $V(\lambda)$ can be described as follows:

Proposition 5.8. *Let* $B(Y_{k,l})$ $(k \le l)$ *be the set of vectors in* $B(Y_k) \otimes B(Y_l)$ *of the form*

$$w = u \otimes v = \begin{array}{|c|}\hline j_1 \\ \vdots \\ \hline j_k \\ \hline\end{array} \otimes \begin{array}{|c|}\hline i_1 \\ \\ \vdots \\ \\ \hline i_l \\ \hline\end{array} = \begin{array}{|c|c|}\hline i_1 & j_1 \\ \vdots & \vdots \\ \hline & j_k \\ \hline \multicolumn{1}{|c|}{} \\ \cline{1-1} \multicolumn{1}{|c|}{i_l} \\ \cline{1-1}\end{array}$$

satisfying the conditions:

(i) $i_r \le j_r$ *for* $1 \le r \le k$ *and* i_r *and* j_r *cannot be 0 simultaneously.*

(ii) *whenever w is in the (a,b)-configuration for some a,b with $1 \le a \le b \le n$, then*

$$(q-p)+(s-r)<b-a,$$

where p,q,r,s correspond to a,b as in Definition 5.7.

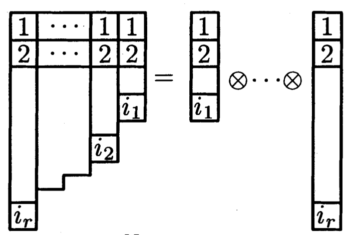
Then for $\lambda = (\epsilon_1 + \cdots + \epsilon_k) + (\epsilon_1 + \cdots + \epsilon_l)$,

$$B(\lambda) \cong B(Y_{k,l}).$$

Now for an arbitrary dominant integral weight λ of $U_q(\mathfrak{osp}(1,2n))$ with

$$\begin{aligned}\lambda &= a_1\omega_1 + \cdots + a_{n-1}\omega_{n-1} + 2a_n\omega_n \\ &= (\epsilon_1 + \cdots + \epsilon_{i_1}) + \cdots + (\epsilon_1 + \cdots + \epsilon_{i_r}),\end{aligned}$$

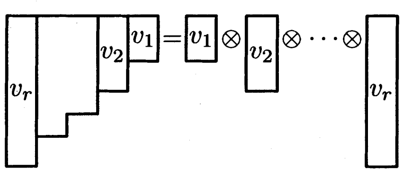
where $1 \le i_1 \le i_2 \le \cdots \le i_r \le n$, let $u_\lambda =$



Then u_λ is a highest weight vector of weight λ in $\mathbf{B}^{\otimes N}$ whose connected component of $\mathbf{B}^{\otimes N}$ is described in the following theorem.

Theorem 5.9. *Let $B(Y)$ be the set of vectors of the form*

$v = v_1 \otimes \cdots \otimes v_r =$



$\in B(Y_{i_1}) \otimes \cdots \otimes B(Y_{i_r})$

such that $v_k \otimes v_{k+1} \in B(Y_{i_k,i_{k+1}})$ for all $k = 1, \cdots, r-1$. Then

$$B(\lambda) \cong B(Y).$$

It is helpful to illlustrate these results with some examples.

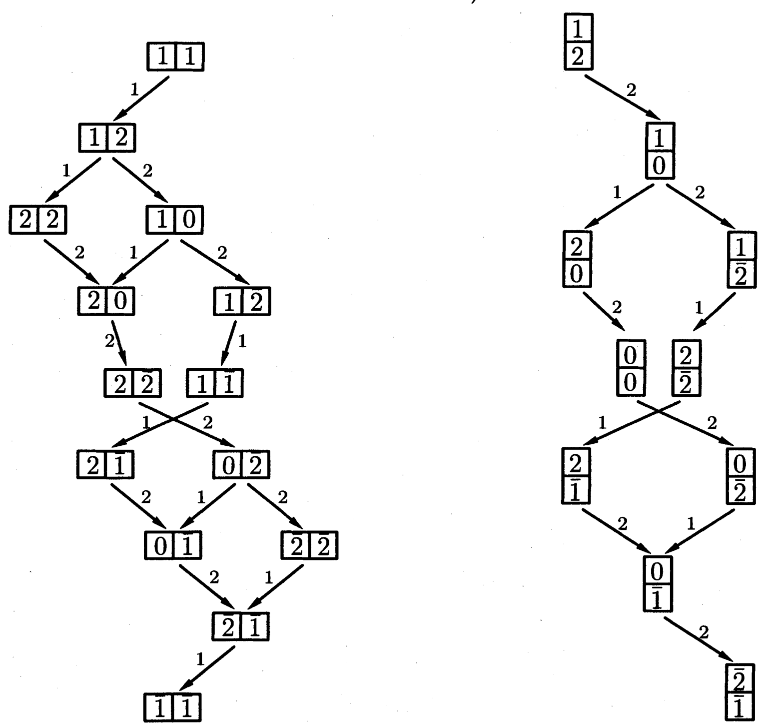
Example 5.10.

(a) The crystal graph $B(2\epsilon_1)$ over $U_q(\mathfrak{osp}(1,4))$.

(b) The crystal graph $B(\epsilon_1 + \epsilon_2)$ over $U_q(\mathfrak{osp}(1,4))$.

a)

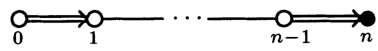
b)



§6. Quantum Affine Superalgebras

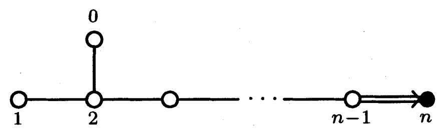
The only families of affine Lie superalgebras that belong to the class of Kac-Moody superalgebras are $B^{(1)}(0,n)$, $A^{(2)}(0,2n-1)$, $C^{(2)}(n+1)$ and $A^{(4)}(0,2n)$. Their Dynkin diagrams are

$B^{(1)}(0,n)\quad (n \geq 2)$



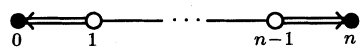
$B^{(1)}(0,1)$

$A^{(2)}(0,2n-1)\quad (n \geq 3)$



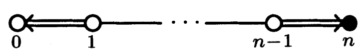
$A^{(2)}(0,3)$

$C^{(2)}(n+1)\quad (n \geq 2)$

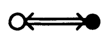


$C^{(2)}(2)$

$A^{(4)}(0,2n)\quad (n \geq 2)$



$A^{(4)}(0,2)$



For simplicity, we restrict our considerations to $B^{(1)}(0,n)$ $(n \geq 2)$, $A^{(2)}(0,2n-1)$ $(n \geq 3)$, and $A^{(4)}(0,2n)$ $(n \geq 2)$ only. For these classes of affine Kac-Moody superalgebras, tableaux bases are not known. However, we can take a different tack and develop the theory of *perfect crystals* as in [KMN1]. To define the notion of perfect crystals, we require some preliminaries.

