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Recent progress of intersection theory for twisted (co)homology groups

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§1. Introduction

Maybe you have ever seen at least one of the following formulae:

$$B(p,q)B(-p,-q) = \frac{2\pi i(p+q)}{pq}\frac{1-e^{2\pi i(p+q)}}{(1-e^{2\pi ip})(1-e^{2\pi iq})},$$

$$\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin \pi p}, \quad \left(\int_{-\infty}^{\infty} e^{-t^2/2}dt\right)^2 = 2\pi,$$

where

$$B(p,q) := \int_0^1 t^p(1-t)^q \frac{dt}{t(1-t)}, \quad \Gamma(p) := \int_0^\infty t^p e^{-t}\frac{dt}{t}$$

are the Gamma and the Beta functions.

In this paper, we give a geometric meaning for these formulae: If one regards such an integral as the dual pairing between a (kind of) cycle and a (kind of) differential form, then the value given in the right hand side of each formula is the product of the *intersection* numbers of the two cycles and that of the two forms appeared in the left-hand side.

Of course the intersection theory is not made only to *explain* these well known formulae; for applications, see [CM], [KM], [Y1].

§2. Twisted (co)homology groups

Let $l_1, \ldots, l_{n+1}$ be polynomials of degree 1 in $t_1, \ldots, t_k$, $(n \geq k \geq 1)$ and $\alpha_1, \ldots, \alpha_{n+1}$ be complex numbers satisfying

Assumption 1. $\alpha_j \notin \mathbb{Z}, \quad \alpha_0 := -\alpha_1 - \cdots - \alpha_{n+1} \notin \mathbb{Z}.$

Put

$$
\begin{aligned}
L_j &= \text{ hyperplane defined by } l_j, \quad j = 1, \dots, n+1, \\
T &= \mathbb{C}^k - \cup_{j=1}^{n+1} L_j \\
&= \mathbb{P}^k - \cup_{j=0}^{n+1} L_j, \quad L_0 : \text{ hyperplane at infinity,} \\
u &= \prod_{j=1}^{n+1} l_j^{\alpha_j} : \quad \text{multi-valued function on } T,
\end{aligned}
$$

$\mathcal{L}$, $\check{\mathcal{L}}$: local systems caused by u^{-1} and u, respectively,

$$
\begin{aligned}
\omega &= \sum_{j=1}^{n} \alpha_j \frac{dl_j}{l_j} : \quad \text{single-valued 1-form on } T, \\
\nabla &= d + \omega\wedge, \quad \check{\nabla} = d - \omega\wedge : \quad \text{derivations.}
\end{aligned}
$$

Assumption 2. No $k+1$ hyperplanes in $\{L_j\}_{j=0}^{n+1}$ intersect in $\mathbb{P}^k$.

Denoting the k-dimensional cohomology groups (with compact support) and the (locally finite) homology groups by the usual symbols, we have the three natural dual parings (explained below):

$$
\begin{array}{ccc}
H_c^k(T, \mathcal{L}) & \longleftrightarrow & H^k(T, \check{\mathcal{L}}) \\
\updownarrow & & \updownarrow \\
H_k^{lf}(T, \check{\mathcal{L}}) & & H_k(T, \mathcal{L}).
\end{array}
$$

All other dimensional (co)homology groups vanish. By de Rham's theorem, cohomology classes can be represented by smooth global forms:

$$
H_c^k(T, \mathcal{L}) \cong H^k(\mathcal{E}_c^\bullet, \nabla), \quad H^k(T, \check{\mathcal{L}}) \cong H^k(\mathcal{E}^\bullet, \check{\nabla}),
$$

where $\mathcal{E}^p$ and $\mathcal{E}_c^p$ are spaces of smooth p-forms on T and those with compact support. Through these isomorphisms, the columns in the above diagram can be realized by the integration

$$
\langle \varphi, \delta \rangle := \int_\delta \varphi u, \quad \text{or} \quad \langle \psi, \gamma \rangle := \int_\gamma \psi u^{-1}
$$

of k-forms along k-cycles, where

$$
\varphi \in H_c^k(T, \mathcal{L}), \quad \delta \in H_k^{lf}(T, \check{\mathcal{L}}), \quad \text{or} \quad \psi \in H^k(T, \check{\mathcal{L}}), \quad \gamma \in H_k(T, \mathcal{L}),
$$

respectively. Such an integration is often called a *hypergeometric integral* (HG integral for short) because if one let the hyperplanes L_j move then the integral defines a hypergeometric function of type $(k+1, n+2)$. When $k = 1, n = 2$ this is indeed the Gauss hypergeometric function.

The first row is the *intersection form for cohomology groups*, and can be represented by the integral

$$\varphi\cdot\psi := \int_T (\varphi u)\wedge(\psi u^{-1}) = \int_T \varphi\wedge\psi$$

of $2k$-forms over T, where $\varphi \in H^k(\mathcal{E}_c^\bullet,\nabla)$, $\psi \in H^k(\mathcal{E}^\bullet,\check{\nabla})$. (N.B. In [KY1], $\psi\wedge\varphi$ is used in place of $\varphi\wedge\psi$.)

Now these three pairings induce the Poincaré isomorphisms:

$$H_c^k(T,\mathcal{L}) \cong H_k(T,\mathcal{L}), \qquad H^k(T,\check{\mathcal{L}}) \cong H_k^{lf}(T,\check{\mathcal{L}}).$$

Thus through these two isomorphisms the intersection form for cohomolog groups induces the dual pairing, called the *intersection form for homology groups*, of the two homology groups. In this way we have the four *compatible* pairings:

$$\begin{array}{ccccl} H_c^k(T,\mathcal{L}) & \longleftrightarrow & H^k(T,\check{\mathcal{L}}) & & \text{intersection form} \cdot \text{for coh.} \\ \updownarrow & & \updownarrow & & \\ H_k^{lf}(T,\check{\mathcal{L}}) & \longleftrightarrow & H_k(T,\mathcal{L}) & & \text{intersection form} \cdot \text{for hom.} \end{array}$$

HG integral HG integral

Let us take bases as

$$\begin{array}{ll} \varphi^i \in H_c^k(T,\mathcal{L}), & \psi^i \in H^k(T,\check{\mathcal{L}}), \\ \delta_i \in H_k^{lf}(T,\check{\mathcal{L}}), & \gamma_i \in H_k(T,\mathcal{L}). \end{array}$$

Denoting the matrix $(\langle\varphi^i,\delta_j\rangle)_{ij}$ by $(\langle\varphi,\delta\rangle)$ and $(\delta_i\cdot\gamma_j)_{i,j}$ by $(\delta\cdot\gamma)$, we have

$$(\varphi\cdot\psi) = (\langle\varphi,\delta\rangle)(\gamma\cdot\delta)^{-1}\,{}^t(\langle\psi,\gamma\rangle),$$

which gives quadratic relations among the HG integrals.

