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Vassiliev Invariants of Braids and Iterated Integrals

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§ Introduction

The notion of finite type invariants of knots was introduced by Vassiliev in his study of the discriminats of function spaces (see [13]). It was shown by Kontsevich [9] that such invariants, which we shall call the Vassiliev invariants, can be expressed universally by iterated integrals of logarithmic forms on the configuration space of distinct points in the complex plane.

In the present paper we focus on the Vassiliev invariants of braids. Our main object is to clarify the relation between the Vassiliev invariants of braids and the iterated integrals of logarithmic forms on the configuration space which are homotopy invariant. A version of such description for pure braids is given in [6]. We denote by B_n the braid group on n strings. Let J be the ideal of the group ring $\mathbf{C}[B_n]$ generated by $\sigma_i - \sigma_i^{-1}$, where $\{\sigma_i\}_{1 \leq i \leq n-1}$ is the set of standard generators of B_n. The vector space of the Vassiliev invariants of B_n of order k with values in $\mathbf{C}$ can be identified with $\mathrm{Hom}(\mathbf{C}[B_n]/J^{k+1}, \mathbf{C})$. Let us stress that such vector space had been studied in terms of the iterated integrals due to K. T. Chen before the work of Vassiliev. We introduce a graded algebra $\tilde{\mathcal{A}}_n$, which is a semi-direct product of the completed universal enveloping algebra of the holonomy Lie algebra of the configuration space and the group algebra of the symmetric group. We construct a homomorphism $\theta : B_n \longrightarrow \tilde{\mathcal{A}}_n$ expressed as an infinite sum of Chen's iterated integrals, which gives a universal expression of the holonomy of logarithmic connections. This homomorphism may be considered as a prototype of the Kontsevich integral for knots. Using this homomorpshim we shall determine all iterated integrals of logarithmic forms which provide invariants of braids (see Theorem 3.1). As a Corollary we recover the isomorphism

$$\tilde{\mathcal{A}}_n \cong \varprojlim \mathbf{C}[B_n]/J^j$$

which also follows from Chen's theory of iterated integrals (see also [11]). Here we notice that the above isomorphism can be shown over $\mathbf{Q}$ by means of the expression of the Vassiliev invariants based on the Drinfel'd associator defined over $\mathbf{Q}$.

The paper is organized in the following way. In Section 1 we discuss in general the situation of the complement of an arrangement of hyperplanes in a complex vector space and recall basic facts on the itegrability of logarithmic connections. In Section 2 we give a brief summary of fundamental results in Chen's theory of iterated integrals. Section 3 is the main part of the present paper. We describe the Vassiliev invariants of braids and their relation with Chen's iterated integrals of logarithmic forms.

§1. Arrangements and integrable connections

Let H_j, $1 \leq j \leq m$, be affine hyperplanes in the complex vector space $\mathbf{C}^n$ and we denote by f_j a linear form defining H_j. We define the logarithmic differential form ω_j by

$$\omega_j = \frac{1}{2\pi\sqrt{-1}}\, d\log f_j = \frac{1}{2\pi\sqrt{-1}} \frac{df_j}{f_j}.$$

We put $X = \mathbf{C}^n \setminus \cup_{j=1}^m H_j$. Let V be a complex vector space and we consider the trivial vector bundle $E = X \times V$ over X. For $A_j \in \mathrm{End}(V)$, $1 \leq j \leq m$, the 1-form $\omega = \sum_{j=1}^m A_j \omega_j$ defines a connection on the vector bundle E. We have the following Lemma.

Lemma 1.1. *The 1-form $\omega = \sum_{j=1}^m A_j \omega_j$ defines an integrable connection if the condition*

$$[A_{j_p}, A_{j_1} + \cdots + A_{j_s}] = 0, \ 1 \leq p \leq s$$

is satisfied for any maximal family of hyperplanes $H_{j_1}, \cdots, H_{j_s}$ such that

$$\mathrm{codim}_{\mathbf{C}}(H_{j_1} \cap \cdots \cap H_{j_s}) = 2.$$

Proof. For each triplet of hyperplanes H_i, H_j, H_k contained in the set of hyperplanes $\{H_{j_p}\}_{1 \leq p \leq s}$ we have the relation

$$\omega_i \wedge \omega_j + \omega_j \wedge \omega_k + \omega_k \wedge \omega_i = 0.$$

To show the integrability of the connection defined by ω it is sufficient to prove $\omega \wedge \omega = 0$ since $d\omega = 0$. This follows from the above quadratic relations among logarithmic forms. Q.E.D.

