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Polytopes, Invariants and Harmonic Functions

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Abstract.

The classical harmonic functions are characterized in terms of the mean value property with respect to the unit ball. Replacing the ball by a polytope, we are led to the notion of polyhedral harmonic functions, i.e., those continuous functions which satisfy the mean value property with respect to a given polytope. The study of polyhedral harmonic functions involves not only analysis but also algebra, including combinatorics of polytopes and invariant theory for finite reflection groups. We give a brief survey on this subject, focusing on some recent results obtained by the author.

§1. Introduction

The harmonic functions are a very important class of functions in mathematics as well as in physics. Let us recall a classical theorem of Gauss and Koebe stating that they are characterized in terms of the mean value property with respect to the unit ball.

Theorem 1.1 (Gauss-Koebe). *Let Ω be a domain in $\mathbb{R}^n$. Any function $f \in C^2(\Omega)$ is harmonic if and only if $f \in C(\Omega)$ satisfies the mean value property with respect to the n-dimensional unit ball B^n with center at the origin:*

$$f(x) = \frac{1}{|B^n|} \int_{B^n} f(x+ry)\,dy \qquad (\forall\, x \in \Omega,\ 0 < \forall\, r < \mathrm{dist}(x, \partial\Omega)),$$

where $|B^n|$ denotes the volume of B^n.

This theorem naturally leads us to the following simple question (see Figure 1).

Problem 1.1. What happens if the ball is replaced by a *polytope* ?

Namely, we are interested in the problem of characterizing those continuous functions which satisfy the mean value property with respect to a given polytope. In this paper we give a brief survey on this subject, focusing on some recent results obtained by the author. See also [22].

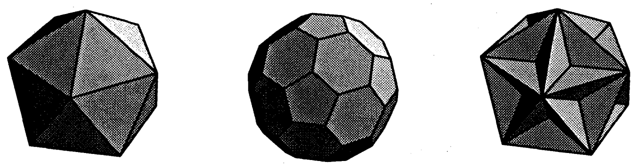

FIG 1. Examples of Polyhedra

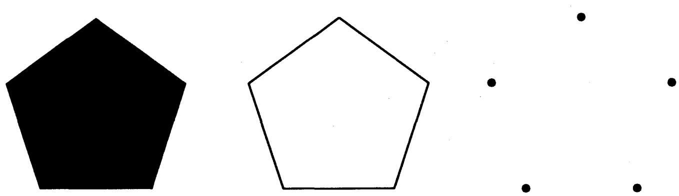

FIG 2. Skeletons of Pentagon

§2. Polyhedral Harmonic Functions

We formulate Problem 1.1 more precisely. Let P be an n-dimensional polytope, and $P(k)$ be the k-skeleton of P for $k = 0, 1, \ldots, n$ (see Figure 2). A continuous function $f \in C(\Omega)$ is said to be *$P(k)$-harmonic* if f satisfies the mean value property with respect to $P(k)$, that is, for any $x \in \Omega$, there exists a positive number $r_x > 0$ such that

$$f(x) = \frac{1}{|P(k)|} \int_{P(k)} f(x + ry)\, d\mu_k(y) \qquad (\forall x \in \Omega,\ 0 < \forall r < r_x),$$

where μ_k is the k-dimensional Euclidean measure and $|P(k)| = \mu_k(P(k))$ is the total measure of $P(k)$. Let $\mathcal{H}_{P(k)}(\Omega)$ denote the linear space of all $P(k)$-harmonic functions on Ω. Then our problem is stated as follows.

Problem 2.1. Characterize the function space $\mathcal{H}_{P(k)}(\Omega)$.

The history of polyhedral harmonics began with the works of Kakutani and Nagumo[24] (1935) and Walsh[28] (1936), who considered the vertex problem ($k = 0$) for a regular convex polygon. Since then several authors have discussed various problems in various settings ([1][2][3][5][6]

[10][11][13][14][25]). See the references in [16] for a more extensive literature. In particular, Friedman and Littman[12] posed a rather surprising question.

Problem 2.2 (Friedman-Littman, 1962). Is $\mathcal{H}_{P(k)}(\Omega)$ finite dimensional ?

This problem had been open until recently when the author was able to settle it in the affirmative.

§3. General Properties

In general the function space $\mathcal{H}_{P(k)}(\Omega)$ satisfies the following properties.