Note that up to now we presented abstract nonsense which is valid for any complex manifold and for any local system. Our task is, for the special T and $\mathcal{L}$ given above, to pick a suitable basis of each (co)homology group and evaluate the intersection numbers.

§3. Intersection form for cohomology groups

To pick an explicit basis of the cohomology groups, holomorphic forms or possibly algebraic forms are better. Recall the isomorphisms, due to comparison theorems,

$$\begin{aligned} H^k(T,\mathcal{L}) &\cong H^k(\mathcal{E}^\bullet,\nabla) \cong H^k(\Omega^\bullet,\nabla) \\ &\cong H^k(\Omega^\bullet(*L),\nabla) \cong H^k(\Omega^\bullet(\log L),\nabla), \end{aligned}$$

where Ω^p, $\Omega^p(*L)$ and $\Omega^p(\log L)$ are spaces of holomorphic forms on T, algebraic forms and logarithmic forms with poles only along $\cup_{j=0}^{n+1} L_j$, respectively.

For a multi-index $I = (i_0, \dots, i_k)$, $0 \le i_0 < \cdots < i_k \le n+1$, we define a logarithmic k-form

$$\varphi_I = \frac{dl_{i_0}}{l_{i_1}} \wedge \cdots \wedge \frac{dl_{i_{k-1}}}{l_{i_k}}.$$

For example, the following $\binom{n}{k}$ forms give a basis of $H^k(\Omega^\bullet(\log L), \nabla)$:

$$\varphi_I, \qquad i_0 = 0 < i_1 < \cdots < i_k \le n.$$

It is known (e.g. [DM]) and easy to prove, under Assumption 1, the isomorphism

$$H_c^k(T, \mathcal{L}) \cong H^k(T, \mathcal{L}).$$

Thus together with the isomorphism $H^k(T,\mathcal{L}) \cong H^k(\Omega^\bullet(\log L), \nabla)$ above, we can let φ_I represent also an element of $H_c^k(T, \mathcal{L})$. We wish to evaluate the intersection numbers of these forms. The key point is to represent the isomorphism

$$\iota : H^k(\Omega^\bullet(\log L), \nabla) \xrightarrow{\cong} H^k(\mathcal{E}_c^\bullet, \nabla) \quad (\cong H_c^k(T, \mathcal{L})),$$

explicit enough so that the $2k$-dimensional integral

$$\int \iota(\varphi_I) \wedge \varphi_J$$

is computable. This can be done and we get

Theorem 1. *The intersection number $\varphi_I \cdot \varphi_J$ of*

$$\varphi_I \in H_c^k(T, \mathcal{L}) \quad \text{and} \quad \varphi_J \in H^k(T, \check{\mathcal{L}}),$$

where $I = \{i_0, \dots, i_k\}$, $0 \le i_0 < \cdots < i_k \le n+1$, $J = \{j_0, \dots, j_k\}$, $0 \le j_0 < \cdots < j_k \le n+1$, is equal to the (I, J)-minor of the tri-diagonal symmetric matrix

$$Int_{coh}(\alpha) = 2\pi\sqrt{-1} \begin{pmatrix} 1/\alpha_0 + 1/\alpha_1 & 1/\alpha_1 & 0 & \cdots \\ 1/\alpha_1 & 1/\alpha_1 + 1/\alpha_2 & 1/\alpha_2 & \\ 0 & 1/\alpha_2 & 1/\alpha_2 + 1/\alpha_3 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Actual value of $\varphi_I \cdot \varphi_J$ is given as follows:

$$(2\pi\sqrt{-1})^k \frac{\sum_{i\in I}\alpha_i}{\prod_{i\in I}\alpha_i} \quad \text{if} \quad I = J,$$
$$(2\pi\sqrt{-1})^k \frac{(-1)^{\mu+\nu}}{\prod_{i\in I\cap J}\alpha_i} \quad \text{if} \quad \#(I\cap J) = k,$$

0, otherwise, where μ and ν are determined by $\{i_\mu\} = I - J$ and $\{j_\nu\} = J - I$.

Though there are technical difficulties for general k, the essential idea of the proof can be seen from that of the case $k = 1$. So we prove this theorem only when $k = 1$, and when $k \geq 2$ we describe where the difficulty lies and how we can manage.

3.1. Proof of Theorem 1 when $k = 1$. We express the image $\iota(\varphi_I)$ explicitly. We find a smooth function f on T such that $\varphi_I - \nabla f$ is compactly supported. This means that $\varphi_I - \nabla f$ represents the class $\iota(\varphi_I)$ of $H^1_c(T, \mathcal{L})$.

We can find a convergent power series f_p centered at the point L_p satisfying $\nabla f_p = \varphi_I$. Let h_p be a smooth real function on $\mathbb{P}^1$ such that $h_p(t) = 0$ $(t \notin U_p)$, $0 < h_p(t) < 1$ $(t \in U_p \setminus V_p)$, $h_p(t) = 1$ $(t \in V_p)$, where $L_p \in V_p \subset U_p$, and U_p is a small neighborhood of L_p. Regarding $f := \sum_{p=0}^{n+1} h_p f_p$ as defined on T, we have

$$\varphi_I - \nabla f = \varphi_I - \sum_{p=0}^{n+1}[h_p\nabla(f_p) + f_p dh_p] = \sum_{p=0}^{n+1}[(1-h_p)\varphi_I - f_p dh_p],$$

which is of compact support on T. The Stokes theorem and the residue theorem yields

$$\int_T \iota(\varphi_I)\wedge\varphi_J = \sum_{p=0}^{n+1}\int_T [(1-h_p)\varphi_I - f_p dh_p]\wedge\varphi_J$$
$$= \sum_{p=0}^{n+1}\int_{U_p\setminus V_p} -f_p dh_p \wedge \varphi_J = \sum_{p=0}^{n+1}\int_{\partial(U_p\setminus V_p)} -h_p f_p \varphi_J$$
$$= \sum_{p=0}^{n+1}\int_{\partial V_p} f_p\varphi_J = 2\pi\sqrt{-1}\sum_{p=0}^{n+1}\mathrm{Res}_{L_p}(f_p\varphi_J).$$

Completion of the proof is now immediate.