Let $\mathfrak{g}$ be an affine Kac-Moody superalgebra corresponding to one of these diagrams, and let $I = \{0, 1, \cdots, n\}$ be the index set for the simple roots. We denote by $U'_q(\mathfrak{g})$ the quantum superalgebra corresponding to the derived subalgebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$. Let $P' = \bigoplus_{i=0}^{n} \mathbf{Z}\Lambda_i$ be the weight lattice of $\mathfrak{g}'$ with dominant integral weights $(P')^+$, and let $(P')^\vee = \bigoplus_{i=0}^{n} \mathbf{Z}h_i$ be the dual weight lattice of $\mathfrak{g}'$. For an element b in a crystal graph B of a $U'_q(\mathfrak{g})$-module V, set

$$\varepsilon(b) = \sum_{i=0}^{n} \varepsilon_i(b)\Lambda_i, \quad \text{and} \quad \varphi(b) = \sum_{i=0}^{n} \varphi_i(b)\Lambda_i. \tag{6.1}$$

Definition 6.2. *Assume B is a crystal graph of a finite dimen-*

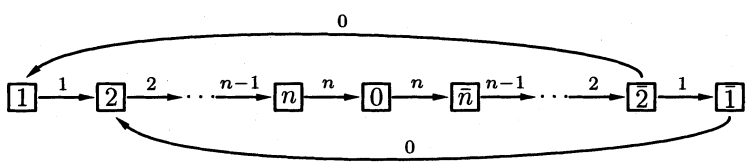
sional $U'_q(\mathfrak{g})$-module and let $l > 0$ be a positive integer. Then B is a **perfect crystal of level** $l > 0$ *if*

(i) *$B \otimes B$ is connected,*
(ii) *there exists a weight $\lambda_0 \in P'$ such that $|B_{\lambda_0}| = 1$ and $wt(B) \subset \lambda_0 + \sum_{i \neq 0} \mathbf{Z}_{\leq 0} \alpha_i$,*
(iii) *$\langle c, \varepsilon(b) \rangle \geq l$ for all $b \in B$, where c denotes the canonical central element for the affine Kac-Moody superalgebra $\mathfrak{g}$,*
(iv) *for each dominant integral weight $\lambda \in (P')^+$ of level l, there exist unique elements b^λ and b_λ in B such that $\varepsilon(b^\lambda) = \lambda$, $\varphi(b_\lambda) = \lambda$.*

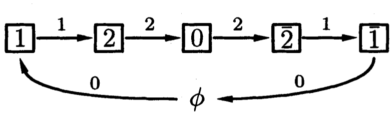
Perfect crystals play a crucial role in realizing the crystal graphs of irreducible highest weight modules over quantum affine superalgebras. We first present some illustrative examples.

Example 6.3.

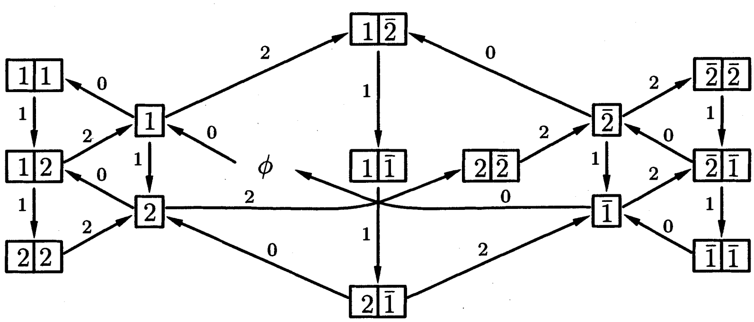
(a) Level 1 perfect crystal for $A^{(2)}(0, 2n-1)$



(b) Level 1 perfect crystal for $A^{(4)}(0, 4)$



(c) Level 2 perfect crystal for $A^{(4)}(0, 4)$



Perfect crystals give rise to the following important crystal isomorphism.

Theorem 6.4. ([KMN2]) *Assume* $\lambda \in (P')_l^+ = \{\mu \in (P')^+ \mid \langle c, \mu \rangle = l\}$ *is a dominant integral weight of level* l, *and let* $B(\lambda)$ *be the crystal graph of the irreducible highest weight module* $V(\lambda)$ *over* $U_q'(\mathfrak{g})$. *Then for any perfect crystal* B *of level* l, *there is an isomorphism of crystals*

$$B(\lambda) \otimes B \cong B(\lambda'), \qquad u_\lambda \otimes b^\lambda \longmapsto u_{\lambda'},$$

where b^λ *is the unique element in* B *such that* $\varepsilon(b^\lambda) = \lambda$ *and* $\lambda' = \lambda + wt(b^\lambda)$.

Thanks to Theorem 6.4, the crystal graph $B(\lambda)$ has a *path realization.* Start with $\lambda = \lambda_0$ a dominant integral weight of $U_q'(\mathfrak{g})$ of level l, and let B be a perfect crystal of level l. By repeating the isomorphism of crystal graphs given in Theorem 6.4, we obtain

$$\begin{aligned} B(\lambda) \otimes B &\cong B(\lambda_1), \quad u_\lambda \otimes b_0 \mapsto u_{\lambda_1}, \\ B(\lambda_1) \otimes B &\cong B(\lambda_2), \quad u_{\lambda_1} \otimes b_1 \mapsto u_{\lambda_2}, \\ B(\lambda_2) \otimes B &\cong B(\lambda_3), \quad u_{\lambda_2} \otimes b_2 \mapsto u_{\lambda_3}, \\ &\cdots\cdots \end{aligned}$$

where $b_k = b^{\lambda_k}$ for $k = 0, 1, 2, \cdots$. Since there are only finitely many dominant integral weights of $U_q'(\mathfrak{g})$ of a given level, we must have $\lambda_N = \lambda_0 = \lambda$ for some $N > 0$. Thus, there is a chain of crystal isomorphisms

$$\begin{aligned} B(\lambda) &\cong B(\lambda_{N-1}) \otimes B \cong B(\lambda_{N-2}) \otimes B \otimes B \\ &\cong \cdots \cong B(\lambda_0) \otimes B \otimes \cdots \otimes B \end{aligned}$$

such that

$$\begin{aligned} u_\lambda &\longmapsto u_{\lambda_{N-1}} \otimes b_{N-1} \longmapsto u_{\lambda_{N-2}} \otimes b_{N-2} \otimes b_{N-1} \\ &\longmapsto \cdots \longmapsto u_\lambda \otimes b_0 \otimes \cdots \otimes b_{N-1}. \end{aligned}$$