The relation among the logarithmic forms used in the proof of Lemma 1.1 is a special case of the relations for describing the structure of the cohomology ring of X given by Orlik and Solomon [12]. We denote by X_n the configuration space of ordered distinct n points in the complex plane. Namely X_n is defined by

$$X_n = \{(z_1, \cdots, z_n) \in \mathbf{C}^n \ ; \ z_i \neq z_j \text{ if } i \neq j\}.$$

Let us consider the action of the symmetric group $\mathcal{S}_n$ on X_n by the permutation of the coordinates. The quotient space $\mathcal{S}_n \setminus X_n$ is denoted by Y_n. We have an unramified covering $\pi : X_n \to Y_n$. We fix basepoints $x \in X_n$ and $y \in Y_n$ satisfying $\pi(x) = y$. The fundamental group of X_n is by definition the braid group on n strings and is denoted by B_n. The fundamental group of Y_n is the pure braid group on n strings and is denoted by P_n. We denote by σ_j, $1 \leq j \leq n-1$, the standard generators of B_n, where σ_j is represented by the braid interchanging the j-th and $(j+1)$-st strings in the positive direction. We put

$$\gamma_{ij} = \sigma_i \cdots \sigma_{j-1} \sigma_j^2 \sigma_{j-1}^{-1} \cdots \sigma_i^{-1}$$

for $1 \leq i < j \leq n$. It is known that P_n is generated by $\gamma_{ij}, 1 \leq i < j \leq n$ (see [1]).

We consider the logarithmic differential forms

$$\omega_{ij} = \frac{1}{2\pi\sqrt{-1}} d \log(z_i - z_j), \quad 1 \leq i < j \leq n$$

defined on X_n. It was shown by Arnold that the cohomology ring of X_n is generated by the de Rham cohomology classes of the above logarithmic forms with the relations

$$\omega_{ij} \wedge \omega_{jk} + \omega_{jk} \wedge \omega_{ki} + \omega_{ki} \wedge \omega_{ij} = 0, \quad i < j < k.$$

Let V be a complex vector space and let A_{ij}, $1 \leq i \neq j \leq n$, be linear transformations of V satisfying $A_{ij} = A_{ji}$. We consider the 1-form $\omega = \sum_{i<j} A_{ij} \omega_{ij}$. As a special case of Lemma 1.1, we see that ω defines an integrable connection if the condition

$$[A_{ik}, A_{ij} + A_{jk}] = 0 \quad i, j, k \text{ distinct}$$
$$[A_{ij}, A_{kl}] = 0 \quad i, j, k, l \text{ distinct}$$

is satisfied. The above relation among A_{ij} is called the infinitesimal pure braid relation.

Now we explain the Knizhnik-Zamolodchikov connection as a typical example. Let $\mathbf{g}$ be a finite dimensional complex semi-simple Lie algebra and we denote by I_μ, $1 \le \mu \le \dim \mathbf{g}$, an orthonormal basis of $\mathbf{g}$ with respect to the Cartan-Killing form. Let $\rho_j : \mathbf{g} \to \mathrm{End}(V_j)$, $1 \le j \le n$, be representations of the Lie algebra $\mathbf{g}$. We put

$$\Omega_{ij} = \sum_\mu 1 \otimes \cdots \otimes 1 \otimes \rho_i(I_\mu) \otimes 1 \otimes \cdots \otimes 1 \otimes \rho_j(I_\mu) \otimes 1 \otimes \cdots \otimes 1$$

for $1 \le i, j \le n$, where $\rho_i(I_\mu)$ stands for the action on the i-th component of the tensor product $V_1 \otimes \cdots \otimes V_n$. By using the fact that the Casimir element $\sum_\mu I_\mu \cdot I_\mu$ lies in the center of the universal enveloping algebra of $\mathbf{g}$ we can show that the above Ω_{ij}, $1 \le i, j \le n$, satisfy the infinitesimal pure braid relation. Therefore the 1-form

$$\omega = \lambda \sum_{i<j} \Omega_{ij} \omega_{ij}$$

defines an integrable connection for any complex parameter λ, which we shall call the Knizhnik-Zamolodchikov connection. As the holonomy of this connecion we obtain linear representations of the pure braid group. We refer the readers to [4] and [7] for a detailed description of these representations.