3.2. Strategy for $k \geq 2$. We prepare some notation. Let L_{P^q} be the intersection of $L_{p_1}, L_{p_2}, \dots, L_{p_q}$, and let U_{P^q} be a small tubular neighborhood of L_{P^q} in $\mathbb{P}^k$, where P^q is a multi-index with cardinality q, say,

$$P^q = \{p_1, p_2 \dots, p_q\}, \quad 0 \leq p_1 < p_2 < \cdots < p_q \leq n+1.$$

For multi-indices P^{q-1} and P^q, if $P^{q-1} \subset P^q$, then we put

$$\delta(P^{q-1}; P^q) = (-1)^r, \quad \text{where} \quad \{p_r\} := P^q \setminus P^{q-1}.$$

Step 1. Construct a system of holomorphic $(k-q)$-forms f_{P^q} on $U_{P^q} \cap T$ such that

$$\begin{aligned} \nabla(f_{P^1}) &= \varphi_I, \\ \nabla(f_{P^q}) &= \sum_{P^{q-1} \subset P^q} \delta(P^{q-1}; P^q) f_{P^{q-1}} \quad (2 \leq q \leq k); \end{aligned}$$

these can be obtained as convergent power series. Complexity lies on the fact that the singularities $\cup L_{p=j}$ are not isolated.
Step 2. By patching f_{P^q} inductively by the help of partition of the unity on $\cup_{j=0}^{n+1} U_j$, we get a smooth $(k-1)$-form f on T such that

$$\nabla f = \varphi_I \quad \text{in} \quad \cup_{j=0}^{n+1} U_j.$$

Since $\varphi_I - \nabla f$ is of compact support on T and is cohomologous to φ_I in $H^k(\mathcal{E}, \nabla)$, it represents $\iota(\varphi_I)$.
Step 3. Repeated use of the Stokes theorem and the residue theorem leads to

$$\begin{aligned} \int_T \iota(\varphi_I) \wedge \varphi_J &= \int_T -df \wedge \varphi_J \\ &= (2\pi\sqrt{-1})^k \sum_{P^k} \mathrm{Res}_{L_{P^k}}(f_{P^k} \varphi_J), \end{aligned}$$

which will imply the theorem.

§4. Intersection form for homology groups

Since we assumed that our hyperplane arrangement is in general position (Assumption 2), we can continuously deform the arrangement, keeping its intersection pattern, into a real arrangement, by which we mean all the linear forms l_j are defined over the real numbers. So we assume that our arrangement is real.

Note that there are many arrangements not in general position that one can not deform into a real one.

Let $T_{\mathbb{R}}$ be the real locus of T. $\binom{n}{k}$ bounded chambers support cycles forming a basis of $H_k^{lf}(T, \check{\mathcal{L}})$. One can load any branch of u on the chambers; too much freedom annoys us. In order to make it in a systematic way, we further deform the arrangement and put the hypersurfaces in a specially nice way. Then the k-dimensional cases can be reduced to the simplest case $k = 1$.

Loaded cycles: We represent elements of $H_p(T, \check{\mathcal{L}})$ by *loaded p-cycles*, which is convenient here and will be indispensable in §8.2. A loaded p-chain is a formal sum of loaded p-simplexes. A loaded p-simplex is a topological simplex on which a branch of u is assigned. The boundary operator is naturally defined. For example, the boundary of a loaded path (1-chain) is given by

(ending point loaded with the value of the function there)

$-$(starting point loaded with the value of the function there).

The boundary of a higher dimensional loaded chain is defined in an obvious way. A loaded p-chain is called a *loaded p-cycle* if its boundary vanishes.

4.1. Case $k = 1$. Let $x_1, \ldots, x_{n+1}$ be distinct real points on $\mathbb{P}^1$ satisfying $x_1 < \cdots < x_{n+1}$. Then the multi-valued function

$$u = \prod_{j=1}^{n} l_j^{\alpha_j}, \qquad l_j = t - x_j$$

is defined on $T = \mathbb{P}^1 - \{x_1, \ldots, x_{n+1}, x_0 = \infty\}$. On each oriented interval $\overrightarrow{(x_p, x_{p+1})}$, we load a branch of the function u determined by

$$\arg(t - x_j) = \begin{cases} 0 & j \leq p, \\ -\pi & p+1 \leq j, \end{cases}$$

and call this loaded path $\check{I}_p$. Note that if you analytically continue the branch of u corresponding to some loaded path $\check{I}_j$ through the *lower* half part of the t-plane T, then you get the branches of u corresponding to other loaded paths $\check{I}_i$. But if you do the same starting from a point in (x_j, x_{j+1}), passing through the *upper* part and ending at a point in (x_{j-1}, x_j), you get

$$c_j := e^{2\pi i \alpha_j}$$

times the branch u corresponding to the loaded paths $\check{I}_{j-1}$.

Anyway, $\check{I}_j$ represent elements of $H_1^{lf}(T, \check{\mathcal{L}})$. For example, n non-compact loaded cycles $\check{I}_1, \dots, \check{I}_n$ form a basis. Loading u^{-1} in place of u, we get non-compact loaded cycles I_j; for example, $I_1, \dots, I_n$ form a basis of $H_1^{lf}(T, \mathcal{L})$.

As we did in §3, to define intersection numbers, we must make a compact counterpart $\text{reg}I_j$, regularization of I_j. This can be done by attaching two circles at the ends:

$$-\frac{c_j}{d_j}\, C_\epsilon^j + \overrightarrow{(x_j+\varepsilon, x_{j+1}-\varepsilon)} + \frac{c_{j+1}}{d_{j+1}}\, C_{-\epsilon}^{j+1}, \quad d_j := c_j - 1,$$

where $C_{\pm\epsilon}^j$ is the positively oriented circle of radius $\epsilon > 0$ center at x_j starting at $x_j \pm \epsilon$ (see Figure 1), and by loading u^{-1} along the three paths, where the branch of u^{-1} at each starting point is that of I_j. Note that $\text{reg}I_j$ is homologous to I_j in $H_1^{lf}(T, \mathcal{L})$. $\text{reg}I_1, \dots, \text{reg}I_n$ form a basis of $H_1(T, \mathcal{L})$.