The sequence

$$\begin{aligned} p_\lambda = (p_\lambda(k))_{k\geq 1} &= \cdots \otimes p_\lambda(k) \otimes \cdots \otimes p_\lambda(2) \otimes p_\lambda(1) \\ &= \cdots \otimes b_0 \otimes \cdots \otimes b_{N-1} \otimes b_0 \otimes \cdots \otimes b_{N-1} \end{aligned}$$

is called the *ground-state path* of weight λ. A *λ-path* is a sequence $p = (p(k))_{k\geq 1} = \cdots \otimes p(k) \otimes \cdots \otimes p(2) \otimes p(1)$ with $p(k) \in B$ such that $p(k) = p_\lambda(k)$ for all k sufficiently large. Let $\mathcal{P}(\lambda, B)$ be the set of all λ-paths in B and define the crystal structure on $\mathcal{P}(\lambda, B)$ using the tensor product rule to obtain the following.

Theorem 6.5.

$$B(\lambda) \cong \mathcal{P}(\lambda, B).$$

Because the structure of perfect crystals for quantum affine Kac-Moody superalgebras is the same as for perfect crystals for quantum affine Kac-Moody algebras, their description can be found in [KMN2] and [KK]. We close this section by giving an example of path realization.

Example 6.6. Let B be a perfect crystal of level 1 given in Example 6.3 (b). By Theorem 6.4, there is an isomorphism of crystal graphs

$$B(\Lambda_0) \otimes B \cong B(\Lambda_0), \qquad u_{\Lambda_0} \otimes \phi \longmapsto u_{\Lambda_0}.$$

Hence the ground-state path p_{Λ_0} is given by

$$p_{\Lambda_0} = \cdots \otimes \phi \otimes \phi \otimes \phi = (\cdots \phi, \phi, \phi),$$

and the path realization of $B(\Lambda_0)$ is

$$
\begin{tikzcd}
 & (\cdots,\phi,\phi,\phi) \arrow[d, "0"'] & \\
 & (\cdots,\phi,\phi,1) \arrow[dl, "1"'] & \\
(\cdots,\phi,\phi,2) \arrow[d, "0"'] \arrow[dr, "2"] & & \\
(\cdots,\phi,1,2) \arrow[dr, "2"] & (\cdots,\phi,\phi,0) \arrow[d, "0"'] \arrow[dr, "2"] & \\
 & (\cdots,\phi,1,0) \arrow[dl, "1"'] \arrow[dr, "2"] & (\cdots,\phi,\phi,\bar{2}) \arrow[dl, "1"'] \arrow[d, "0"'] \\
(\cdots,\phi,2,0) & (\cdots,\phi,\phi,\bar{1}) & (\cdots,\phi,1,\bar{2}) \\
\vdots & \vdots & \vdots
\end{tikzcd}
$$

§7. The Quantum Superalgebra $U_q(\mathfrak{gl}(m,n))$

In this section we focus on the q-analogue of one of the basic Lie superalgebras – the general linear Lie superalgebra $\mathfrak{gl}(m,n)$.

Suppose $V = V_0 \oplus V_1$ is a $\mathbf{Z}_2$-graded complex vector space such that dim $V_0 = m$ and dim $V_1 = n$. For $i = 0, 1$, let $\mathrm{End}(V)_i = \{x \in \mathrm{End}(V) \mid xV_j \subseteq V_{i+j}\}$ (subscripts are read mod 2). Then $\mathfrak{gl}(m,n)$ is $\mathrm{End}(V) = \mathrm{End}(V)_0 \oplus \mathrm{End}(V)_1$ regarded as a Lie superalgebra under the supercommutator product

$$[x,y] = xy - (-1)^{ij}yx, \qquad x \in \mathrm{End}(V)_i, \quad y \in \mathrm{End}(V)_j.$$

Set $\mathbf{B} = \mathbf{B}_+ \sqcup \mathbf{B}_-$, where $\mathbf{B}_+ = \{\overline{m}, \ldots, \overline{1}\}$, and $\mathbf{B}_- = \{1, \ldots, n\}$. We can think of V_0 (resp. V_1) as having a basis indexed by the elements of $\mathbf{B}_+$ (resp. $\mathbf{B}_-$), so that $\mathfrak{gl}(m,n)$ can be viewed as matrices having rows and columns indexed by $\mathbf{B}$. The diagonal matrices in $\mathfrak{gl}(m,n)$ can be taken to be a Cartan subalgebra for $\mathfrak{gl}(m,n)$. Let $P = \bigoplus_{b\in\mathbf{B}} \mathbf{Z}\epsilon_b$ be the lattice of integral weights and $P^\vee = \bigoplus_{b\in\mathbf{B}} \mathbf{Z}E_{b,b}$ the dual weight lattice of $\mathfrak{gl}(m,n)$, where ϵ_b denotes the projection of a matrix onto its (b,b)-entry, and $E_{b,b}$ is the standard matrix unit. Then the symmetric bilinear form on P is given by

$$(\epsilon_a, \epsilon_{a'}) = \begin{cases} 1 & \text{if } a = a' \in \mathbf{B}_+, \\ -1 & \text{if } a = a' \in \mathbf{B}_-, \\ 0 & \text{otherwise.} \end{cases}$$

We assume that the index set for the simple roots of $\mathfrak{g} = \mathfrak{gl}(m,n)$ is $I = I^{\text{even}} \sqcup I^{\text{odd}}$ where

$$\begin{aligned} I^{\text{even}} &= \{\overline{m-1}, \dots, \overline{1}, 1, \dots, n-1\}, \\ I^{\text{odd}} &= \{0\}, \end{aligned}$$

and the simple roots are given by

$$\alpha_i = \begin{cases} \epsilon_{\overline{a+1}} - \epsilon_{\overline{a}} & \text{if } i = \overline{a} \;\text{ and }\; a = m-1, \dots, 1, \\ \epsilon_{\overline{1}} - \epsilon_1 & \text{if } i = 0, \\ \epsilon_i - \epsilon_{i+1} & \text{if } i = 1, \dots, n-1. \end{cases}$$