§2. Review of Chen's iterated integrals

We recall the definition and basic properties of Chen's iterated integrals. Let M be a smooth manifold and we fix two points a and b in M. We denote by $\mathcal{P}_{a,b}(M)$ the set of smooth paths $\gamma : [0,1] \to M$. Let Δ_q denote the simplex defined by

$$\Delta_q = \{(t_1, \cdots, t_q) \in \mathbf{R}^q \ ; \ 0 \le t_1 \le \cdots \le t_q \le 1\}.$$

Let us consider the map

$$\phi : \mathcal{P}_{a,b}(M) \times \Delta_q \to M^q$$

defined by

$$\phi(\gamma, (t_1, \cdots, t_q)) = (\gamma(t_1), \cdots, \gamma(t_q))$$

where M^q stands for the q-fold Cartesian product of the manifold M. Let ω be a differential form of degree p on M. Then integrating the pull-back $\phi^*\omega$ along the fiber of the projection map $\pi : \mathcal{P}_{a,b}(M) \times \Delta_q \to \mathcal{P}_{a,b}(M)$, we obtain

$$\pi_* \phi^* \omega = \int_{\Delta_q} \phi^* \omega,$$

which is considered to be a differential form of degree $p-q$ on the path space $\mathcal{P}_{a,b}(M)$. For differential forms ω_j, $1 \leq j \leq q$, on M we denote by $\omega_1 \times \cdots \times \omega_q$ the differential form on M^q given by $\pi_1^*\omega_1 \wedge \cdots \wedge \pi_q^*\omega_q$ where $\pi_j : M^q \to M$ denotes the projection on the j-th factor. By applying the above construction we obtain the differential form $\pi_*\phi^*(\omega_1 \times \cdots \times \omega_q)$ on the path space $\mathcal{P}_{a,b}(M)$. The value of $\pi_*\phi^*(\omega_1 \times \cdots \times \omega_q)$ at $\gamma \in \mathcal{P}_{a,b}(M)$ is also denoted by

$$\int_\gamma \omega_1 \cdots \omega_q.$$

which is by definition Chen's iterated integral of $\omega_1, \cdots, \omega_q$ along the path γ. We can show that the differential form $d(\pi_*\phi^*\omega)$ on the path space $\mathcal{P}_{a,b}(M)$ is written as the sum of $\pi_*\phi^*(d\omega)$ and

$$\int_{\partial\Delta_q} \phi^*\omega$$

with a suitable sign convention. This leads us to define the following double complex.

We denote by $A^p(M)$ the vector space of smooth differential forms of deree p on M. We define $\mathcal{C}^{p,-q}(M)$ to be the direct sum

$$\oplus_{p_1+\cdots+p_q=p,\ p_1>0,\cdots,p_q>0}\ [A^{p_1}(M) \otimes \cdots \otimes A^{p_q}(M)].$$

Let us introduce the differentials

$$d_1 : \mathcal{C}^{p,-q} \to \mathcal{C}^{p+1,-q}, \quad d_2 : \mathcal{C}^{p,-q} \to \mathcal{C}^{p,-q+1}$$

by

$$\begin{aligned} d_1(\omega_1 \otimes \cdots \otimes \omega_q) &= \sum_{i=1}^{q} (-1)^i (J\omega_1 \otimes \cdots \otimes J\omega_{i-1} \otimes d\omega_i \otimes \cdots \otimes \omega_q) \\ d_2(\omega_1 \otimes \cdots \otimes \omega_q) &= \sum_{i=1}^{q-1} (-1)^{i-1} (J\omega_1 \otimes \cdots \otimes J\omega_{i-1} \otimes (J\omega_i \wedge \omega_{i+1}) \\ &\quad \otimes \omega_{i+2} \otimes \cdots \otimes \omega_q) \end{aligned}$$

where $J\omega$ stands for $(-1)^{\deg \omega}\omega$. Putting $\mathcal{C}^n = \oplus_{p-q=n}\mathcal{C}^{p,-q}$ and $d = d_1 + d_2$, we obtain the associated total complex $\mathcal{C}^\bullet = \oplus_{n\in\mathbf{Z}}\ \mathcal{C}^n$, which we shall call the bar complex.