Let us evaluate the intersection number $\text{reg}I_i \cdot \check{I}_j$. As is explained in §2, the definition is made through the intersection number of cohomology groups; it is a, so to speak, indirect *analytic* definition. In the following, we give a direct it topological definition, by which one can evaluate intersection numbers explicitly. These two definitions agree (see [KY1]); this fact will be referred to the *compatibility of intersection forms for homology and cohomology groups.*

Deform the support of $\check{I}_j$ so that it intersects transversally with that of $\text{reg}I_i$; any deformation will do. At each intersection point of the two supports, multiply the values of the two functions loaded to make the local intersection number at this point. Then sum up all the local intersection numbers, and finally change the sign to get $\text{reg}I_i \cdot \check{I}_j$ (see Figure 1). Here is an actual computation:

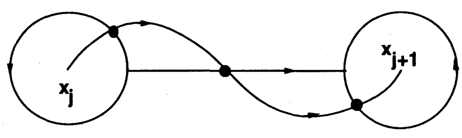

FIG 1. Intersection of $regI_j$ and I_j

$$(regI_j)\cdot \check{I}_j = -\left(\frac{c_j}{d_j} - 1 + \frac{c_{j+1}}{d_{j+1}}\right) = -\left(\frac{d_{j,j+1}}{d_j d_{j+1}}\right),$$

$$(regI_j)\cdot \check{I}_{j-1} = \frac{1}{d_j}, \qquad (regI_{j-1})\cdot \check{I}_j = \frac{c_j}{d_j},$$

0, otherwise, where $d_{ij} = c_i c_j - 1$. Therefore the *intersection matrix* $Int_{hom}(\alpha) = (regI_i \cdot \check{I}_j)_{ij}$ is given by the following tri-diagonal matrix

$$Int_{hom}(\alpha) = -\begin{pmatrix} d_{12}/d_1 d_2 & -c_2/d_2 & 0 & \cdots \\ -1/d_2 & d_{23}/d_2 d_3 & -c_3/d_3 & \ddots \\ 0 & -1/d_3 & d_{34}/d_3 d_4 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

(N.B. The intersection matrix in [KY1] is given by $-{}^t Int_{hom}(\alpha) = Int_{hom}(-\alpha)$ according to the definition of the intersection form for cohomology groups made there (cf. §2).)

4.2. Case $k \geq 2$. For given $n+1$ real points on $\mathbb{C}$

$$x_1 < \cdots < x_j < \cdots < x_{n+1}, \quad x_0 = \infty,$$

we define $n+1$ real hyperplanes $L_1, \ldots, L_n$ in $t = (t_1, \ldots, t_k)$-space by

$$l_j := t_r + (-x_j)t_{r-1} + \cdots + (-x_j)^{r-1} t_1 + (-x_j)^r, \quad 1 \leq j \leq n,$$

and L_0 the hyperplane at infinity. This arrangent $\{L_0, \ldots, L_n\}$ is called a Veronese arrangement, since an embedding of $\mathbb{P}^1$ into $\mathbb{P}^k$ by

$$t_0 = s^k,\ t_1 = s^{k-1}, \ldots, t_{k-1} = s,\ t_k = 1$$

is called the Veronese embedding. When $k = 2$ and $n = 4$, the arrangement is illustrated in Figure 2. Set

$$U = \prod_{j=1}^{n} l_j(t)^{\alpha_j},$$

where $l_j(t)$ is the linear form in t just defined above. For a multi-index,

$$I = (i_1, \ldots, i_k), \quad 1 \leq i_1 < \cdots < i_k \leq n,$$

we define loaded cycles $D_I \in H_k^{lf}(T, \mathcal{L})$ and $\check{D}_I \in H_k^{lf}(T, \check{\mathcal{L}})$ with support on the chamber (see Figure 2)

$$|D_I| = \{t \in T_{\mathbb{R}} \mid (-1)^{P(j)} l_j(t) > 0, \quad 1 \leq j \leq n\},$$

loaded with U^{-1} and U, respectively, with

$$\arg l_j = -P(j)\pi, \quad 1 \leq j \leq n,$$

where $P(j)$ denotes the cardinality of $\{p \mid i_p < j\}$. Since each loaded

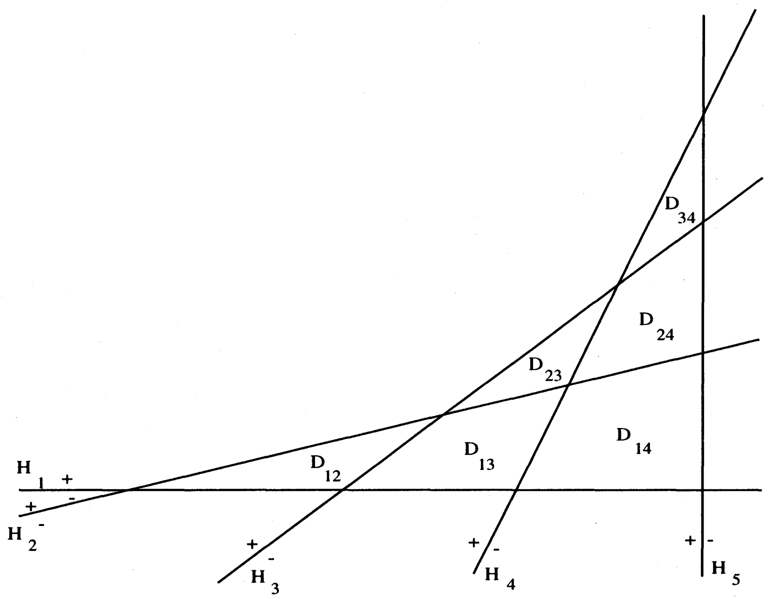

FIG 2. A Veronese arrangement ($k = 2,\ n = 4$) and the chambers

cycle is locally a direct product of 1-dimensional cycles, the regularizations $\mathrm{reg}D_I \in H_k(T, \mathcal{L})$ are naturally defined. We now state the result, which is very similar to Theorem 1.

Theorem 2. *For multi-indices $I = (i_1 \dots i_k)$, $1 \le i_1 < \dots < i_k \le n$, $J = (j_1 \dots j_k)$, $1 \le j_1 < \dots < j_k \le n$, the intersection number $regD_I \cdot \check{D}_J$ is equal to the (I, J)-minor of the matrix $Int_{hom}(\alpha)$.*

For rigorous proofs, see [KY2]. This theorem can be naturally understood if you write

$$D_J = I_{j_1} \wedge \cdots \wedge I_{j_k}, \quad J = (j_1, \dots, j_k)$$

which is justified in [IK2].

§5. Quadratic relations

As we pointed out at the end of §2 (see also the middle of §4.1), the compatibility of the intersection forms for homology groups and cohomology groups, which is a general, universal and abstract equality, produces explicit quadratic relations among hypergeometric integrals — twisted analogues of the Riemann equality for periods.