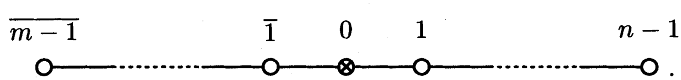
The coroot corresponding to α_i is the unique $h_i \in P^\vee$ satisfying

$$\ell_i \langle h_i, \lambda \rangle = (\alpha_i, \lambda) \quad \text{for any} \lambda \in P,$$

where

$$\ell_i = \begin{cases} 1 & \text{if } i = \overline{m-1}, \dots, \overline{1} \text{ or } 0, \\ -1 & \text{if } i = 1, \dots, n-1. \end{cases}$$

Relative to this indexing of simple roots, the Dynkin diagram is given by



This diagram corresponds to the matrix

(7.1)

$$A = (a_{i,j})_{i,j \in I} = \begin{pmatrix} 2 & -1 & 0 & \cdots & & & & & & \\ -1 & 2 & -1 & \cdots & & & & & & \\ 0 & -1 & \ddots & & & & & & & \\ & & & 2 & -1 & & & & & \\ & & & -1 & 0 & 1 & & & & \\ & & & & -1 & 2 & -1 & & & \\ & & & & & -1 & 2 & -1 & & \\ & & & & & & & & \ddots & -1 \\ & & & & & & & & -1 & 2 \end{pmatrix}$$

with rows and columns indexed by the elements of $I = \{\overline{m-1}, \ldots, \overline{1}, 0, 1, \ldots, n-1\}$. Note that $a_{0,0} = 0$ and $a_{0,1} = 1$ so that A is not a generalized Cartan matrix of the type considered in Section 2.

As in [K1], we can construct the contragredient Lie superalgebra $\mathfrak{g} = \mathfrak{g}(A, I^{\text{odd}})$, $I^{\text{odd}} = \{0\}$, associated with the Cartan data (A, I^{odd}). The Lie superalgebra $\mathfrak{g} = \mathfrak{g}(A, I^{\text{odd}})$ is isomorphic to the special linear Lie superalgebra of matrices of supertrace zero:

$$\mathfrak{sl}(m,n) = \{x \in \mathfrak{gl}(m,n) | \ \text{str}(x) = 0\},$$

where for an $(m+n) \times (m+n)$ matrix $x = (x_{b,b'})_{b,b' \in \mathbf{B}}$, its *supertrace* is given by

$$\text{str}(x) = \sum_{b \in \mathbf{B}_+} x_{b,b} - \sum_{b \in \mathbf{B}_-} x_{b,b}.$$

The general linear Lie superalgebra $\mathfrak{gl}(m,n)$ is a 1-dimensional central extension of $\mathfrak{sl}(m,n)$.

Definition 7.2. ([KT], [Y]) *The* **quantum superalgebra** *$U_q(\mathfrak{gl}(m,n))$ is the associative algebra over $\mathbf{Q}(q)$ with 1 generated by the elements e_i, f_i $i \in I = I^{\text{even}} \cup I^{\text{odd}}$ and q^h $(h \in P^\vee)$ with defining relations*

(7.3)

$$\begin{aligned}
&q^0 = 1, \quad q^h q^{h'} = q^{h+h'} \quad \textit{for } h, h' \in P^\vee, \\
&q^h e_i q^{-h} = q^{\alpha_i(h)} e_i, \quad q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i \quad \textit{for} \ \ h \in P^\vee, i \in I, \\
&e_i f_j - (-1)^{p(i)p(j)} f_j e_i = \delta_{i,j}(K_i - K_i^{-1})/(q_i - q_i^{-1}) \quad \textit{for } i, j \in I, \\
&(ad_q e_i)^{1-a_{i,j}}(e_j) = (ad_q f_i)^{1-a_{i,j}}(f_j) = 0 \\
&\qquad\qquad \textit{if } i \neq j \in I^{\text{even}}, \textit{ or if } i \in I^{\text{even}} \textit{ and } j = 0, \\
&e_0^2 = f_0^2 = 0, \\
&e_0 e_{\overline{1}} e_0 e_1 + e_{\overline{1}} e_0 e_1 e_0 + e_0 e_1 e_0 e_{\overline{1}} \\
&\qquad\qquad + e_1 e_0 e_{\overline{1}} e_0 - (q + q^{-1}) e_0 e_{\overline{1}} e_1 e_0 = 0, \\
&f_0 f_{\overline{1}} f_0 f_1 + f_{\overline{1}} f_0 f_1 f_0 + f_0 f_1 f_0 f_{\overline{1}} \\
&\qquad\qquad + f_1 f_0 f_{\overline{1}} f_0 - (q + q^{-1}) f_0 f_{\overline{1}} f_1 f_0 = 0.
\end{aligned}$$

Here, p denotes the parity map with $p(i) = 0$ if $i \neq 0$ and $p(0) = 1$, $q_i = q^{\ell_i}$, and $K_i = q^{\ell_i h_i}$ for $i \in I$.

The Hopf superalgebra structure on $U_q(\mathfrak{gl}(m,n))$ has comultiplication Δ, counit ϵ, and antipode S specified by the formulas in (2.7)-(2.9).

It follows from (7.3) that the subalgebra $U_q(\mathfrak{gl}(m,n))_i$ generated by e_i, f_i, $K_i^{\pm 1}$ is isomorphic to $U_q(\mathfrak{sl}_2)$ for $i \neq 0$ and to the quantum superalgebra $U_q(\mathfrak{sl}(1,1))$ for $i = 0$.

We now define the category of $U_q(\mathfrak{gl}(m,n))$-modules for which the crystal base theory is developed in [BKK].

Definition 7.4. *The category $\mathcal{O}_{\text{int}}$ is the category of $\mathbf{Z}_2$-graded finite dimensional $U_q(\mathfrak{gl}(m,n))$-modules M and $U_q(\mathfrak{gl}(m,n))$-module homomorphisms which satisfy the following constraints:*

(i) *M has a weight decomposition $M = \bigoplus_{\lambda \in P} M_\lambda$, where* $M_\lambda = \{u \in M \mid q^h u = q^{\langle h, \lambda \rangle} u \quad \text{for all } h \in P^\vee\}$.

(ii) *if $M_\mu \neq 0$, then $\mu(h_0) \geq 0$.*

(iii) *if $\mu(h_0) = 0$, then $e_0 M_\mu = f_0 M_\mu = 0$. Thus M is a direct sum of 1-dimensional or 2-dimensional irreducible modules over $U_q(\mathfrak{gl}(m,n))_0 \cong U_q(\mathfrak{sl}(1,1))$.*

The category $\mathcal{O}_{\text{int}}$ is stable under taking subquotients and tensor products. In [BKK] it is conjectured that the modules in $\mathcal{O}_{\text{int}}$ are completely reducible.

