Advanced Studies in Pure Mathematics 27, 2000
Arrangements - Tokyo 1998
pp. 257–272

On the fundamental group of the complement of a complex hyperplane arrangement

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Dedicated to Professor Peter Orlik on his 60th birthday

§1. Introduction

Let $\mathbf{K}$ be a field, and let $V = \mathbf{K}^l$ be a finite dimensional vector space over $\mathbf{K}$. An *arrangement of hyperplanes* in V is a finite family $\mathcal{A}$ of affine hyperplanes of V. The *complement* of $\mathcal{A}$ is defined by

$$M(\mathcal{A}) = V \setminus \bigcup_{H \in \mathcal{A}} H \ .$$

If $\mathbf{K}$ is $\mathbf{C}$, then the complement $M(\mathcal{A})$ is an open and connected subset of V.

The present paper is concerned with fundamental groups of complements of complex arrangements of hyperplanes.

The most popular such a group is certainly the pure braid group; it appears as the fundamental group of the complement of the "braid arrangement" (see [OT]). So, $\pi_1(M(\mathcal{A}))$ can be considered as a generalization of the pure braid group, and one can expect to show that many properties of the pure braid group also hold for $\pi_1(M(\mathcal{A}))$. However, the only general known results on this group are presentations [Ar], [CS1], [Ra], [Sa1]. Many interesting questions remain, for example, to know whether such a group is torsion free.

We focus in this paper on two families of arrangements of hyperplanes, to the fundamental group of which many well-known results on the pure braid group can be extended. Both of them, of course, contain the braid arrangement. These families are the "simplicial arrangements" and the "supersolvable arrangements". Note that there is another well-understood family of arrangements, the "reflection arrangements" (see

2000 *Mathematics Subject Classification.* Primary: 32S22.

[OT, Ch. 6] and [BMR]), which contains the braid arrangement, and which is not treated in the present paper.

The methods to approach each of these two families are completely different. The first method, which applies to simplicial arrangements, consists on associating with a real arrangement $\mathcal{A}$ a groupoid $G(\mathcal{A})$ that we call Deligne groupoid. Any vertex group of $G(\mathcal{A})$ is isomorphic to $\pi_1(M(\mathcal{A}_{\mathbf{C}}))$, where $\mathcal{A}_{\mathbf{C}}$ is the complexification of $\mathcal{A}$. If $\mathcal{A}$ is a simplicial arrangement, then it is shown that there exists an "automatic structure" on $G(\mathcal{A})$. Then, follow many properties of $\pi_1(M(\mathcal{A}))$. This is the object of Section 2. The second method, which applies to supersolvable arrangements, consists on proving the existence of certain fibrations. In the case of supersolvable arrangements, these fibrations give rise to a presentation of $\pi_1(M(\mathcal{A}))$ as an "iterated semidirect product" of free groups. This is the object of Section 3.

§2. The Deligne groupoid

Throughout this section, $\mathbf{K}$ is $\mathbf{R}$, and $\mathcal{A}$ is a (real) arrangement of hyperplanes in V. The *complexification* of V is $V_{\mathbf{C}} = \mathbf{C}^l$. The *complexification* of a hyperplane H is the hyperplane $H_{\mathbf{C}}$ of $V_{\mathbf{C}}$ having the same equation as H. The *complexification* of $\mathcal{A}$ is the arrangement $\mathcal{A}_{\mathbf{C}} = \{H_{\mathbf{C}}; H \in \mathcal{A}\}$ in $V_{\mathbf{C}}$.

DEFINITION. A *groupoid* is a category such that there is a morphism between any two objects, and such that each morphism is invertible.

A group is a groupoid with exactly one object. An object of a groupoid G is called *vertex* of G. For any vertex x, the set of morphisms from x to itself forms a group called *vertex group* and denoted by G_x.

Now, in order to define the Deligne groupoid $G(\mathcal{A})$ associated with a real arrangement of hyperplanes $\mathcal{A}$, we first give some terminology on oriented graphs.

DEFINITION. An *oriented graph* Γ is the following data:

1) a set $V(\Gamma)$ of *vertices*,

2) a set $A(\Gamma)$ of *arrows*,

3) a mapping $s : A(\Gamma) \to V(\Gamma)$ called *source*, and a mapping $t : A(\Gamma) \to V(\Gamma)$ called *target*.

Consider the abstract set $A(\Gamma)^{-1} = \{a^{-1}; a \in A(\Gamma)\}$ in one-to-one correspondence with $A(\Gamma)$, and set $s(a^{-1}) = t(a)$ and $t(a^{-1}) = s(a)$, for a in $A(\Gamma)$. A *path* of Γ is an expression