The simplest example is the one in §1

$$B(p,q)B(-p,-q) = \frac{2\pi i(p+q)}{pq} \cdot \frac{1-e^{2\pi i(p+q)}}{(1-e^{2\pi ip})(1-e^{2\pi iq})}.$$

Now we know the meaning of the right-hand side: It is the product of the intersection number of the forms

$$\frac{dt}{t(1-t)} \in H^1(\Omega^\bullet(\log L), \nabla) \quad \text{and}$$
$$\frac{dt}{t(1-t)} \in H^1(\Omega^\bullet(\log L), \check{\nabla}), \quad L = \{0,1,\infty\}$$

and that of the cycles

$$(0,1)\otimes u^{-1} \in H_1(T,\mathcal{L}) \quad \text{and} \quad (0,1)\otimes u \in H_1(T,\check{\mathcal{L}}), \quad u := t^p(1-t)^q.$$

Here is another example due to Gauss:

$$\begin{aligned} &F(a,b,c;x)F(1-a,1-b,2-c;x) \\ &\quad = \; F(a+1-c,b+1-c,2-c;x)F(c-a,c-b,c;x), \end{aligned}$$

where F is the hypergeometric function (cf. [CM], [Mat1]).

Twisted analogues of Riemann inequality. When $\alpha_j \in \mathbb{R}$, we can speak about the Hodge structure on the cohomology groups, and get twisted analogues of Riemann inequality. [HY] studies these when $k=1$.

§6. Further study

So far, we worked on the projective spaces $\mathbb{P}^k$, linear forms l_j, function $u = \prod l_j^{\alpha_j}$, 1-form $\omega = du/u$, etc, under Assumption 1: $\alpha_j \notin \mathbb{Z}$, and Assumption 2: no $k+1$ hyperplanes in $\{L_j\}$ intersect.

For a general arrangement, without Assumption 2 but with a genericity for α_j corresponding to Assumption 1, the structure of the cohomology group can be described in terms of the so-called Orlik-Solomon algebra, and an explicit basis of the homology group is known, if the arrangement is *real.* By successive blowing-up one can make the proper transform of the arrangement normally crossing — there is a systematic way to do this — then one can, in principle, evaluate the intersection numbers (cf. [KY2], [Yos2]). We expect that these intersection numbers can be expressed combinatorially in a closed form.

For imaginary arrangements, $k \geq 2$, or non-linear arrangements (cf. [KY2]), little is known about explicit cycles.

Motivated by an integral whose integrand involves hypergeometric functions, Hanamura, Ohara and Takayama study intersection theory when the rank of the local system $\mathcal{L}$ is larger than 1 (cf. [Oha1,2], [OT]). They use hyperplane-section method, which is expected to be effective also to the previous problem.

Recall the famous limit formula:

$$(1+\lambda t)^{1/\lambda} \longrightarrow e^t, \quad \text{as} \quad \lambda \longrightarrow 0$$

and a less famous one

$$(1+\lambda t)^{1/\lambda(\mu-\lambda)}(1+\mu t)^{1/\mu(\lambda-\mu)} \longrightarrow e^{t^2/2}, \quad \text{as} \quad \lambda,\ \mu \longrightarrow 0.$$

In §1, starting from the Beta integral you find two 'limit' integrals, one of them is the Gamma function. These formulae suggest another direction of generalization of the theories stated above, that is, to consider for example

$$u = \prod_{j=1}^{m}(t-x_j)^{\alpha_j}\exp f, \quad \omega = d\log u = \sum_{j=1}^{m}\alpha_j\frac{dt}{t-x_j}+df, \quad \nabla = d+\omega\wedge,$$

where f is a polynomial in t. The corresponding hypergeometric integrals represent various *confluent* hypergeometric functions; the extreme ones are those without l_j; such integrals are called *generalized Airy integrals*, because

$$\int \exp(-t^3/3+xt)dt$$

represents the Airy function.

In the following sections we study the confluent cases. Since the above limit formulae are delicate, if you know what I mean, the above theories in §§2 – 5 do *not* directly imply those for confluent cases; we must establish it independently. Of course you can expect some limit relations among them (see [KHT2], [Ha2]).

§7. Confluent cases, general frame

Let $n_1 \geq \cdots \geq n_m$ be natural numbers and L_j $(1 \leq j \leq m)$ be hyperplanes in $\mathbb{P}^k$ defined by linear forms l_j of $t_1, \ldots, t_k$; put $T = \mathbb{P}^k \setminus \cup_{j=1}^m L_j$. We define a rational exact 1-form ω'_j with n_j-fold poles along L_j; this is explicitly given in §9. Put

$$\omega = \sum_{j=1}^{m} \left(\alpha_j \frac{dl_j}{l_j} + \omega'_j \right), \quad \nabla = d + \omega\wedge$$

and consider the following complex

$$0 \longrightarrow \Omega^0(*L) \xrightarrow{\nabla} \Omega^1(*L) \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Omega^k(*L) \longrightarrow 0.$$

We want to define the intersection pairing between $H^k(\Omega^\bullet(*L), \nabla)$ and $H^k(\Omega^\bullet(*L), \check{\nabla})$ as we did in non-confluent cases. However, we can easily see that

$$H^k(\mathcal{E}^\bullet, \nabla) \not\simeq H^k(\Omega^\bullet(*L), \nabla) \not\simeq H^k(\mathcal{E}_c^\bullet, \nabla)$$

in general. So we need to introduce a reasonable cohomology theory on which a perfect pairing can be naturally defined. We also want to have a suitable homology theory and Poincaré isomorphisms to get intersection numbers for homology groups. Up to now only two extreme cases are studied:

Case $k = 1$,

Case $T = \mathbb{C}^k$, i.e. ω admites poles only along the hyperplane at infinity.

§8. Confluent cases $k = 1$

8.1. Twisted de Rham cohomology groups.

A smooth function f defined in a neighborhood U of the point x is said to be rapidly decreasing at x if f satisfies

$$\frac{\partial^{p+q}}{\partial t^p \partial \bar{t}^q} f(x) = 0, \quad p, q = 0, 1, 2, \ldots.$$

Let $\mathcal{S}^p$ be the vector space of smooth p-forms on $\mathbb{P}^1$ which are rapidly decreasing at $x_i (= L_i)$ for every i. A smooth function f defined in $U \backslash \{x\}$ is said to be polynomially growing at x if there exists $r \in \mathbb{N}$ such that $(t - x)^r f$ is smooth on U. Let $\mathcal{P}^p$ be the vector space of smooth p-forms f on T which are polynomially growing at x_i for every i.