We define a linear map $\mu : \mathcal{C}^\bullet \to A^\bullet(\mathcal{P}_{a,b}(M))$ by

$$\mu(\omega_1 \otimes \cdots \otimes \omega_q) = \pi_*\phi^*(\omega_1 \times \cdots \times \omega_q).$$

We have the following Proposition.

Proposition 2.1 (K. T. Chen [2]). *The map*

$$\mu : \mathcal{C}^\bullet \to A^\bullet(\mathcal{P}_{a,b}(M))$$

defines a homomorphism of graded differential algebras.

We fix a basepoint $x \in M$ and we denote by $\Omega_x(M)$ the loop space of M based at x. Namely $\Omega_x(M)$ is by definition the space of paths $\mathcal{P}_{x,x}(M)$. The following is a fundamental result due to K. T. Chen in the case M is simply connected.

Theorem 2.2 (K. T. Chen [2]). *Let M be a simply connected manifold. The above map μ induces an isomorphism of cohomology*

$$H^j(\mathcal{C}^\bullet) \cong H^j_{DR}(\Omega_x(M))$$

where $H^j_{DR}(\Omega_x(M))$ denotes the de Rham cohomology of the loop space $\Omega_x(M)$.

Let us describe the relation between the fundamental group of M and the 0-th cohomology $H^0(\mathcal{C}^\bullet)$ of the bar complex $\mathcal{C}^\bullet$. The iterated integration map

$$\iota : \mathcal{C}^0 \times \Omega_x(M) \to \mathbf{R}$$

defined by $\iota(\omega_1 \otimes \cdots \otimes \omega_q, \gamma) = \int_\gamma \omega_1 \cdots \omega_q$ induces a natural paring map

$$H^0(\mathcal{C}^\bullet) \times \pi_1(M, x) \to \mathbf{R}$$

which gives a bilinear map

$$H^0(\mathcal{C}^\bullet) \times \mathbf{R}[\pi_1(M, x)] \to \mathbf{R}.$$

Here $\mathbf{R}[\pi_1(M,x)]$ stands for the group algebra of $\pi_1(M,x)$ over $\mathbf{R}$. We denote by I the kernel of the augmentation homomorphism $\varepsilon : \mathbf{R}[\pi_1(M,x)] \to \mathbf{R}$. Let us introduce the increasing filtration $\mathcal{F}_k\mathcal{C}^n$, $k \geq 0$, on the bar complex $\mathcal{C}^\bullet$ defined by

$$\mathcal{F}_k\mathcal{C}^n = \oplus_{p-q=n,\ q \leq k}\ \mathcal{C}^{p,-q}.$$

The above filtration is preserved by the differential and induces a filtration on the cohomology of the bar complex $\mathcal{C}^\bullet$. The following Theorem is due to K. T. Chen.

Theorem 2.3 (K. T. Chen [3]). *The iterated integration map induces an isomorphism*

$$\mathcal{F}_k H^0(\mathcal{C}^\bullet) \cong \mathrm{Hom}_{\mathbf{R}}(\mathbf{R}[\pi_1(M,x)]/I^{k+1}, \mathbf{R}).$$

Let us denote by $\mathrm{Lib}(H_1(M,\mathbf{Q}))$ the free Lie algebra over $\mathbf{Q}$ generated by the first homology $H_1(M,\mathbf{Q})$. We consider the dual of the cup product homomorphism

$$\alpha : H_2(M,\mathbf{Q}) \to H_1(M,\mathbf{Q}) \wedge H_1(M,\mathbf{Q})$$

and the ideal in $\mathrm{Lib}(H_1(M,\mathbf{Q}))$ generated by Im α is denoted by $\mathcal{I}$. Here we identify the wedge product with the Lie bracket. The holonomy Lie algebra of M over $\mathbf{Q}$ is defined to be

$$\mathbf{g}(M)_{\mathbf{Q}} = \mathrm{Lib}(H_1(M,\mathbf{Q}))/\mathcal{I}.$$

We have the filtration

$$\mathbf{g}(M)_{\mathbf{Q}} = \Gamma_0 \supset \Gamma_1 \cdots \supset \Gamma_j \supset \cdots$$