$$g = a_1^{\varepsilon_1} \dots a_d^{\varepsilon_d} ,$$

where $a_i \in A(\Gamma)$, $\varepsilon_i \in \{\pm 1\}$, and $t(a_i^{\varepsilon_i}) = s(a_{i+1}^{\varepsilon_{i+1}})$ for all $i = 1, \dots, d-1$. The vertex $s(a_1^{\varepsilon_1})$ is called *source* of g and is denoted by $s(g)$, and the vertex $t(a_d^{\varepsilon_d})$ is called *target* of g and is denoted by $t(g)$. The integer d is the *length* of g. Any vertex is assumed to be a path of length 0. For a path $f = a_1^{\varepsilon_1} \dots a_d^{\varepsilon_d}$, we write $f^{-1} = a_d^{-\varepsilon_d} \dots a_1^{-\varepsilon_1}$. For two paths $f = a_1^{\varepsilon_1} \dots a_d^{\varepsilon_d}$ and $g = b_1^{\mu_1} \dots b_k^{\mu_k}$ with $t(f) = s(g)$, we write $fg = a_1^{\varepsilon_1} \dots a_d^{\varepsilon_d} b_1^{\mu_1} \dots b_k^{\mu_k}$. A *positive path* is a path $f = a_1^{\varepsilon_1} \dots a_d^{\varepsilon_d}$ with $\varepsilon_1 = \varepsilon_2 = \dots = \varepsilon_d = 1$. The *distance* between two vertices x and y is defined to be the minimal length of a path connecting x and y. Any path which achieves this minimum is called *minimal path.*

Call an oriented graph *connected* if there is a path connecting any two vertices.

DEFINITION. Let Γ be an oriented connected graph. A *congruence* on Γ is an equivalence relation $\sim$ on the set of paths of Γ, which satisfies the following conditions:

1) if $f \sim g$, then $s(f) = s(g)$ and $t(f) = t(g)$,

2) $ff^{-1} \sim s(f)$ for any path f,

3) if $f \sim g$, then $f^{-1} \sim g^{-1}$,

4) if $f \sim g$, h_1 is a path with $t(h_1) = s(f) = s(g)$, and h_2 is a path with $s(h_2) = t(f) = t(g)$, then $h_1 f h_2 \sim h_1 g h_2$.

A congruence $\sim$ on a connected oriented graph Γ determines a groupoid $G(\Gamma, \sim)$: the objects of $G(\Gamma, \sim)$ are the vertices, and the morphisms of $G(\Gamma, \sim)$ are the equivalence classes of paths.

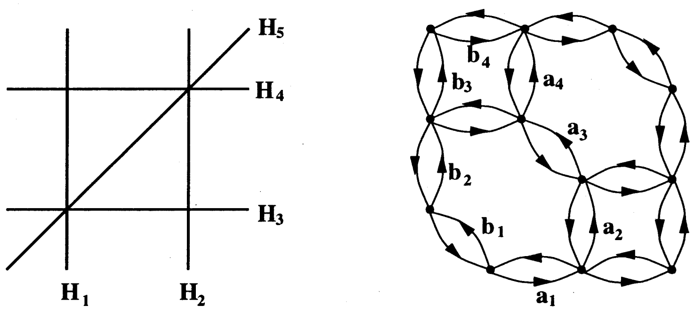
Let $\mathcal{A}$ be a (real) arrangement of hyperplanes. Now, we associate with $\mathcal{A}$ a connected oriented graph $\Gamma(\mathcal{A})$ and a congruence $\sim$ on $\Gamma(\mathcal{A})$, and we define the Deligne groupoid $G(\mathcal{A})$ associated with $\mathcal{A}$ to be $G(\Gamma(A), \sim)$.

DEFINITION. A *chamber* of $\mathcal{A}$ is a connected component of $M(\mathcal{A}) = V \backslash (\cup_{H \in \mathcal{A}} H)$. Call two chambers C and D *adjacent* if there exists exactly one hyperplane in $\mathcal{A}$ which separates C and D. Let $\Gamma(\mathcal{A})$ be the oriented graph whose vertices are the chambers, and whose arrows are the pairs (C, D) of adjacent chambers. Note that (C, D) and (D, C) are distinct arrows of $\Gamma(\mathcal{A})$, if C, D are adjacent chambers. Let $\sim$ be the smallest congruence on $\Gamma(\mathcal{A})$ satisfying: if α and β are both positive minimal paths with the same source and the same target, then $\alpha \sim \beta$. The *Deligne groupoid* of $\mathcal{A}$ is defined to be the groupoid $G(\mathcal{A}) = G(\Gamma(\mathcal{A}), \sim)$ associated with $\Gamma(\mathcal{A})$ and $\sim$. Note that, for two chamber C, D, there is a unique equivalence class of positive minimal paths with source C and target D. This class will be denoted by $\delta(C, D)$.

EXAMPLE. Consider the arrangement of lines $\mathcal{A} = \{H_1, \ldots, H_5\}$ drawn in Figure 1. Then $\Gamma(\mathcal{A})$ is the oriented graph also drawn in Figure 1. Let

$$\alpha = a_1 a_2 a_3 a_4 \ , \quad \beta = b_1 b_2 b_3 b_4 \ .$$

Then α and β are both positive minimal paths with the same source and the same target, thus $\alpha \sim \beta$.



A Γ(A)

FIGURE 1

THEOREM 2.1 (PARIS [Pa1], SALVETTI [Sa1]). *Let $\mathcal{A}$ be a (real) arrangement of hyperplanes. Then any vertex group of $G(\mathcal{A})$ is isomorphic to the fundamental group of $M(\mathcal{A}_{\mathbf{C}})$.*

The Deligne groupoid was first introduced, and Theorem 2.1 was proved, in [De] for simplicial arrangements.