We consider two complexes with differential ∇ :

$$\begin{aligned}(\mathcal{S}^\bullet,\nabla) \quad &: \quad \mathcal{S}^0 \xrightarrow{\nabla} \mathcal{S}^1 \xrightarrow{\nabla} \mathcal{S}^2 \xrightarrow{\nabla} 0,\\ (\mathcal{P}^\bullet,\nabla) \quad &: \quad \mathcal{P}^0 \xrightarrow{\nabla} \mathcal{P}^1 \xrightarrow{\nabla} \mathcal{P}^2 \xrightarrow{\nabla} 0.\end{aligned}$$

The cohomology groups $H^k(\mathcal{S}^\bullet,\nabla)$ and $H^k(\mathcal{P}^\bullet,\nabla)$ are called rapidly decreasing and polynomially growing twisted de Rham cohomology groups with respect to ∇, respectively. The inclusions

$$(\Omega^\bullet(*L),\nabla) \subset (\mathcal{P}^\bullet,\nabla), \quad (\mathcal{S}^\bullet,\nabla) \subset (\mathcal{P}^\bullet,\nabla)$$

of complexes induce the following isomorphisms among twisted de Rham cohomology groups.

Theorem 3. $H^p(\Omega^\bullet(*L),\nabla) \simeq H^p(\mathcal{P}^\bullet,\nabla) \simeq H^p(\mathcal{S}^\bullet,\nabla)$, $p = 0,1,2.$

The first isomorphism can be proved by the help of $\bar{\partial}$-calculus. Since the injectivity of the natural map $H^p(\mathcal{S}^\bullet,\nabla) \to H^p(\mathcal{P}^\bullet,\nabla)$ is easy, we mention briefly its surjectivity when $p = 1$. For a ∇-closed form $\varphi \in \Omega^1(*L)$, there exists a unique formal meromorphic Laurent series F_i around x_i satisfying $\nabla F_i = \varphi$. If $n_i \geq 2$, F_i is divergent in general, however, there exists a polynomially growing smooth function f_i with the same expansion as F_i. Thus the form

$$\varphi - \sum_{i=0}^{m} \nabla(h_i f_i)$$

is in $\mathcal{S}^1$, where h_i is a smooth function defined in §3.1. This implies the surjectivity.

8.2. Twisted homology groups. Let Δ be a singular p-simplex in T, define a function u_Δ on Δ by

$$u_\Delta(t) = \exp\left(\int^t \omega\right),$$

where the path of the integration is in Δ . We consider only chains ρ such that if x_i belongs to the closure of $\rho = \sum_j b_j\ \Delta_j$ in $\mathbb{P}^1$ then

$$\lim_{t\to x_i,\ t\in\rho} (t - x_i)^r u_\rho(t) = 0, \quad r = 0,1,2,\ldots,$$

where $u_\rho(t) = u_{\Delta_j}(t)$ $(t \in \Delta_j)$. Let $C_p(T,\omega)$ be the space of loaded p-chains $\sum_j b_j\ \Delta_j \otimes u_{\Delta_j}$ for all such p-chains $\rho = \sum_j b_j\ \Delta_j$. The boundary operator ∂_ω on $C_\bullet(T,\omega)$ is naturally defined, and we get the p-th homology group $H_p(C_\bullet(T,\omega),\partial_\omega)$ as we did in §4. There is a natural pairing between $H^1(\mathcal{S}^\bullet,\nabla)$ and $H_1(C_\bullet(T,\omega),\partial_\omega)$ through the (confluent) hypergeometric integral

$$\langle\varphi,\gamma\rangle=\sum_j b_j\int_{\Delta_j}u_{\Delta_j}(t)\varphi,$$

where $\varphi\in\mathcal{S}^1,\gamma=\sum_j b_j\ \Delta_j\ \otimes u_{\Delta_j}(t)\in C_1(T,\omega)$.

Theorem 4. *The pairing between $H^1(\mathcal{S}^\bullet,\nabla)$ and $H_1(C_\bullet(T,\omega),\partial_\omega)$ is perfect.*

8.3. Intersection pairings.

There is a natural pairing between $\mathcal{S}^1$ and $\mathcal{P}^1$ by

$$\int_{\mathbb{P}^1}\varphi\wedge\psi,\quad \varphi\in\mathcal{S}^1,\ \psi\in\mathcal{P}^1.$$

This pairing descends to the perfect pairing $\cdot$ between $H^1(\mathcal{S}^\bullet,\nabla)$ and $H^1(\mathcal{P}^\bullet,\check{\nabla})$. Theorem 3 yields the isomorphism $\iota: H^1(\Omega^\bullet(*L),\nabla)\to H^1(S^\bullet(*L),\nabla)$, which induces the intersection pairing of $H^1(\Omega^\bullet(*L),\nabla)$ and $H^1(\Omega^\bullet(*L),\check{\nabla})$ by

$$\varphi\cdot\psi=\int_{\mathbb{P}^1}\iota(\varphi)\wedge\psi.$$

Theorem 5. *The intersection number $\varphi\cdot\psi$ of $\varphi\in H^1(\Omega^\bullet(*L),\nabla)$ and $\psi\in H^1(\Omega^\bullet(*L),\check{\nabla})$ is given by*

$$\varphi\cdot\psi=2\pi i\sum_{j=0}^{m}\mathrm{Res}_{t=x_j}(F_j\psi),$$

where F_j is the meromorphic formal Laurent series around x_j satisfying $\nabla F_j=\varphi$.

Note that we can evaluate the intersection number $\varphi\cdot\psi$ by this theorem; see examples in the next subsection.

So far in this section, we defined three pairings:

$$\begin{array}{ccc} H^1(\Omega^\bullet(*L),\nabla)\cong H^1(\mathcal{S}^\bullet,\nabla) & \longleftrightarrow & H^1(\mathcal{P}^\bullet,\check{\nabla})\cong H^1(\Omega^\bullet(*L),\check{\nabla}) \\ \updownarrow & & \updownarrow \\ H_1(C_\bullet(T,\omega),\partial_\omega) & & H_1(C_\bullet(T,-\omega),\partial_{-\omega}). \end{array}$$

These pairings define a pairing between the two homology groups.

Theorem 6. *Suppose for two loaded cycles*

$$\rho^+ = \sum_i b_i \, \Delta_i^+ \otimes u_{\Delta_i^+}(t) \in H_1(C_\bullet(T,\omega), \partial_\omega)$$

and

$$\rho^- = \sum_j b_j \, \Delta_j^- \otimes u_{\Delta_j^-}^{-1}(t) \in H_1(C_\bullet(T,-\omega), \partial_{-\omega}),$$

Δ_i^+ and Δ_j^- meet transversally at finitely many points. Then the intersection number $\rho^+ \cdot \rho^-$ is equal to

$$\rho^+ \cdot \rho^- = \sum_{i,j} \sum_{v \in \Delta_i^+ \cap \Delta_j^-} b_i b_j [u_{\Delta_i^+}(t)]_{t=v} [u_{\Delta_j^-}^{-1}(t)]_{t=v} I_v(\Delta_i^+, \Delta_j^-),$$

where $I_v(\Delta_i^+, \Delta_j^-)$ is the topological intersection number of Δ_i^+ and Δ_j^- at $v \in T$.