Theorem 3.1 ([16]). *Let P be any n-dimensional polytope and $k \in \{0, 1, \ldots, n\}$. Then,*

(1) *$\mathcal{H}_{P(k)}(\Omega)$ is independent of the domain Ω, namely, the restriction map $\mathcal{H}_{P(k)} := \mathcal{H}_{P(k)}(\mathbb{R}^n) \to \mathcal{H}_{P(k)}(\Omega)$ is an isomorphism;*
(2) *$\mathcal{H}_{P(k)}$ is a finite-dimensional linear space of polynomials;*
(3) *Let $G \subset O(n)$ be the symmetry group of P. Then $\dim \mathcal{H}_{P(k)} \geq |G|$;*
(4) *If G is irreducible, then $\mathcal{H}_{P(k)}$ is a finite-dimensional linear space of harmonic polynomials;*
(5) *$\mathcal{H}_{P(k)}$ is an $\mathbb{R}[\partial]$-module, where $\mathbb{R}[\partial]$ is the ring of linear partial differential operators with constant coefficients.*

This theorem shows that the space $\mathcal{H}_{P(k)}(\Omega)$ of polyhedral harmonic functions is completely different from the space $\mathcal{H}(\Omega)$ of classical harmonic functions. A summary of comparisons between them is given in Table 1. The most remarkable contrast is their dimensionality; $\mathcal{H}_{P(k)}(\Omega)$ is finite dimensional, while $\mathcal{H}(\Omega)$ is infinite dimensional. The finite-dimensionality of $\mathcal{H}_{P(k)}(\Omega)$ gives rise to the problem of computing $\dim \mathcal{H}_{P(k)}(\Omega)$ and, moreover, that of constructing a natural basis of it. In view of (5) of Theorem 3.1, investigating the structure of $\mathcal{H}_{P(k)}(\Omega)$ as an $\mathbb{R}[\partial]$-module is also an interesting problem. Some results in these directions will be presented in Sections 4 and 5. But these problems are yet to be considered more extensively. Hereafter we put $\mathcal{H}_{P(k)} = \mathcal{H}_{P(k)}(\Omega)$, since it is independent of the domain Ω.

§4. Regular Convex Polytopes

Our problem is of particular interest when P admits ample symmetry. A typical instance is the case where P is a *regular convex polytope*

	$\mathcal{H}(\Omega)$: classical	$\mathcal{H}_{P(k)}(\Omega)$: polyhedral
domain Ω	depends on Ω (natural boundary)	independent of Ω
dimension	$\dim \mathcal{H}(\Omega) = \infty$	$\dim \mathcal{H}_{P(k)}(\Omega) < \infty$
functions	transcendental in general	only polynomials
PDEs	$\Delta f = 0$ (single equation)	an infinite system (holonomic)

TABLE 1. Classical vs. Polyhedral Harmonic Functions

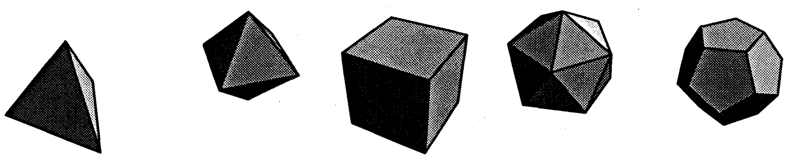

FIG 3. Platonic Solids (Regular Convex Polyhedra)

with center at the origin. We refer to [4] for the classification of regular convex polytopes (see Figure 3 for $n = 3$). In this case it is known that the symmetry group of P is a *finite reflection group*. So we can apply invariant theory for finite reflection groups to characterize the function space $\mathcal{H}_{P(k)}$.

We recall some basic definitions. A finite reflection group is a finite group generated by reflections. Here a *reflection* is an orthogonal transformation $g \in O(n)$ that takes a nonzero vector $v \in \mathbb{R}^n$ to its negative $-v$, while keeping the orthogonal complement H to v pointwise fixed. The hyperplane $H = H_g$ is called the reflecting hyperplane of g. Let $\alpha_g : \mathbb{R}^n \to \mathbb{R}$ be a linear form such that $\operatorname{Ker} \alpha_g = H_g$, (such an α_g is unique up to a nonzero constant multiple). Given a finite reflection group G, let R be the set of all reflections in G. Then the *fundamental* *alternating polynomial* for G is defined by

$$\Delta_G(x) = \prod_{g \in R} \alpha_g(x).$$

It is uniquely determined up to a nonzero constant multiple. We give an example.