DEFINITION. Let $\mathcal{A}$ be a (real) arrangement of hyperplanes. We say that $\mathcal{A}$ is *central* if all the hyperplanes of $\mathcal{A}$ contain the origin. We say further that $\mathcal{A}$ is *essential* if the intersection of all the elements of $\mathcal{A}$ is $\{0\}$. Call $\mathcal{A}$ *simplicial* if it is central and essential, and if all the chambers of $\mathcal{A}$ are cones over simplices.

Two results on simplicial arrangements are particularly interesting. The first one (Theorem 2.3) is due to Deligne [De], and the second one (Theorem 2.5) is due to Charney [Ch]. Many properties of the fundamental group of $M(\mathcal{A}_{\mathbf{C}})$ are derived from these theorems. The proofs of both are very close from the work of Garside [Ga] and Thurston [Th] on braid groups. They are both strongly based on the following lemma 2.2. Note that, by [Pa2], the conclusion of Lemma 2.2 is true if and only if $\mathcal{A}$ is a simplicial arrangement.

Let $\mathcal{A}$ be a (real) arrangement of hyperplanes. Let f, g be two positive paths of $\Gamma(\mathcal{A})$ with $s(f) = s(g)$. We say that f *begins* with g

if there exists a positive path h such that $s(h) = t(g)$, $t(h) = t(f)$, and $f \sim gh$. For a positive path f, let $\mathrm{Begin}(f)$ denote the set of positive minimal paths α such that f begins with α.

LEMMA 2.2 (DELIGNE [De]). *Let $\mathcal{A}$ be a simplicial arrangement of hyperplanes. For every positive path f of $\Gamma(\mathcal{A})$, there exists a (unique up to equivalence) positive minimal path α such that* $\mathrm{Begin}(f) = \mathrm{Begin}(\alpha)$. *In particular, f begins with α.*

A space M is called an *Eilenberg-MacLane space* if its universal cover is contractible. Such a space is specially interesting to study its fundamental group because the homologies of M and $\pi_1(M)$ are equal and, consequently, many topological properties of M reflect on $\pi_1(M)$. We refer to [Br] for more details on the subject.

THEOREM 2.3 (DELIGNE [De]). *Let $\mathcal{A}$ be a simplicial arrangement of hyperplanes. Then $M(\mathcal{A}_{\mathbf{C}})$ is an Eilenberg-MacLane space.*

COROLLARY 2.4. *Let $\mathcal{A}$ be a simplicial arrangement of hyperplanes.*
i) *$\pi_1(M(\mathcal{A}_{\mathbf{C}}))$ is torsion free.*
ii) *$\pi_1(M(\mathcal{A}_{\mathbf{C}}))$ has finite cohomological dimension.*
iii) *$H_*(\pi_1(M(\mathcal{A}_{\mathbf{C}})), \mathbf{Z})$ is torsion free (by* [OS]*).*

Automatic groups form a large class of groups which contains all finite groups, abelian groups, free groups, fundamental groups of compact hyperbolic manifolds, and, more generally, hyperbolic groups in Gromov's sense [GH]. On the other hand, if an automatic group is nilpotent, then it is virtually abelian. More generally, if a subgroup of a biautomatic group is nilpotent, then it is virtually abelian [GS]. Briefly, an automatic group is a group provided with an extra combinatorial structure which "controls" the words and their lengths in the group. Such a structure allows to compute the growth function of the group, gives isoperimetric inequalities, and furnishes algorithms to solve the word problem and, if the structure is biautomatic, to solve the conjugacy problem. A finite index subgroup of an automatic group "inherits" the automatic structure from the group. Conversely, if a finite index subgroup is automatic, then the automatic structure of the subgroup can be extended to the whole group. The theory of (bi)automatic groupoids is identical to the theory of (bi)automatic groups. In particular, an automatic groupoid has finitely many vertices, and every vertex group inherits the automatic structure from the groupoid. We refer to [ECH] for a general exposition on the subject.

A natural question is whether the Deligne groupoid $G(\mathcal{A})$ of a real arrangement $\mathcal{A}$ admits an automatic structure. This question has been solved by Charney [Ch] in the case of simplicial arrangements. This is the subject of the remainder of the section.

Now, we give a precise definition of a (bi)automatic groupoid and, after stating Charney's theorem, we show the automatic structure on $G(\mathcal{A})$ when $\mathcal{A}$ is a simplicial arrangement. We will notice that the definition of this automatic structure highly depends on Lemma 2.2 above.

Let A be a finite set (of letters). We write A^* for the free monoid generated by A. The elements of A will be called *words*.

DEFINITION. A *finite state automaton* is a quintuple $\mathcal{F} = (V, A, \mu, Y, v_0)$, where V is a finite set called *state set*, A is a finite set called the *alphabet*, $\mu : V \times A \to V$ is a function called the *transition function*, Y is a subset of V called the *accept state set*, and v_0 is an element of V called *start state*. For $v \in V$ and $f = x_1 \dots x_n \in A^*$ we define the state $\mu(v, f)$ inductively on n by:

$$\mu(v,f) = \begin{cases} v & \text{if } n = 0 \\ \mu(\mu(v, x_1 \dots x_{n-1}), x_n) & \text{if } n \geq 1 \end{cases}$$

Then

$$L_{\mathcal{F}} = \{f \in A^* \ ; \ \mu(v_0, f) \in Y\}$$

is called the *language recognized* by $\mathcal{F}$. A *regular language* is a language recognized by a finite state automaton.