8.4. Examples. The compatibility of the parings yields quadratic relations among confluent hypergeometric functions.

Let $\omega = -tdt$, so $u(t) = e^{-t^2/2}$. The (co)homology groups in question are 1-dimensional. Put

$$\rho^+ = [-\infty, \infty] \otimes e^{-t^2/2}, \quad \rho^- = [-i\infty, i\infty] \otimes e^{t^2/2}.$$

Let us compute the intersection number $dt \cdot dt$ applying Theorem 5. Since the pole of ω is at ∞ only, we solve the equation $\nabla F = dt$ at ∞. By a straightforward calculation, we have

$$F = -s + s^3 - 2s^5 + 2 \cdot 4s^7 - 2 \cdot 4 \cdot 6s^9 + \cdots, \quad s = 1/t.$$

Since $\mathrm{Res}_{s=0}(F(s)(-ds/s^2)) = 1$, $dt \cdot dt$ equals $2\pi i$. One can easily see that Theorem 6 implies $\rho^+ \cdot \rho^- = 1$. Since

$$\langle dt, \gamma^+ \rangle = \int_{-\infty}^{+\infty} e^{-t^2/2} dt, \quad \langle dt, \gamma^- \rangle = \int_{-i\infty}^{+i\infty} e^{t^2/2} dt = i \int_{-\infty}^{+\infty} e^{-t^2/2} dt,$$

we have the formula announced in §1:

$$\left(\int_{-\infty}^{\infty} e^{-t^2/2} dt \right) \cdot 1 \cdot \left(i \int_{-\infty}^{\infty} e^{-t^2/2} dt \right) = 2\pi i.$$

We present two more examples: the inversion formula for the gamma function

$$\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin \pi\alpha}$$

and Lommel's formula

$$J_a(z)J_{-a+1}(z) + J_{a-1}(z)J_{-a}(z) = \frac{2\sin(\pi a)}{\pi z},$$

which holds for the Bessel function with parameter $a \in \mathbb{C} \setminus \mathbb{Z}$

$$J_a(z) = \left(\frac{z}{2}\right)^a \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(a+k+1)} \left(\frac{z}{2}\right)^k,$$

where $z \in \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$ and the argument of z is in $(-\pi/2, \pi/2)$. For details including proofs, refer to [MMT].

[Ha2] shows that such quadratic relations are indeed obtained from these in §5 by confluence process.

§9. Confluent cases, generalized Airy $k \geq 2$

Let ω be an exact 1-form on $T = \mathbb{C}^k$, with parameters $\alpha_1, \ldots, \alpha_n$, defined as

$$\omega = d\theta_{n+1}(t) + \sum_{j=1}^{n} \alpha_j d\theta_j(t),$$

where θ_j are polynomials in $t = (t_1, \ldots, t_k)$ of degree j defined by

$$\log(1 + t_1 X + t_2 X^2 + \cdots + t_k X^k) = \sum_{j \geq 1} \theta_j(t) X^j;$$

for example, $\theta_1(t) = t_1$, $\theta_2(t) = t_2 - t_1^2/2$, $\theta_3(t) = t_3 - t_1 t_2 + t_1^3/3$. Note that the form ω has poles of order $n+2$ along the hyperplane L at infinity. Let $H^p(\Omega^\bullet, \nabla)$ be the p-th cohomology group of the complex

$$(\Omega^\bullet, \nabla) : 0 \longrightarrow \Omega^0 \xrightarrow{\nabla} \Omega^1 \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Omega^k \longrightarrow 0,$$

where Ω^p the vector space of polynomial p-forms. In [Kim2], it is shown that only $H^k(\Omega^\bullet, \nabla)$ survives and is $\binom{n}{k}$-dimensional, further it is conjectured that there exists a basis expressed in terms of Schur polynomials. This conjecture is established in [IM]. In order to state this, we consider the map

$$\phi : \mathbb{C}^k \ni s = (s_1, \ldots, s_k) \mapsto t = (t_1, \ldots, t_k) = (e_1(s), \ldots, e_k(s)) \in \mathbb{C}^k,$$

where $e_j(s)$ is the elementary symmetric polynomial of degree j.

Theorem 7. *$H^k(\Omega^\bullet, \nabla)$ can be spanned by*

$$\Theta_I = d\theta_{i_1} \wedge \cdots \wedge d\theta_{i_k}, \quad I = (i_1, \ldots, i_k),\ 1 \le i_1 < \cdots < i_k \le n.$$

The pull buck $\phi^*(\Theta_I)$ of Θ_I by ϕ is given by

$$\phi^*(\Theta_I) = Sc_\lambda(s)\Delta(s) ds_1 \wedge \cdots \wedge ds_k,$$

where $Sc_\lambda(s)$ is the Schur polynomial attached to the Young digram $\lambda = (i_k - k, \ldots, i_1 - 1)$ and $\Delta(s)$ is the difference product of $s_1, \ldots, s_k$.

Let us define the intersection pairing $H^k(\Omega^\bullet, \nabla)$ and $H^k(\Omega^\bullet, \check{\nabla})$. Note that the map ϕ induces the biholomorphic map from the quotient variety $(\mathbb{P}^1)^k/S_k$ to $\mathbb{P}^k$. We can easily see that

$$\phi^*(d\theta_i(t_1, \ldots, t_k)) = \sum_{j=1}^{k} d\theta_i(s_j, 0, \ldots, 0).$$

We regard $\phi^*(\omega)$ as a meromorphic 1-form on $(\mathbb{P}^1)^k/S_k$. We can deform $\phi^*(\Theta_I)$ into a S_k-invariant rapidly decreasing k-form $\iota(\phi^*(\Theta_I))$ on $\mathbb{C}^k$ by adding $dF + \phi^*(\omega) \wedge F$, where F is a S_k-invariant polynomially growing $(k-1)$-form on $\mathbb{C}^k$. Since $(\mathbb{P}^1)^k$ is the $k!$-fold covering of $(\mathbb{P}^1)^k/S_k$, we define the intersection number $\Theta_I \cdot \Theta_J$ for $\Theta_I \in H^k(\Omega^\bullet, \nabla)$ and $\Theta_J \in H^k(\Omega^\bullet, \check{\nabla})$ as

$$\langle \Theta_I, \Theta_J \rangle = \frac{1}{k!} \int_{\mathbb{C}^k} \iota(\phi^*(\Theta_I)) \wedge \phi^*(\Theta_J).$$

Theorem 8. *The intersection number $\langle \Theta_I, \Theta_J \rangle$ is equal to the skew Schur polynomial $Sc_{\lambda/\tilde{\mu}}(\alpha)$ with elementary symmetric polynomials as variables, where $\tilde{\mu}$ is the complement of the Young diagram $\mu = (j_k - k, \ldots, j_1 - 1)$ in the $k \times (n-k)$ rectangle, and $\lambda/\tilde{\mu}$ is the skew Young diagram of $\lambda = (i_k - k, \ldots, i_1 - 1)$ and $\tilde{\mu}$.*

The cohomology theory introduced in this section will be presented in full in [IM]. The homological counter part is still unsettled.