Example 4.1. If P is a regular n-simplex with center at the origin, then G is the symmetric group $\mathfrak{S}_n$ acting on $\mathbb{R}^n$ by permuting the coordinates $x_1, \ldots, x_n$. In this case,

$$\Delta_G(x) = \prod_{i<j} \langle p_i - p_j, x \rangle,$$

where $p_0, p_1, \ldots, p_n$ are the vertices of P and $\langle \cdot, \cdot \rangle$ is the Euclidean inner product on $\mathbb{R}^n$.

Theorem 4.1 ([6][10][17][20][23]). *Let P be any n-dimensional regular convex polytope that is not a measure polytope. Let $G \subset O(n)$ be the symmetry group of P, and $\Delta_G(x)$ be the fundamental alternating polynomial for the finite reflection group G. Then,*

(1) $\mathcal{H}_{P(k)}$ *is independent of* $k = \dim P(k)$;
(2) *The dimension of* $\mathcal{H}_{P(k)}$ *is equal to the order of* G: $\dim \mathcal{H}_{P(k)} = |G|$;
(3) $\mathcal{H}_{P(k)}$ *is generated by* $\Delta_G(x)$ *as an* $\mathbb{R}[\partial]$*-module*: $\mathcal{H}_{P(k)} = \mathbb{R}[\partial]\Delta_G(x)$.

The author believes that the same result holds for the measure polytope, although he does not have a complete proof as yet. (This was proved in [13] only for $k = 0$.) The dimension of $\mathcal{H}_{P(k)}$ for each regular convex polytope P is given in Table 2, (the value for the measure polytope is still conjectural).

§5. Triangle Mean Value Property

We explicitly determine $\mathcal{H}_{\Delta(k)}$ for any *triangle* Δ in $\mathbb{R}^2$ and $k = 0, 1, 2$. To state the result we introduce some notations. Let A_1, A_2, A_3 be the vertices of the triangle Δ. (A point A in $\mathbb{R}^2$ is identified with the vector $\overrightarrow{OA}$, where O is the origin in $\mathbb{R}^2$.) The indices i, j, k stand for any permutation of $1, 2, 3$. Let A_i' be the mid-point of the side $\overline{A_j A_k}$:

$$A_i' = \frac{A_j + A_k}{2}.$$

$\dim P$	P : regular solids	$\dim \mathcal{H}_{P(k)}$
2	regular m-gon	$2m$
n	regular n-simplex (tetrahedron)	$(n+1)!$
n	cross polytope (octahedron)	$2^n n!$
n	measure polytope (cube)	$2^n n!$
3	icosahedron	120
3	dodecahedron	120
4	24-cell	1152
4	600-cell	14400
4	120-cell	14400

TABLE 2. Dimension of $\mathcal{H}_{P(k)}$ for Regular Solids

The *reciprocal* triangle Δ' of Δ is defined to be the triangle having A_1', A_2', A_3' as its vertices. Let $B := (1/3)(A_1 + A_2 + A_3)$ be the *barycenter* of Δ, and I' be the *incenter* of Δ' (see Figure 4). Then the center of gravity C_k for $\Delta(k)$ is defined by

$$C_k = \begin{cases} B & (k = 0, 2), \\ I' & (k = 1). \end{cases}$$

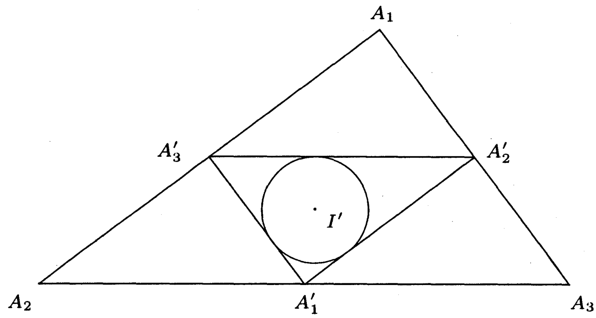

FIG 4. Reciprocal Triangle and Its Incenter

Theorem 5.1 ([21]). *The dimension of the linear space $\mathcal{H}_{\Delta(k)}$ is given by*

$$\dim \mathcal{H}_{\Delta(k)} = \begin{cases} 6 & (C_k = O), \\ 2 & (C_k \neq O). \end{cases}$$