Proposition 7.5. ([BKK]) *Let $V(\lambda)$ be an irreducible highest weight $U_q(\mathfrak{gl}(m,n))$-module with highest weight $\lambda = \sum_{b \in \mathbf{B}} \lambda_b \epsilon_b \in P$, where $\mathbf{B} = \{\overline{m} < \overline{m-1} < \cdots < \overline{2} < \overline{1} < 1 < 2 < \cdots < n-1 < n\}$. If $V(\lambda)$ belongs to category $\mathcal{O}_{\text{int}}$, then we have*

(i) *$\lambda_b \geq \lambda_{b'}$ for $b < b'$.*

(ii) *if $\lambda_b > 0$ for some $b = 1, \cdots, n$, then $\lambda_0 \geq b$.*

§8. Crystal bases for $U_q(\mathfrak{gl}(m,n))$

Whenever M is in the category $\mathcal{O}_{\text{int}}$ for $U_q(\mathfrak{gl}(m,n))$ and $i \in I^{\text{even}}$, then for any $u \in M$ of weight $\lambda \in P$, there is a unique expression

$$u = \sum_{k \geq 0, -\langle h_i, \lambda \rangle} f_i^{(k)} u_k$$

with $e_i u_k = 0$ for each k. For $U_q(\mathfrak{gl}(m,n))$ (and other contragredient Lie superalgebras) we use the divided powers

$$f_i^{(k)} = \frac{1}{[k]_i!} f_i^k,$$

where

$$(8.1) \qquad \begin{aligned} &[k]_i = (q_i^k - q_i^{-k})/(q_i - q_i^{-1}), \\ &[k]_i! = \prod_{n=1}^{k} [n]_i \quad \text{for } k \geq 1, \quad \text{and} \quad [0]! = 1. \end{aligned}$$

It is convenient to adopt the convention that $f_i^{(k)} = 0$ for $k < 0$.

Then the Kashiwara operators are defined by

Case (1): for $i = \overline{m-1}, \cdots, \overline{1}$,

$$(8.2) \qquad \tilde{e}_i u = \sum_k f_i^{(k-1)} u_k, \qquad f_i u = \sum_k f_i^{(k+1)} u_k.$$

Case (2): for $i = 1, \cdots, n-1$,

$$(8.3) \qquad \tilde{e}_i u = \sum_k q_i^{\lambda(h_i)+1} f_i^{(k-1)} u_k, \quad \tilde{f}_i u = \sum_k q_i^{-\lambda(h_i)+1} f_i^{(k+1)} u_k.$$

Case (3): for $i = 0$,

$$(8.4) \qquad \tilde{e}_i u = q_i^{-1} K_i e_i u, \quad \tilde{f}_i u = f_i u.$$

As before, let $\mathbf{A}$ denote the subring of $\mathbf{Q}(q)$ consisting of all rational functions $f/g \in \mathbf{Q}(q)$ such that $g(0) \neq 0$. Assume M is a $U_q(\mathfrak{gl}(m,n))$-module in the category $\mathcal{O}_{\text{int}}$. A free $\mathbf{A}$-submodule L of M is a **crystal lattice** if it satisfies the conditions in Definition 2.10 (but using the Kashiwara operators in (8.4)). A **crystal base** of M is a pair (L, B), where B is a subset of L/qL for which (i)-(v) of Definition 2.11 hold. The associated crystal of (L, B) consists of $B/\{\pm 1\}$ with the structure of a colored oriented graph where b, $b' \in B/\{\pm 1\}$ are joined by the i-arrow, $b \xrightarrow{i} b'$, if $\tilde{f}_i b = b'$.

Lemma 8.5. (See Lemma 2.7 of [BKK].) *Let M be a $U_q(\mathfrak{gl}(m,n))$-module in $\mathcal{O}_{\text{int}}$ with a crystal base (L, B). Assume that*

(a) *the associated crystal is connected, and*
(b) *there is a weight λ such that* $\dim M_\lambda = 1$.

Then

(i) *L/qL is an irreducible module over the algebra generated by the $\tilde{e}_i$'s and the $\tilde{f}_i$'s.*
(ii) *M is an irreducible $U_q(\mathfrak{g})$-module.*
(iii) *$L'_\lambda = L_\lambda$ for L' a crystal lattice implies $L' = L$.*
(iv) *The crystal base of M is unique up to a constant multiple.*

The antiautomorphism η of $U_q(\mathfrak{gl}(m,n))$ defined by

$$\begin{aligned}\eta(q^h) &= q^h\\ \eta(e_i) &= q_i f_i K_i^{-1}\\ \eta(f_i) &= q_i^{-1} K_i e_i\end{aligned}$$

satisfies $\eta^2 = \mathrm{id}$. We say that a symmetric bilinear form $(\cdot,\cdot)$ on a $U_q(\mathfrak{gl}(m,n))$-module M is a *polarization* if $(au, v) = (u, \eta(a)v)$ holds for any $u, v \in M$ and $a \in U_q(\mathfrak{gl}(m,n))$.

It is an easy consequence of the relation $\Delta \circ \eta = (\eta \otimes \eta) \circ \Delta$ that the following holds:

Lemma 8.6. ([BKK]) *Let M_1 and M_2 be two $U_q(\mathfrak{gl}(m,n))$-modules with polarizations. Then the symmetric bilinear form $(\cdot,\cdot)$ on $M_1 \otimes M_2$ defined by $(u_1 \otimes u_2, v_1 \otimes v_2) = (u_1, v_1)(u_2, v_2)$ is a polarization.*

The Kashiwara operators $\tilde{e}_i$ and $\tilde{f}_i$ on the modules M in $\mathcal{O}_{\mathrm{int}}$ are defined so that $\tilde{e}_i$ and $\tilde{f}_i$ are adjoints of each other at $q = 0$ with respect to a polarization. More precisely:

Proposition 8.7. ([BKK]) *Let M be a $U_q(\mathfrak{gl}(m,n))$-module in $\mathcal{O}_{\mathrm{int}}$ with a crystal lattice L, and let $(\,\cdot\,,\,\cdot\,)$ be a polarization of M. Assume $(L, L) \subset \mathbf{A}$. Then the induced $\mathbf{Q}$-valued symmetric bilinear form $(\,\cdot\,,\,\cdot\,)_0$ on L/qL satisfies $(\tilde{e}_i u, v)_0 = (u, \tilde{f}_i v)_0$ for any u, $v \in L/qL$.*

Definition 8.8. ([BKK]) *A crystal base (L, B) for a $U_q(\mathfrak{gl}(m,n))$-module M is said to be polarizable if there is a polarization $(\cdot,\cdot)$ of M such that $(L, L) \subset \mathbf{A}$, and the induced $\mathbf{Q}$-valued symmetric bilinear form $(\cdot,\cdot)_0$ on L/qL satisfies*

$$(b, b')_0 = \begin{cases} \pm 1 & \textit{if } b' = \pm b, \\ 0 & \textit{otherwise} \end{cases}$$

for all $b, b' \in B$.