defined inductively by $\Gamma_{j+1} = [\Gamma_0, \Gamma_j]$ for $j \geq 0$. As the quotient $\mathbf{g}(M)_{\mathbf{Q}}/\Gamma_j$ we obtain a nilpotent Lie algebra whose universal enveloping algebra is denoted by $U(\mathbf{g}(M)_{\mathbf{Q}}/\Gamma_j)$. We consider the $\mathbf{Q}$-algebra $\mathcal{A}(M)_{\mathbf{Q}}$ defined by the inverse limit

$$\mathcal{A}(M)_{\mathbf{Q}} = \varprojlim U(\mathbf{g}(M)_{\mathbf{Q}}/\Gamma_j).$$

In the case when the manifold M is the complement of hyperlpanes $X = \mathbf{C}^n \setminus \cup_{j=1}^m H_j$ we have an isomorphism

$$U(\mathbf{g}(X)_{\mathbf{Q}}/\Gamma_j) \cong \mathbf{Q}[\pi_1(X,x)]/I^{j+1}$$

which induces an isomorphism of complete Hopf algebras

$$\mathcal{A}(X)_{\mathbf{Q}} \cong \varprojlim \mathbf{Q}[\pi_1(X,x)]/I^j.$$

We refer the readers to [6] for the above isomorphisms. In this case the algebra $\mathcal{A}(X)_{\mathbf{Q}}$ has the following explicit description. We take basis $X_j, 1 \leq j \leq m$, whose dual basis consists of the logarithmic forms $\omega_j, 1 \leq j \leq m$. Let us denote by $\mathbf{Q}\langle\langle X_1, \cdots, X_m\rangle\rangle$ the algebra of formal power series in the non-commutative indeterminates $X_j, 1 \leq j \leq m$, and let $\mathcal{J}$ be its ideal generated by

$$[X_{j_p}, X_{j_1} + \cdots + X_{j_s}], \quad 1 \leq p \leq s$$

for any maximal family of hyperplanes $H_{j_1}, \cdots, H_{j_s}$ such that

$$\mathrm{codim}_{\mathbf{C}}(H_{j_1} \cap \cdots \cap H_{j_s}) = 2.$$

Then we have an isomorphism

$$\mathcal{A}(X)_{\mathbf{Q}} \cong \mathbf{Q}\langle\langle X_1, \cdots, X_m \rangle\rangle / \mathcal{J}$$

as complete Hopf algebras.

For a field k containing $\mathbf{Q}$ we put $\mathcal{A}(X)_k = \mathcal{A}(X)_{\mathbf{Q}} \otimes k$. We are going to construct a homomorphism

$$\theta : \pi_1(X, x) \to \mathcal{A}(X)_{\mathbf{C}}$$

which gives a universal expression of the holonomy of the connection $\omega = \sum_{j=1}^m A_j \omega_j$ in Section 1. We put $\tilde{\omega} = \sum_{j=1}^m X_j \otimes \omega_j$ and we define the map θ as the infinite sum of iterated integrals given by

$$\theta(\gamma) = 1 + \int_\gamma \tilde{\omega} + \cdots + \int_\gamma \underbrace{\tilde{\omega} \cdots \tilde{\omega}}_{k} + \cdots$$

for $\gamma \in \pi_1(X, x)$. Here by definition

$$\int_\gamma \underbrace{\tilde{\omega} \cdots \tilde{\omega}}_{k} = \sum_{j_1, \cdots, j_k} \left(\int_\gamma \omega_{j_1} \cdots \omega_{j_k} \right) X_{j_1} \cdots X_{j_k}.$$

We see that θ induces a homomorphism of algebras

$$\tilde{\theta} : \mathbf{C}[\pi_1(X, x)] \to \mathcal{A}(X)_{\mathbf{C}}.$$

§3. Iterated integrals and invariants of braids

First we describe the notion of Vassiliev invariants of braids by means of the group algebra of the braid group. Let J be the kernel of the natural homomorphism $\pi : \mathbf{C}[B_n] \to \mathbf{C}\mathcal{S}_n$. It turns out that J is the ideal generated by $\sigma_i - \sigma_i^{-1}$, where $\{\sigma_i\}_{1 \leq i \leq n-1}$ is the set of standard generators of B_n. Let $v : B_n \to \mathbf{C}$ be an invariant of braids with values in $\mathbf{C}$. Extending v linearly we obtain a linear map $\tilde{v} : \mathbf{C}[B_n] \to \mathbf{C}$. We shall say that v is a Vassiliev invariant of order k if $\tilde{v}$ factors through $\mathbf{C}[B_n]/J^{k+1}$. We denote by $\mathcal{V}_k(B_n)$ the complex vector space of the Vassiliev invariants of order k for B_n. We have an identification