References

[AK] K. Aomoto and M. Kita, *Hypergeometric functions* (*in Japanese*), Springer-Verlag, Tokyo, 1994.

[AKOT] K. Aomoto, M. Kita, P. Orlik and H. Terao, Twisted de Rham cohomology groups of logarithmic forms, Adv. Math. **128** (1997), 119–152.

[Cho] K. Cho, A generalization of Kita and Noumi's vanishing theorems of cohomology groups of local system, Nagoya Math. J. **147** (1997), 63–69.

[CM] K. Cho and K. Matsumoto, Intersection theory for twisted cohomologies and twisted Riemann's period relation I, Nagoya Math. J. **139** (1995), 67–86.

[DM] P. Deligne and G.D. Mostow, Monodromy of hypergeometric functions and non-lattice integral monodromy, I.H.E.S. Publ. Math. **63** (1986), 5–89.

[De] P. Deligne, *Équations Différentielles à Points Singuliers Réguliers,* Lecture Notes in Math. **163**, Springer-Verlag, 1970.

[FT] M. Falk and H. Terao, β**nbc**-bases for cohomology of local systems on hyperplane complements, Trans. Amer. Math. Soc., **349** (1997), 189–202.

[Ha1] Y. Haraoka, Confluence of cycles for hypergeometric functions on $Z_{2,n+1}$, Transac. AMS **349** (1997), 675–712.

[Ha2] Y. Haraoka, Quadratic relations for confluent hypergeometric functions on $Z_{2,n+1}$, to appear in Funkcial. Ekvac.

[HY] M. Hanamura and M. Yoshida, Hodge structure on twisted cohomologies and twisted Riemann inequalities, Nagoya Math. J. **154** (1999), 123–139.

[IM] K. Iwasaki and K. Matsumoto, Intersection theory for generalized Airy integrals, (in preparation).

[IK1] K. Iwasaki and M. Kita, Exterior power structure on the twisted de Rham cohomology of the complements of real Veronese arrangements, J. de Math. Pures et Appl. **75** (1995), 69–84.

[IK2] K. Iwasaki and M. Kita, Twisted homology of the configuration spaces of n-points with application to the hypergeometric functions, Kumamoto J. Math. **12** (1999), 9–72.

[Kim1] H. Kimura, On rational de Rham cohomology associated with the generalised confluent hypergeometric functions I, $\mathbb{P}^1$ case, Proc. Roy. Soc. Edinburgh, **127A** (1997), 145–155.

[Kim2] H. Kimura, On the homology group associated with the general Airy integral, Kumamoto J. Math. **10** (1997), 11–29.

[Kim3] H. Kimura, On rational de Rham cohomology associated with the generalized Airy function, Ann. Sc. Norm. Sup. Pisa XXIV(1997), 351–366.

[KT] H. Kimura and M. Taneda, Analogue of flat basis and cohomological intersection number for general hypergeometric functions, preprint (1998).

[KHT1] H. Kimura, Y. Haraoka and K. Takano, The generalized confluent hypergeometric functions, Proc. Japan Acad. **68A** (1992), 290–295.

[KHT2] —, On confluence of the general hypergeometric systems, Proc. Japan Acad. **69A** (1993), 99–104.

[KHT3] —, On contiguity relations of the confluent hypergeometric systems, Proc. Japan Acad. **70A** (1994), 47–49.

[KM] M. Kita and K. Matsumoto, Duality for Hypergeometric Functions and Invariant Gauss-Manin Systems, Compositio Math., **108** (1997), 77–106.

[KMM] N. Kachi, K. Matsumoto and M. Mihara, The perfectness of the intersection pairings of twisted cohomology and homology groups with respect to rational 1-forms, Kyushu J. Math. **53** (1999), 163–188.

[KY1] M. Kita and M. Yoshida, Intersection theory for twisted cycles I, Math. Nachr. **166** (1994), 287–304.

[KY2] —, ibid. II, Math. Nachr. **168** (1994), 171–190.

[MMT] H. Majima, K. Matsumoto and N. Takayama, Intersection theory for confluent hypergeometric functions, preprint (1998).

[Maj] H. Majima, *Asymptotic Analysis for Integrable Connections with Irregular Singular Points*, Lecture Notes in Math. **1075**, Springer-Verlag, 1984.

[Mal] B. Malgrange, Remarques sur les equations differentielles a points singuliers irreguliers, *Equations Différentielles et Systèmes de Pfaff dans le Champ Complexe,* Lecture notes in Math. **712**, Springer-Verlag, 1979, 77–86.

[Mat1] K. Matsumoto, Quadratic identities for hypergeometric series of type (k, l), Kyushu J. Math. 48(1994), 335–345.

[Mat2] —, Intersection numbers for logarithmic k-forms, Osaka J. Math., **35** (1998), 873–893.

[Mat3] —, Intersection numbers for 1-forms associated with confluent hypergeometric functions, Funkcial. Ekvac., **41** (1998), 291–308.

[Oha1] K. Ohara, Computation of the monodromy for the generalized hypergeometric function ${}_pF_{p-1}(a_1, \ldots, a_p; b_1, \ldots, b_p; z)$, Kyushu J. Math. 51(1997), 101–124.

[Oha2] K. Ohara, Intersection numbers of twisted cohomology groups associated with Selberg-type integrals, preprint (1998).

[OrT] P. Orlik and H. Terao, *Arrangements of Hyperplanes,* Springer-Verlag, 1991.

[OhT] K. Ohara and N. Takayama, Evaluation of intersection numbers of twisted homology groups of locally constant sheaves of more than 1 dimension, preprint (1998).

[Sa] C. Sabbah, On the comparison theorem for elementary irregular $\mathcal{D}$-modules, Nagoya Math. J. **141** (1996), 107–124.

[Y1] M. Yoshida, *Hypergeometric Functions, My Love,* Vieweg, 1997.

[Y2] M. Yoshida, Intersection theory for twisted cycles III — Determinant formulae, to appear in Math. Nachr.

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