Assume M_1 and M_2 are $U_q(\mathfrak{gl}(m,n))$-modules in the category $\mathcal{O}_{\text{int}}$, and let (L_1, B_1) and (L_2, B_2) be their crystal bases. Proposition 8.9 below, which is proved in [BKK], says that $M_1 \otimes M_2$ has a crystal base given by $L = L_1 \otimes_{\mathbf{A}} L_2$ and $B = B_1 \otimes B_2 \subset (L_1/qL_1) \otimes (L_2/qL_2) = L/qL$. To describe the action of the Kashiwara operators on B we require ε_i and φ_i, which are defined exactly as in (4.5).

Proposition 8.9. *Suppose* (L_ν, B_ν) *is a crystal base of* M_ν, $\nu = 1, 2$. *Then*

(i) (L, B) *is a crystal base of* $M_1 \otimes M_2$.

(ii) *The actions of* $\tilde{e}_i$ *and* $\tilde{f}_i$ *on* $b_1 \otimes b_2$ ($b_1 \in B_1$ *and* $b_2 \in B_2$) *are as follows:*

(a) *If* $i = \overline{m-1}, \cdots, \bar{1}$, *then*

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i(b_1) \otimes b_2 & \textit{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i(b_2) & \textit{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i(b_1) \otimes b_2 & \textit{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i(b_2) & \textit{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases}$$

(b) *If* $i = 1, \cdots, n-1$, *then*

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} b_1 \otimes \tilde{e}_i(b_2) & \textit{if } \varphi_i(b_2) \geq \varepsilon_i(b_1), \\ \tilde{e}_i(b_1) \otimes b_2 & \textit{if } \varphi_i(b_2) < \varepsilon_i(b_1), \end{cases}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} b_1 \otimes \tilde{f}_i(b_2) & \textit{if } \varphi_i(b_2) > \varepsilon_i(b_1), \\ \tilde{f}_i(b_1) \otimes b_2 & \textit{if } \varphi_i(b_2) \leq \varepsilon_i(b_1). \end{cases}$$

(c) *If* $i = 0$, *then*

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i(b_1) \otimes b_2 & \textit{if } \langle h_i, wt(b_1) \rangle > 0, \\ \pm b_1 \otimes \tilde{e}_i(b_2) & \textit{if } \langle h_i, wt(b_1) \rangle = 0, \end{cases}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i(b_1) \otimes b_2 & \textit{if } \langle h_i, wt(b_1) \rangle > 0, \\ \pm b_1 \otimes \tilde{f}_i(b_2) & \textit{if } \langle h_i, wt(b_1) \rangle = 0. \end{cases}$$

The sign in part (c) depends on the parity of b_1 and i.

Part (i) of the next theorem is an immediate consequence of Definition 8.8 and Proposition 8.9. Part (ii) uses the polarization on a module M to show that the orthogonal subspace $N^\perp$ of any submodule N of M forms a complement. Part (iii) then follows directly from (i) and (ii).

Theorem 8.10.

(i) *Let (L_ν, B_ν) be a polarizable crystal base of $M_\nu \in \mathcal{O}_{\text{int}}$ $(\nu = 1, 2)$. Then $(L_1 \otimes_A L_2, B_1 \otimes B_2)$ is a polarizable crystal base of $M_1 \otimes M_2$.*

(ii) *If M is a $U_q(\mathfrak{gl}(m,n))$-module in $\mathcal{O}_{\text{int}}$ with a polarizable crystal base, then M is completely reducible.*

(iii) *If M_ν $(\nu = 1, \dots, k)$ is a $U_q(\mathfrak{gl}(m,n))$-module in $\mathcal{O}_{\text{int}}$ with a polarizable crystal base, then $M_1 \otimes \cdots \otimes M_k$ is completely reducible.*

§9. Young Tableaux and Crystal Graphs for $U_q(\mathfrak{gl}(m,n))$

The result in (ii) of Theorem 8.10 is particularly striking because most modules for contragredient Lie superalgebras and their quantized enveloping algebras are not completely reducible. In this section we study the natural $(m+n)$-dimensional module $\mathbf{V}$ of $U_q(\mathfrak{gl}(m,n))$ and its tensor powers. Critical to the discussion will be the fact that $\mathbf{V}$ belongs to $\mathcal{O}_{\text{int}}$ and has a polarizable crystal base. Then its tensor powers $\mathbf{V}^{\otimes k}$ have a polarizable crystal base and are completely reducible by (iii). In the nonquantum setting Berele and Regev [BR] have studied the tensor powers $V^{\otimes k}$ of the $(m+n)$-dimensional module V for $\mathfrak{gl}(m,n)$. They have shown that $V^{\otimes k}$ is completely reducible, and its summands have a combinatorial basis indexed by certain tableaux. Our idea in [BKK] was to exploit that tableau basis to describe a crystal base for the summands of $\mathbf{V}$. This line of attack follows the papers [KN], [KM], [L], and [MM], which construct crystal bases for the finite dimensional simple Lie algebras of types A_n, B_n, C_n, D_n, E_6, and G_2, and for the fundamental representation of the affine Lie algebra $\widehat{\mathfrak{sl}}(n)$ using tableaux. In earlier work (not using tableaux) Zou introduced a crystal base for the Lie superalgebra $\mathfrak{sl}(2,1)$ and studied its properties. However, Zou's notion of a crystal base in [Z2] differs from the one in Definition 2.11 since his base is invariant under some but not all of the Kashiwara operators. Zou's recent paper [Z3] has followed the approach of [BKK] to produce crystal bases for the family of simple Lie superalgebras $\Gamma(\sigma_1, \sigma_2, \sigma_3)$.