$$\mathcal{V}_k(B_n) \cong \mathrm{Hom}_{\mathbf{C}}(\mathbf{C}[B_n]/J^{k+1}, \mathbf{C})$$

as complex vector spaces. We have an increasing sequence of vector spaces

$$\mathcal{V}_0(B_n) \subset \cdots \subset \mathcal{V}_k(B_n) \subset \mathcal{V}_{k+1}(B_n) \subset \cdots$$

whose union $\mathcal{V}(B_n) = \cup_{k \geq 0} \mathcal{V}_k(B_n)$ is called the vector space of Vassiliev invariants of braids.

For the configuration space X_n defined in Section 1, we set $\mathcal{A}_n = \mathcal{A}(X_n)_{\mathbf{C}}$. The algebra $\mathcal{A}_n$ is described as the quotient $\mathbf{C}\langle\langle X_{ij} \rangle\rangle / \mathcal{J}$ where $\mathbf{C}\langle\langle X_{ij} \rangle\rangle$ is the algebra of formal non-commutative power series with indeterminates X_{ij}, $1 \leq i \neq j \leq n$, and $\mathcal{J}$ is the ideal generated by $X_{ij} - X_{ji}$ and the infinitesimal pure braid relations

$$[X_{ik}, X_{ij} + X_{jk}] \quad i, j, k \text{ distinct}$$
$$[X_{ij}, X_{kl}] \quad i, j, k, l \text{ distinct.}$$

Let us notice that $\mathcal{A}_n$ has a structure of a graded algebra with $\deg X_{ij} = 1$. We denote by $\mathcal{A}_n^p$ the degree p part of $\mathcal{A}_n$. We put $\mathcal{A}_n^{\leq k} = \oplus_{p \leq k} \mathcal{A}_n^p$. We introduce an extension $\tilde{\mathcal{A}}_n$ of the algebra $\mathcal{A}_n$. As a vector space $\tilde{\mathcal{A}}_n$ is defined to be the tensor product $\mathcal{A}_n \otimes \mathbf{C}[\mathcal{S}_n]$. We introduce a structure of an algebra for $\tilde{\mathcal{A}}_n$ by

$$X_{ij} s = s X_{s(i)s(j)}.$$

for $X_{ij} \in \mathcal{A}_n$ and $s \in \mathcal{S}_n$. The algebra $\tilde{\mathcal{A}}_n$ is the semi-direct product of $\mathcal{A}_n$ and $\mathbf{C}[\mathcal{S}_n]$ defined by the above relation. This algebra has a structure of a graded algebra with $\deg X_{ij} = 1$ and $\deg s = 0$ for $s \in \mathcal{S}_n$. As in the case of $\mathcal{A}_n$, we denote by $\tilde{\mathcal{A}}_n^p$ the degree p part of $\tilde{\mathcal{A}}_n$. We put $\tilde{\mathcal{A}}_n^{\leq k} = \oplus_{p \leq k} \tilde{\mathcal{A}}_n^p$.

Our next object is to define a linear map

$$w : \mathcal{V}_k(B_n) \to \mathrm{Hom}_{\mathbf{C}}(\tilde{\mathcal{A}}_n^{\leq k}, \mathbf{C}).$$

First we observe that an element of $\tilde{\mathcal{A}}_n^{\leq k}$ is written as a linear combination of elements of the form

$$X_{i_1 j_1} \cdots X_{i_p j_p} s, \quad s \in \mathcal{S}_n, \ p \leq k.$$

We choose $\sigma \in B_n$ such that $p(\sigma) = s$. The map w is defined by

$$\begin{aligned} &w(v)(X_{i_1 j_1} \cdots X_{i_p j_p} s) \\ &= \tilde{v}((\gamma_{i_1 j_1} - 1) \cdots (\gamma_{i_p j_p} - 1)\ \sigma). \end{aligned}$$

We see that since v is of order k the map w is well-defined. The above $w(v)$ is called the weight system for v.