DEFINITION. Let G be a groupoid. A set S of morphisms is called a *generating set* if every morphism of G is the composition of finitely many elements of S. The *length* of a morphism f (with repect to S), denoted by $\lg_S(f)$, is the shortest length of a word in S^* which represents f. Let f, g be two morphisms with the same source. The *distance* between f and g, denoted by $d_S(f, g)$, is the length of $f^{-1}g$.

REMARK. Let $\mathcal{A}$ be a real arrangement of hyperplanes and $\Gamma = \Gamma(\mathcal{A})$. Then $G(\mathcal{A})$ has a natural generating set: $A(\Gamma) \cup A^{-1}(\Gamma)$. However, we will see later that this is not the generating set used to define the automatic structure on $G(\mathcal{A})$ when $\mathcal{A}$ is a simplicial arrangement.

DEFINITION. Let G be a groupoid and S a generating set of G. For $f \in S^*$, we denote by $\bar{f}$ the morphism of G represented by f if it exists. A language L in S^* *represents* G if every element of L represents a morphism and every morphism is represented by an element of L. For $f = x_1 \dots x_n \in L$ and a positive integer t, we write $\bar{f}(t) = \overline{x_1 \dots x_t}$ if $1 \leq t \leq n$ and $\bar{f}(t) = \bar{f}$ if $t \geq n$. Let κ be a positive integer. We say that L has the *κ-fellow traveller property* if, for all $f, g \in L$ such that $\bar{f}$ and $\bar{g}$ have the same source, we have:

$$d_S(\bar{f}(t), \bar{g}(t)) \leq \kappa \cdot d_S(\bar{f}, \bar{g})$$

for all integer $t \geq 1$.

DEFINITION. A groupoid G is *automatic* if there exist a finite generating set S of G, a constant $\kappa \geq 1$, and a regular language L in S^*, such that L represents G and has the κ-fellow traveller property. If, in addition, the language L^{-1} in $(S^{-1})^*$, obtained by formally inverting the elements of L, also has the κ-fellow traveller property, then G is called *biautomatic.*

THEOREM 2.5 (CHARNEY [Ch]). *Let $\mathcal{A}$ be a simplicial arrangement of hyperplanes, and let $G(\mathcal{A})$ be the Deligne groupoid of $\mathcal{A}$. Then $G(\mathcal{A})$ is biautomatic.*

COROLLARY 2.6. *Let $\mathcal{A}$ be a simplicial arrangement of hyperplanes.*
i) $\pi_1(M(\mathcal{A}_{\mathrm{C}}))$ *is biautomatic.*
ii) $\pi_1(M(\mathcal{A}_{\mathrm{C}}))$ *has the conjugacy problem solvable.*
iii) $\pi_1(M(\mathcal{A}_{\mathrm{C}}))$ *has a quadratic isoperimetric inequality.*

Let $\mathcal{A}$ be a simplicial arrangement. We turn now to give the definition of the finite state automaton $\mathcal{F} = (V, S, \mu, Y, v_0)$ which determines the automatic structure on $G(\mathcal{A})$. We refer to [Ch] for the proof that both, the language L recognized by $\mathcal{F}$ and its inverse L^{-1}, have the 6 -fellow traveller property.

Let $\Gamma = \Gamma(\mathcal{A})$. Recall that the vertex set $V(\Gamma)$ is the set of chambers of $\mathcal{A}$. For $C, D \in V(\Gamma)$, we denote by $\delta(C, D)$ the (unique) equivalence class of positive minimal paths with source C and target D. We write $\Delta_C = \delta(C, -C)$ for all $C \in V(\Gamma)$. Then

$$S = \{\delta(C, D) \ ; \ C, D \in V(\Gamma)\} \cup \{\Delta_C^{-1} \ ; \ C \in V(\Gamma)\} \ .$$

It is true but non trivial that S generates $G(\mathcal{A})$. Note also that $\delta(C, C)$ is the identity morphism on C and lies in S for all $C \in V(\Gamma)$. We set

$$V = S \cup \{v_0, v_1\} \ , \quad Y = S \ ,$$

where v_0, v_1 are "abstract" states. v_0 is the start state and v_0, v_1 are the only non accept states. So, it remains to define the transition function $\mu : V \times S \to V$. We say that $\delta(C_0, C_1)\delta(C_1, C_2)$ is *normal* if, according to Lemma 2.2, $\delta(C_0, C_1)$ is the (unique) class of positive minimal paths such that

$$\mathrm{Begin}(\delta(C_0, C_1)\delta(C_1, C_2)) = \mathrm{Begin}(\delta(C_0, C_1)).$$