The simplest representation of $U_q(\mathfrak{gl}(m,n))$ is its $(m+n)$-dimensional vector representation $\mathbf{V} = \mathbf{V}_+ \oplus \mathbf{V}_-$, where $\mathbf{V}_\pm = \bigoplus_{b\in\mathbf{B}_\pm} \mathbf{Q}(q)v_b$, (where $\mathbf{B}_\pm$ is as in Section 6), and the action is specified by

$$q^h v_b = q^{\epsilon_b(h)} v_b,$$

$$e_i v_b = \begin{cases} v_{\overline{k+1}} & \text{if } i = \overline{k} \text{ and } b = \overline{k} \text{ with } k = 1,\dots,m-1 \\ v_{\overline{1}} & \text{if } i = 0 \text{ and } b = 1, \\ v_k & \text{if } i = k \text{ and } b = k+1 \text{ with } k = 1,\dots,n-1, \\ 0 & \text{otherwise,} \end{cases}$$

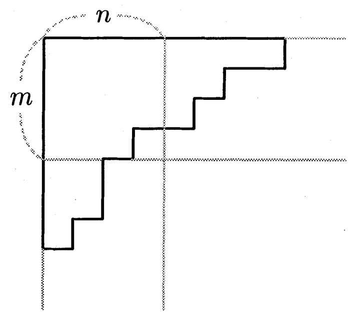
$$f_i v_b = \begin{cases} v_{\overline{k}} & \text{if } i = \overline{k} \text{ and } b = \overline{k+1} \;\text{ with } k = 1,\dots,m-1, \\ v_1 & \text{if } i = 0 \;\text{ and } b = \overline{1}, \\ v_{k+1} & \text{if } i = k \text{ and } b = k \text{ with } k = 1,\dots,n-1, \\ 0 & \text{otherwise.} \end{cases}$$

The $U_q(\mathfrak{gl}(m,n))$-module $\mathbf{V}$ belongs to the category $\mathcal{O}_{\text{int}}$, and $\mathbf{L} = \bigoplus_{b\in\mathbf{B}} \mathbf{A}v_b$ is a crystal lattice. The set $\{\pm v_b \mod q\mathbf{L} \mid b \in \mathbf{B}\}$ determines a crystal base of $\mathbf{V}$ with associated crystal:

$$\boxed{\overline{m}} \xrightarrow{\overline{m-1}} \boxed{\overline{m-1}} \xrightarrow{\overline{m-2}} \cdots$$
$$\cdots \xrightarrow{\overline{2}} \boxed{\overline{2}} \xrightarrow{\overline{1}} \boxed{\overline{1}} \xrightarrow{0} \boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots$$
$$\cdots \xrightarrow{n-2} \boxed{n-1} \xrightarrow{n-1} \boxed{n}$$

With respect to the symmetric bilinear form on $\mathbf{V}$ which has $\{v_b \mid b \in \mathbf{B}\}$ as an orthonormal basis, $(\mathbf{L}, \mathbf{B} \sqcup (-\mathbf{B}))$ is a polarizable crystal base. As we have mentioned before, Theorem 8.10 (iii) says that $\mathbf{V}^{\otimes k}$ is completely reducible for all $k \geq 1$. Moreover, $(\mathbf{L}^{\otimes k}, (\mathbf{B} \sqcup (-\mathbf{B}))^{\otimes k})$ is a polarizable crystal base for $\mathbf{V}^{\otimes k}$.

A *Young diagram* is a collection of boxes arranged in left-justified rows with a weakly decreasing number of boxes in each row. A diagram obtained by removing a smaller Young diagram from a larger one containing it is a *skew Young diagram.* We say that a Young diagram is an (m,n)-hook Young diagram if the number of boxes in the $(m+1)$st row is less than or equal to n. Thus, an (m,n)-hook Young diagram fits inside the (m,n)-hook as displayed below.



We assign an order to the elements of $\mathbf{B}$ by saying

$$\overline{m} < \overline{m-1} < \cdots < \overline{2} < \overline{1} < 1 < 2 < \cdots < n-1 < n.$$

Then a *semistandard tableau* is obtained by filling a skew Young diagram with elements of $\mathbf{B}$ such that

(i) the entries in each row are increasing, allowing repetition of the elements of $\mathbf{B}_+ = \{\overline{m}, \overline{m-1}, \ldots, \overline{2}, \overline{1}\}$ but not of the elements of $\mathbf{B}_- = \{1, 2, \ldots, n-1, n\}$.

(ii) the entries in each column are increasing, permitting repetition of the elements of $\mathbf{B}_-$ but not of $\mathbf{B}_+$.

It is not difficult to see that a Young diagram can be made into a semistandard tableau with entries in $\mathbf{B}$ if and only if it is an (m,n)-hook Young diagram. These semistandard tableaux were introduced by Berele and Regev. In [BR] they show that the irreducible summands of the tensor powers $V^{\otimes k}$ of the $(m+n)$-dimensional module V for $\mathfrak{gl}(m,n)$ can be indexed by the (m,n)-hook Young diagrams. A basis for the summand indexed by Y is in one-to-one correspondence with the semistandard tableaux of shape Y. The weight of a tableau is $\sum_{b \in \mathbf{B}} \omega_b \epsilon_b$ where ω_b is the number of its entries which are equal to b.

If N is the number of boxes contained in a Young diagram Y, then we can embed the set $B(Y)$ of semistandard tableaux of shape Y into $\mathbf{B}^{\otimes N}$ by reading the entries $\{b_1, \ldots, b_N\}$ of the tableau and identifying the tableau with $b_1 \otimes \cdots \otimes b_N$. There are many different ways this reading can be done, and we single out certain special ones.

Suppose β and β' are boxes in a skew Young diagram with β lying in position (i,j) (row i and column j) and β' in position (i',j'). Then we say β is strictly higher than β' if $\beta \neq \beta'$ and $i \leq i'$ and $j \geq j'$. This just amounts to saying that β lies northeast of β'. In an *admissible reading*, box β is read before box β' whenever β is strictly higher than β'. For example, if we start with the rightmost column and read the entries from top to bottom, and then read the next column from top to bottom, and continue until the bottom entry in the leftmost column is read, we obtain an admissible reading, which we term a *Japanese (or Chinese) reading.* Similarly, reading the rows from right to left starting with the rightmost entry in the top row and proceeding to the bottommost row gives an admissible reading, which we call an *Arabic (or Hebrew) reading.* These particular admissible readings are illustrated in the following figure.

$$\begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & \bar{3} & \bar{2} \\ \hline \bar{4} & \bar{1} & 1 \\ \hline 1 & 3 & \multicolumn{1}{c}{} \\ \cline{1-2} 2 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \quad = \quad \boxed{\bar{2}} \otimes \boxed{1} \otimes \boxed{\bar{3}} \otimes \boxed{\bar{1}} \otimes \boxed{3} \otimes \boxed{\bar{4}} \otimes \boxed{1} \otimes \boxed{2}$$