Let us consider the loop space $\Omega_y(Y_n)$ with basepoint $y \in Y_n$. An element of $\Omega_y(Y_n)$ is called a geometric braid. We define $\mathcal{C}^0(\Omega_y(Y_n))$ to be the subspace of $\mathcal{A}^0(\Omega_y(Y_n))$ spanned by the functions whose values at $\gamma \in \Omega_y(Y_n)$ are given by the iterated integrals of the form

$$\int_{\tilde{\gamma}} \omega_{i_1 j_1} \cdots \omega_{i_k j_k}$$

with some k, where $\tilde{\gamma}$ is the lift of γ in X_n starting at the basepoint $x \in X_n$. The 0-th cohomology $H^0(\mathcal{C}^\bullet(\Omega_y(Y_n)))$ consists of the iterated integrals of logarithmic forms depending only on the homotopy classes of loops $\gamma \in \Omega_y(Y_n)$. This cohomology group has the increasing filtration $\mathcal{F}_k H^0(\mathcal{C}^\bullet(\Omega_y(Y_n)),\ k \geq 0$, defined by the length of the iterated integrals. The following Theorem permits us to determine all such iterated integrals combinatorially in terms of the algebra $\tilde{\mathcal{A}}_n$.

Theorem 3.1. *We have isomorphisms*

$$\mathcal{F}_k H^0(\mathcal{C}^\bullet(\Omega_y(Y_n)) \cong \mathcal{V}_k(B_n) \cong \mathrm{Hom}_{\mathbf{C}}(\tilde{\mathcal{A}}_n^{\leq k}, \mathbf{C}).$$

Proof. We defined the map

$$w : \mathcal{V}_k(B_n) \to \mathrm{Hom}_{\mathbf{C}}(\tilde{\mathcal{A}}_n^{\leq k}, \mathbf{C})$$

by taking the associated weight system. To construct the inverse map we consider the universal holonomy homomorphism

$$\theta : B_n \to \tilde{\mathcal{A}}_n$$

defined in the following way. We put $\tilde{\omega} = \sum_{i<j} X_{ij} \otimes \omega_{ij}$ and we define the map θ as the infinite sum of iterated integrals given by

$$\theta(\gamma) = \left(1 + \int_{\tilde{\gamma}} \tilde{\omega} + \cdots + \int_{\tilde{\gamma}} \underbrace{\tilde{\omega} \cdots \tilde{\omega}}_{k} + \cdots \right) p(\gamma)$$

for B_n, where $\tilde{\gamma}$ is the lift of γ in X_n starting at the basepoint $x \in X_n$ as above and $p : B_n \to \mathcal{S}_n$ is the narutal homomorphism. We denote by $\tau : \tilde{\mathcal{A}}_n \to \tilde{\mathcal{A}}_n^{\leq k}$ the truncation map. For $\beta \in \mathrm{Hom}_{\mathbf{C}}(\tilde{\mathcal{A}}_n^{\leq k}, \mathbf{C})$ we obtain a Vassiliev invariant of order k of B_n as the composition $\beta \circ \tau \circ \theta$. One can check that this construction gives the inverse of the map w. We observe that each element in $\mathcal{F}_k H^0(\mathcal{C}^\bullet(\Omega_y(Y_n))$ defines a Vassiliev invariant of braids of order k. Conversely, given a Vassiliev invariant v of braids of order k, we consider the associated weight system $w(v)$. Then as the composition $w(v) \circ \tau \circ \theta$ we recover the iterated integral expression of the Vassiliev invariant v. This shows the first isomorphism. Q.E.D.

Corollary 3.2. *We have an isomorphism*

$$\tilde{\mathcal{A}}_n^{\leq k} \cong \mathbf{C}[B_n]/J^{k+1}$$

which induces an isomorphism of complete Hopf algebras

$$\tilde{\mathcal{A}}_n \cong \varprojlim \mathbf{C}[B_n]/J^j$$

Remark 3.3. It can been shown that the universal Vassiliev invariants with values in $\mathbf{Q}$ can be defined by means of the Drinfel'd associator defined over $\mathbf{Q}$ (see [4], [5] and [10]). Using this expression we can establish the isomorphism in the above Corollary over $\mathbf{Q}$.

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