Now:

$$
\begin{array}{lll}
\mu(v_0, x) & = x & \text{for all } x \in S \\
\mu(v_1, x) & = v_1 & \text{for all } x \in S \\
\mu(\Delta_C^{-1}, \Delta_D^{-1}) & = \begin{cases} \Delta_D^{-1} \\ v_1 \end{cases} & \begin{array}{l} \text{if } D = -C \\ \text{otherwise} \end{array} \\
\mu(\Delta_C^{-1}, \delta(D_0, D_1)) & = \begin{cases} \delta(D_0, D_1) \\ v_1 \end{cases} & \begin{array}{l} \text{if } D_0 = C \text{ and } D_1 \neq -C \\ \text{otherwise} \end{array} \\
\mu(\delta(C_0, C_1), \Delta_D^{-1}) & = v_1 & \\
\mu(\delta(C_0, C_1), \delta(D_0, D_1)) & = \begin{cases} \delta(D_0, D_1) \\ \\ v_1 \end{cases} & \begin{array}{l} \text{if } C_1 = D_0 \text{ and } \delta(C_0, C_1)\delta(D_0, D_1) \\ \quad \text{is normal} \\ \text{otherwise} \end{array}
\end{array}
$$

§3. Fibration theorem

Let $\mathcal{A}$ be an arrangement of hyperplanes. The *intersection poset* of $\mathcal{A}$ is the set $L(\mathcal{A})$ of nonempty intersections of elements of $\mathcal{A}$, partially ordered by reverse inclusion. It admits a *rank* function defined by $r(X) = \mathrm{Codim} X$, for X in $L(\mathcal{A})$. The space V is the unique minimal element of $L(\mathcal{A})$, and, by [Sa1], all the maximal elements have the same rank. Call the arrangement $\mathcal{A}$ *essential* if the maximal elements are points. Let X, Y in $L(\mathcal{A})$. Their *meet* is defined to be $X \wedge Y = \cap\{H \in \mathcal{A}; X \cup Y \subseteq H\}$. If $X \cap Y \neq \emptyset$, their *join* is defined to be $X \vee Y = X \cap Y$.

Throughout this section, $\mathbf{K}$ is $\mathbf{C}$, $\mathcal{A}$ is a (complex) arrangement of hyperplanes in V, and X is a linear subspace which is not necessarily in $L(\mathcal{A})$.

We say that a hyperplane H of $\mathcal{A}$ is *parallel* to X if either $H \cap X = \emptyset$ or $X \subseteq H$. Consider the projection $p_X : V \to V/X$. If H is parallel to X, then $p_X(H)$ is a hyperplane of V/X. Let $\mathcal{A}/X = \{p_X(H); H \in \mathcal{A} \text{ and } H \text{ parallel to } X\}$. Then the projection p_X induces a projection $p_X : M(\mathcal{A}) \to M(\mathcal{A}/X)$.

Proposition 3.1 (Paris [Pa3]). *The projection $p_X : M(\mathcal{A}) \to M(\mathcal{A}/X)$ admits a cross-section $s_X : M(\mathcal{A}/X) \to M(\mathcal{A})$.*

Definition. Call $Y \in L(\mathcal{A})$ *horizontal* with respect to X if $p_X(Y) = V/X$. Let Hor_X denote the set of horizontal elements of $L(\mathcal{A})$. The *bad set* of $M(\mathcal{A}/X)$ is

$$B_X = \cup\{p_X(Y) \cap M(\mathcal{A}/X)\ ; Y \in L(\mathcal{A}) \setminus \mathrm{Hor}_X\}\ .$$

Theorem 3.2 (Paris [Pa3]). *Let*

$$N_X = M(\mathcal{A}/X) \setminus B_X\ , \quad M_X = p_X^{-1}(N_X) \cap M(\mathcal{A})\ .$$

Then the restriction $p_X : M_X \to N_X$ of p_X to M_X is a locally trivial C^∞ fibration.

REMARK. i) The restriction of s_X to N_X determines a cross-section $s_X : N_X \to M_X$ of the fibration.

ii) Let $y_0 \in N_X$, and let $z_0 = s_X(y_0) \in M_X$. The fiber of p_X containing z_0 is the complement of the arrangement $\mathcal{A}_{z_0}^X$ in $(z_0 + X)$ defined by

$$\mathcal{A}_{z_0}^X = \{(z_0 + X) \cap H \ ; \ H \in \mathcal{A} \text{ and } H \text{ not parallel to } X\} \ .$$

So, by [Hu, Ch. V, Prop. 6.2]:

COROLLARY 3.3. *The following sequence is exact and splits.*

$$1 \to \pi_1(M(\mathcal{A}_{z_0}^X), z_0) \to \pi_1(M_X, z_0) \to \pi_1(N_X, y_0) \to 1$$

Another direct consequence of Theorem 3.2 is, by [Pa3]:

COROLLARY 3.4. *The following sequence is exact and splits.*

$$\pi_1(M(\mathcal{A}_{z_0}^X), z_0) \to \pi_1(M(\mathcal{A}), z_0) \to \pi_1(M(\mathcal{A}/X), y_0) \to 1$$

Note that the morphism $\pi_1(M(\mathcal{A}_{z_0}^X), z_0) \to \pi_1(M(\mathcal{A}), z_0)$ is not injective in general.