Japanese reading

$$\begin{array}{|c|c|c|} \cline{2-3} \multicolumn{1}{c|}{} & \bar{3} & \bar{2} \\ \hline \bar{4} & \bar{1} & 1 \\ \hline 1 & 3 & \multicolumn{1}{c}{} \\ \cline{1-2} 2 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \quad = \quad \boxed{\bar{2}} \otimes \boxed{\bar{3}} \otimes \boxed{1} \otimes \boxed{\bar{1}} \otimes \boxed{\bar{4}} \otimes \boxed{3} \otimes \boxed{1} \otimes \boxed{2}$$

Arabic reading

Theorem 9.1. *(Compare Thm. 4.4 of* [BKK].*) Let Y be a skew Young diagram and $B(Y)$ be the set of semistandard tableaux of shape Y. Then*

(i) *For any admissible reading $\psi : B(Y) \to \mathbf{B}^{\otimes N}$ of Y, the image $\psi(B(Y))$ is stable under the operators $\tilde{e}_i$ and $\tilde{f}_i$ for all $i \in I$.*

(ii) *The induced crystal structure on $B(Y)$ does not depend on the admissible reading.*

For a crystal base (L, B) of a $U_q(\mathfrak{gl}(m,n))$-module, we say that an element $b \in B_\lambda$ is a *genuine highest weight vector* of B if $B_\lambda = \{b\}$ and $\mathrm{Wt}(B) \subset \lambda - Q_+$, where $\mathrm{Wt}(B)$ is the set of weights of the crystal B and $Q_+ = \sum_{\alpha \in \Delta^+} \mathbf{Z}_{\geq 0}\alpha$. Analogously, $b \in B_\mu$ is a *genuine lowest weight vector* of B if $B_\mu = \{b\}$ and $\mathrm{Wt}(B) \subset \mu + Q_+$. A genuine highest (resp. lowest) weight vector is unique whenever it exists. Moreover a genuine highest (resp. lowest) weight vector satisfies $\tilde{e}_i b = 0$ (resp. $\tilde{f}_i b = 0$) for all $i \in I$. It is possible for an element $b \in B$ to satisfy one of these properties without being a genuine highest or lowest weight vector. Such vectors we term *fake highest (or lowest) weight vectors.* The existence of fake highest (or lowest) weight vectors complicates the question of the connectedness of the crystal graph. However, we have

Proposition 9.2. ([BKK]) *The crystal $B(Y)$ associated with any (m,n)-hook Young diagram Y is connected.*

Suppose Y_0 is an (m, n)−hook Young diagram. In [BKK] we have developed a combinatorial procedure for decomposing the tensor product $B(Y_0) \otimes \mathbf{B}$ into connected components $B(Y)$ corresponding to diagrams Y obtained from Y_0 by adding a box. A box in a diagram is a *corner* if there are no boxes in the diagram to its right or beneath it. A place where a box can be adjoined to a diagram to create a corner of a larger diagram is said to be a *co-corner.*

Theorem 9.3. ([BKK]) *Assume Y_0 is an (m, n)-hook Young diagram, and let $B(Y_0)$ be the set of all semistandard tableaux of shape Y_0 endowed with a crystal structure by an admissible reading. Then the tensor product of crystals $B(Y_0) \otimes \mathbf{B}$ has the following decomposition into connected components:*

$$B(Y_0) \otimes \mathbf{B} \cong \bigoplus_{Y} B(Y),$$

where Y runs over the set of all (m, n)-hook Young diagrams obtained from Y_0 by adding a box to a co-corner of Y_0.

As an immediate consequence we obtain

Corollary 9.4. *Any connected component of $\mathbf{B}^{\otimes k}$ of k copies of $\mathbf{B}$ is isomorphic (as a crystal) to $B(Y)$ for some (m, n)-hook Young diagram Y having k-boxes. Moreover, for any skew Young diagrams Y_1 and Y_2, the connected components of the tensor product of crystals $B(Y_1) \otimes B(Y_2)$ have the form $B(Y)$, where Y is an (m, n)-hook Young diagram.*

Consider the set $\widetilde{P}$ of weights $\lambda \in \bigoplus_{b \in \mathbf{B}} \mathbf{Z}\epsilon_b$ satisfying

(i) $\langle h_i, \lambda \rangle \geq 0$ for all $i \in I$,
(ii) $\langle h_0 - h_1 - \cdots - h_k, \lambda \rangle \geq k$ for all $k \in \{1, \ldots, n-1\}$ such that $\langle h_k, \lambda \rangle > 0$.

These conditions exactly translate to the ones encountered in Proposition 7.5. The weights in $\widetilde{P}$ play a distinguished role because of the following.

Proposition 9.5. ([BKK]) *If $V(\lambda)$ is an irreducible $U_g(\mathfrak{gl}(m,n))$-module in $\mathcal{O}_{\text{int}}$ with highest weight λ, then $\lambda \in \widetilde{P}$.*

Suppose that $\lambda \in \widetilde{P}^+ = \widetilde{P} \bigcap \bigoplus_{b \in \mathbf{B}} \mathbf{Z}_{\geq 0} \epsilon_b$ and write $\lambda = a_1 \epsilon_{\overline{m}} + a_2 \epsilon_{\overline{m-1}} + \cdots + a_m \epsilon_{\overline{1}} + d_1 \epsilon_1 + \cdots + d_n \epsilon_n$. Then we can create an (m,n)-hook tableau H_λ by this procedure:

(1) row i has a_i boxes all filled with the entry $\overline{i}$ for $i = 1, \ldots, m$,
(2) starting below row m, column j has d_j boxes that are filled with j for $j = 1, \ldots, n$.

The weight of the tableau H_λ is λ, and its shape is the Young diagram that we denote by Y_λ. Any semistandard tableau with shape Y_λ has weight in $\lambda - Q_+$. Thus, H_λ is a genuine highest weight vector in $B(Y_\lambda)$.

Theorem 9.6. ([BKK]) *If $\lambda \in \widetilde{P}$, then the irreducible $U_q(\mathfrak{gl}(m,n))$-module $V(\lambda)$ with highest weight λ is in $\mathcal{O}_{\text{int}}$, and it has a polarizable crystal base. If $\lambda \in \widetilde{P}^+$, then the associated crystal is isomorphic to $B(Y_\lambda)$.*

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