DEFINITION. Assume that $\mathcal{A}$ is a (complex) central arrangement of hyperplanes. Call X in $L(\mathcal{A})$ *modular* if $X \wedge Y = X + Y$ for all Y in $L(\mathcal{A})$. Call $\mathcal{A}$ *supersolvable* if it is essential and there exists a chain $0 > X_1 > \cdots > X_l = V$ in $L(\mathcal{A})$ such that X_μ is modular and $\dim X_\mu = \mu$ for all $\mu = 1, \ldots, l$.

Theorem 3.2 is particularly interesting if X is modular, because of the following theorem.

THEOREM 3.5 (TERAO [Te]). *Let $\mathcal{A}$ be a central arrangement of hyperplanes, and let X be a modular element of $L(\mathcal{A})$. Then the bad set B_X is empty.*

COROLLARY 3.6. *Let $\mathcal{A}$ be a central arrangement of hyperplanes, and let X be a modular element of $L(\mathcal{A})$. Then $p_X : M(\mathcal{A}) \to M(\mathcal{A}/X)$ is a locally trivial C^∞ fibration.*

COROLLARY 3.7. *Let $\mathcal{A}$ be a central arrangement of hyperplanes, and let X be a modular element of $L(\mathcal{A})$. Then the following sequence is exact and splits.*

$$1 \to \pi_1(M(\mathcal{A}_{z_0}^X), z_0) \to \pi_1(M(\mathcal{A}), z_0) \to \pi_1(M(\mathcal{A}/X), y_0) \to 1$$

Falk and Proudfoot [FP] have independently proved Corollary 3.6 using the same argument as sketched below. Corollary 3.6 is a classical and well-known result in the case of a modular element of dimension 1 [Te]. One can easily verify in this particular case that each fiber is diffeomorphic to $\mathbf{C}$ minus $|\mathcal{A} \setminus \mathcal{A}_X|$ points; however, the existence of trivializing neighborhoods is not explicitly proved in [Te]. To prove the existence of trivializing neighborhoods, one can apply the techniques shown below or, maybe, use simpler arguments like those given by Fadell and Neuwirth in [FN].

The proof of Theorem 3.2 is an application of Thom's first isotopy lemma that we state now.

Let M be a C^∞ manifold, and let A be a subset of M. A C^∞ *Whitney prestratification* of A is a partition $\mathcal{P}$ of $\mathcal{A}$ into subsets, that are called *strata*, satisfying the following conditions:

1) each stratum is a C^∞ submanifold of M;

2) $\mathcal{P}$ is locally finite;

3) if $U, V \in \mathcal{P}$ are such that $\bar{U} \cap V \neq \emptyset$, then $V \subseteq \bar{U}$ (in that case we write $V < U$);

4) if $U, V \in \mathcal{P}$ are such that $V < U$, then (U, V) satisfies the *Whitney Condition (b)* defined in [Ma].

THEOREM 3.8 (MATHER [Ma]). *Let M, N be two C^∞ manifolds, let $f : M \to N$ be a C^∞ function, let A be a subset of M, and let $\mathcal{P}$ be a Whitney prestratification of A. Assume that the restriction $f|_A : A \to N$ is a proper map, and that the restriction $f|_U : U \to N$ is a submersion for every $U \in \mathcal{P}$. Then $f|_A : A \to N$ is a locally trivial C^0 fibration, and $f|_U : U \to N$ is a locally trivial C^∞ fibration for all $U \in \mathcal{P}$.*

We turn now to the proof of Theorem 3.2. Let X be a linear subspace. We assume that $X = \mathbf{C}^d$, $V/X = \mathbf{C}^{l-d}$, and $p_X : \mathbf{C}^d \times \mathbf{C}^{l-d} \to \mathbf{C}^{l-d}$ is the projection on the second coordinate. Let $\mathbf{P}^d$ denote the complex projective space of dimension d. Consider the embedding of $\mathbf{C}^d$ in $\mathbf{P}^d$, and still denote by $p_X : \mathbf{P}^d \times \mathbf{C}^{l-d} \to \mathbf{C}^{l-d}$ the projection on the second coordinate. The proof of Theorem 3.2 given in [FP] and [Pa3] consists on defining a Whitney prestratification on $\mathbf{P}^d \times N_X$ so that M_X is a stratum and the restriction of the projection $p_X : \mathbf{P}^d \times N_X \to N_X$ on each stratum is a submersion. $p_X : \mathbf{P}^d \times N_X \to N_X$ being obviously a proper map, by Theorem 3.8, it follows that $p_X : M_X \to N_X$ is a locally trivial C^∞ fibration.

We focus now on the family of supersolvable arrangements. So, from now on, $\mathcal{A}$ is supposed to be a (complex) supersolvable arrangement of hyperplanes.

First, notice that, iterating Corollaries 3.6 and 3.7, one obtains the following two theorems.

THEOREM 3.9 (TERAO [Te]). *$M(\mathcal{A})$ is an Eilenberg-MacLane space.*

THEOREM 3.10 (FALK and RANDELL [FR1]). *$\pi_1(M(\mathcal{A}))$ can be presented as*

$$\pi_1(M(\mathcal{A})) = F_1 \rtimes (F_2 \rtimes (\ldots (F_{l-1} \rtimes \mathbf{Z}) \ldots)) \ ,$$

where $F_1, \ldots, F_{l-1}$ are free groups.

Like for complexifications of simplicial arrangements, Theorem 3.9 implies:

COROLLARY 3.11. i) *$\pi_1(M(\mathcal{A}))$ is torsion free.*
ii) *$\pi_1(M(\mathcal{A}))$ has finite cohomological dimension.*
iii) *$H_*(\pi_1(M(\mathcal{A})), \mathbf{Z})$ is torsion free (by [OS]).*

It is known not only that $\pi_1(M(\mathcal{A}))$ can be written as an iterated semidirect product of free groups, but also that the succesive actions on the free groups are trivial at the homology level:

LEMMA 3.12. *Let $\pi_1(M(\mathcal{A})) = F_1 \rtimes (F_2 \rtimes (\ldots (F_{l-1} \rtimes \mathbf{Z}) \ldots))$ be the decomposition of Theorem 3.10. Then $F_{\mu+1} \rtimes (\ldots (F_{l-1} \rtimes \mathbf{Z}) \ldots)$ acts trivially on the homology of F_μ for all $\mu = 1, \ldots, l-1$.*

This last lemma is the key of the proof of many properties of $\pi_1(M(\mathcal{A}))$. Cohen and Suciu [CS2] have recently proved that the action of $F_{\mu+1} \rtimes (\ldots (F_{l-1} \rtimes \mathbf{Z}) \ldots)$ on F_μ is actually a "pure braid action", which is a stronger statement.

We focus now on one of these properties, the biordering, which is not so well-known, and refer to [FR2] and [FR3] for an exposition on the other properties of $\pi_1(M(\mathcal{A}))$ (LCS formula, rational $K(\pi, 1)$ property, Koszul property, etc...).

DEFINITION. Call a group G *biorderable* if there exists a total ordering $<$ on G such that $f < g$ implies $h_1 f h_2 < h_1 g h_2$ for all $h_1, h_2, f, g \in G$.

Recall that $\mathcal{A}$ denotes a complex supersolvable arrangement of hyperplanes. We turn now to show that $\pi_1(M(\mathcal{A}))$ is biorderable and explain some consequences of this fact. We refer to [MR] for a general exposition on biorderable groups.

Let G be a biorderable group. Say that $g \in G$ is *positive* if $g > 1$, and denote by P the set of positive elements. Then one has the disjoint union $G = P \sqcup P^{-1} \sqcup \{1\}$. Moreover, $P \cdot P \subseteq P$, and $gPg^{-1} = P$ for all $g \in G$. Conversely, these conditions imply that G is biorderable, namely:

PROPOSITION 3.13. *Let G be a group, and let $P \subseteq G$ be a subset such that $G = P \sqcup P^{-1} \sqcup \{1\}$ is a disjoint union, $P \cdot P \subseteq P$, and $gPg^{-1} = P$ for all $g \in G$. Then G is biorderable, the ordering being given by $g > f$ if $gf^{-1} \in P$.*

Now, consider an exact sequence

$$1 \to K \to G \xrightarrow{\phi} H \to 1 ,$$

and assume that K and H are both biorderable. Let P_K and P_H denote the sets of positive elements of K and H, respectively. Say that $g \in G$ is positive if either $\phi(g) \in P_H$, or $\phi(g) = 1$ (namely, $g \in K$) and $g \in P_K$. Let P denote the set of positive elements. We clearly have the disjoint union $G = P \sqcup P^{-1} \sqcup \{1\}$ and the inclusion $P \cdot P \subseteq P$. Moreover, we have $gPg^{-1} = P$ for all $g \in G$ if and only if we have $gP_Kg^{-1} = P_K$ for all $g \in G$. This last condition holds if K is central, so:

PROPOSITION 3.14. *Let*

$$1 \to K \to G \to H \to 1$$

be an exact sequence such that K and H are both biorderable and K is central in G. Then G is also biorderable.

DEFINITION. Let G be a group. For two subgroups A, B of G, let $[A, B]$ denote the subgroup generated by $\{aba^{-1}b^{-1}; a \in A \text{ and } b \in B\}$. The subgroups G_n of the *lower central series* of G are defined recursively by

$$G_1 = G, \quad G_{n+1} = [G_n, G] \quad n = 1, 2, \dots$$

A group G for which $\cap_{n=1}^{\infty} G_n = \{1\}$ and G_n/G_{n+1} is torsion free for all n, is called *residually nilpotent without torsion.*

An important result obtained with Lemma 3.12 is:

THEOREM 3.15 (FALK and RANDELL [FR1]). *$\pi_1(M(\mathcal{A}))$ is residually nilpotent without torsion.*

Now, $\pi_1(M(\mathcal{A}))$ is biorderable because of the following.

PROPOSITION 3.16. *Let G be a residually nilpotent without torsion group. Then G is biorderable.*

Proof. We first prove that G/G_n is biorderable by induction on n. Consider the exact sequence

$$1 \to G_n/G_{n+1} \to G/G_{n+1} \to G/G_n \to 1 .$$

The group G_n/G_{n+1} is a free abelian group thus is biorderable (take, for example, the lexicographic order), G/G_n is biorderable by induction hypothesis, and G_n/G_{n+1} is central in G/G_{n+1}, thus G/G_{n+1} is biorderable by Proposition 3.14.

Now, call $g \in G$ positive if there exists some $n \geq 1$ such that the class $[g] \in G/G_n$ of g is not the identity and is positive. By the definition of the ordering on G/G_n, this definition does not depend on the choice of n. Let P denote the set of positive elements. The condition $\cap_{n=1}^{\infty} G_n = \{1\}$ implies that $G = P \sqcup P^{-1} \sqcup \{1\}$ is a disjoint union. Moreover, one can easily verify that $P \cdot P \subseteq P$ and $gPg^{-1} = P$ for all $g \in G$. Q.E.D.

An alternative proof of the fact that $\pi_1(M(\mathcal{A}))$ is biorderable can be found in [KR]. There exists a "natural" ordering on any finitely generated free group called *Magnus ordering.* The key of the proof of Kim and Rolfsen is the following lemma.

LEMMA 3.17 (KIM and ROLFSEN [KR]). *Let F be a finitely generated free group, let P be the set of positive elements of F with respect to the Magnus ordering, and let $\alpha \in \mathrm{Aut}(F)$ which acts trivially on the homology of F. Then $\alpha(P) = P$.*

We turn now to give this alternative proof. Let $\pi_1(M(\mathcal{A})) = F_1 \rtimes (F_2 \rtimes (\ldots(F_{l-1} \rtimes \mathbf{Z})\ldots))$ be the decomposition of $\pi_1(M(\mathcal{A}))$ of Theorem 3.10. Write $H_\mu = F_\mu \rtimes (\ldots(F_{l-1} \rtimes \mathbf{Z})\ldots)$ for $\mu = 1, \ldots, l$. It clearly suffices to show that $H_{\mu-1}$ is biorderable if H_μ is biorderable. Let $g \in H_{\mu-1}$. Since $H_{\mu-1} = F_{\mu-1} \rtimes H_\mu$, g can be uniquely written in the form $g = g_1 g_2$, where $g_1 \in F_{\mu-1}$ and $g_2 \in H_\mu$. Say that g is positive if either g_2 is positive, or $g_2 = 1$ and g_1 is positive with repect to the Magnus ordering. Let P denote the set of positive elements of $H_{\mu-1}$. The fact that H_μ acts trivially on the homology of $F_{\mu-1}$ (Lemma 3.12) and Lemma 3.17 imply that $gPg^{-1} = P$ for all $g \in H_{\mu-1}$. The disjoint union $H_{\mu-1} = P \sqcup P^{-1} \sqcup \{1\}$ and the inclusion $P \cdot P \subseteq P$ are obvious.

We turn now to investigate two properties of biorderable groups. The first one says that a biorderable group has no generalized torsion, and the second one says that the group ring of a biorderable group has no zero divisors.

DEFINITION. Let G be a group. An element $g \in G$ is said to be a *generalized torsion element* if there exist $h_1 \ldots, h_r \in G$ such that

$$(h_1 g h_1^{-1})(h_2 g h_2^{-1}) \ldots (h_r g h_r^{-1}) = 1 \ .$$

PROPOSITION 3.18. *A biorderable group contains no generalized torsion elements.*

Proof. Let $g \in G$, $g \neq 1$. Let $h_1, \dots, h_r \in G$. Either $g > 1$ or $g < 1$. Assume $g > 1$. Then $(h_i g h_i^{-1}) > 1$ for all $i = 1, \dots, r$, thus

$$1 < (h_1 g h_1^{-1})(h_2 g h_2^{-1}) \dots (h_r g h_r^{-1}) \neq 1 \ . \qquad \text{Q.E.D.}$$

Proposition 3.18 says in particular that biorderable groups are torsion free.

PROPOSITION 3.19. *Let G be a biorderable group. Then $\mathbf{Z}G$ contains no zero divisors.*

Proof. Let α, β be non zero elements of $\mathbf{Z}G$. We write

$$\alpha = a_1 g_1 + \dots + a_p g_p \ , \quad \beta = b_1 f_1 + \dots + b_q f_q \ ,$$

where $a_1, \dots, a_p, b_1, \dots, b_q \in \mathbf{Z} \setminus \{0\}$, $g_1, \dots, g_p, f_1, \dots, f_q \in G$, $g_1 < g_2 < \dots < g_p$, and $f_1 < f_2 < \dots < f_q$. Then

$$\begin{aligned} &\alpha\beta = \sum_{i,j} (a_i b_j)(g_i f_j) \ , \\ &g_1 f_1 < g_i f_j \quad \text{if } i \neq 1 \text{ or } j \neq 1 \ , \\ &a_1 b_1 \neq 0 \ , \end{aligned}$$

thus $\alpha\beta \neq 0$. Q.E.D.

Let G be a biorderable group. We pointed out before that G is then torsion free because it has no generalized torsion elements. The fact that $\mathbf{Z}G$ has no zero divisors also implies that G is torsion free. Indeed, if g is a torsion element of a group G (say of order k), then

$$(1 - g)(1 + g + g^2 + \dots + g^{k-1}) = 1 - g^k = 0 \ ,$$

thus $(1 - g)$ is a zero divisor. It is not known whether $\mathbf{Z}G$ has no zero divisor in general if G is torsion free.

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