As an $\mathbb{R}[\partial]$-module, $\mathcal{H}_{\Delta(k)}$ is generated by a single homogeneous polynomial $F_k(x)$:

$$\mathcal{H}_{\Delta(k)} = \mathbb{R}[\partial]F_k(x).$$

Explicitly, $F_k(x)$ is given as follows: If $C_k = O$, then

$$F_k(x) = \begin{cases} \prod_{i=1}^{3} \langle A_i'', x\rangle & (k = 0, 2), \\ \sum_{i=1}^{3} \dfrac{\langle A_i'', x\rangle^3}{[a_i(a_j + a_k)]^2} & (k = 1), \end{cases}$$

where a_i is the side-length of $\overline{A_j A_k}$, and A_1'', A_2'', A_3'' are the (unique) vectors satisfying

$$\langle A_i'', A_i'\rangle = 0, \quad \langle A_i'', A_j'\rangle = \frac{1}{a_j} \quad \textit{for} \quad (i,j) = (1,2),\ (2,3),\ (3,1).$$

If $C_k \neq O$, then $F_k(x) = \langle C_k', x\rangle$, where C_k' is a nonzero vector perpendicular to C_k.

§6. Differential Equations

The classical harmonic functions are characterized as the solutions of the Laplace equation $\Delta f = 0$. Note that the Laplace equation is a *single* equation. The $P(k)$-harmonic functions can also be characterized in terms of partial differential equations, though, not by a single equation, but by an *infinite system.* This system is described in terms of some combinatorial data on $P(k)$.

To describe the system we introduce some notations (see also Figure 5). For $j = 0, 1, \ldots, n$, let $\{P_{i_j}\}_{i_j \in I_j}$ be the set of all j-dimensional faces of P, where I_j is an index set; H_{i_j} be the j-dimensional affine subspace of $\mathbb{R}^n$ containing P_{i_j}; and $\pi_{i_j} : \mathbb{R}^n \to H_{i_j}$ be the orthogonal projection from $\mathbb{R}^n$ down to the subspace H_{i_j}. Let $p_{i_j} \in \mathbb{R}^n$ be the vector (or point) in $\mathbb{R}^n$ defined by

$$p_{i_j} = \pi_{i_j}(0) \in H_{i_j}.$$

For $i_j \in I_j$ and $i_{j+1} \in I_{j+1}$, we write $i_j \prec i_{j+1}$ if P_{i_j} is a face of $P_{i_{j+1}}$. For $i_j \prec i_{j+1}$, let $\mathbf{n}_{i_j i_{j+1}}$ be the outer unit normal vector of $\partial P_{i_{j+1}}$ in $H_{i_{j+1}}$ at the face P_{i_j}. It is easy to see that the vector $p_{i_j} - p_{i_{j+1}}$ is parallel to $\mathbf{n}_{i_j i_{j+1}}$, so that one can define the incidence number $[i_j : i_{j+1}] \in \mathbb{R}$ by the relation

$$p_{i_j} - p_{i_{j+1}} = [i_j : i_{j+1}]\mathbf{n}_{i_j, i_{j+1}}.$$

For each $k = 0, 1, \ldots, n$, let $I(k)$ be the set of *k-flags* defined by

$$I(k) = \{i = (i_0, i_1, \ldots, i_k)\,;\, i_j \in I_j, i_0 \prec i_1 \prec \cdots \prec i_k\}.$$

For each k-flag $i = (i_0, i_1, \ldots, i_k) \in I(k)$, we set

$$[i] = [i_0 : i_1][i_1 : i_2] \cdots [i_{k-1} : i_k] \qquad (k = 1, \ldots, n),$$

with the convention $[i] = 1$ for $k = 0$. Note that $[i]$ is the *signed* volume of the k-simplex having $p_{i_0}, p_{i_1}, \ldots, p_{i_k}$ as its vertices. Let $h_m^{(j)}(\xi)$ denote the complete symmetric polynomial of degree m in j-variables:

$$h_m^{(j)}(\xi_1, \ldots, \xi_j) = \sum_{m_1 + \cdots + m_j = m} \xi_1^{m_1} \xi_2^{m_2} \cdots \xi_j^{m_j},$$

where the summation is taken over all j-tuples $(m_1, \ldots, m_j)$ of nonnegative integers satisfying the indicated condition. Finally we set $\langle \xi, \eta \rangle = \xi_1\eta_1 + \cdots + \xi_n\eta_n$ for two vectors $\xi = (\xi_1, \ldots, \xi_n), \eta = (\eta_1, \ldots, \eta_n) \in \mathbb{C}^n$. The following theorem gives a characterization of the $P(k)$-harmonic functions in terms of a system of partial differential equations.

Theorem 6.1 ([16]). *Any $f \in \mathcal{H}_{P(k)}(\Omega)$ is real analytic and satisfies the system of partial differential equations:*

$$\tau_m^{(k)}(\partial) f = 0 \qquad (m = 1, 2, 3, \ldots), \tag{6.1}$$

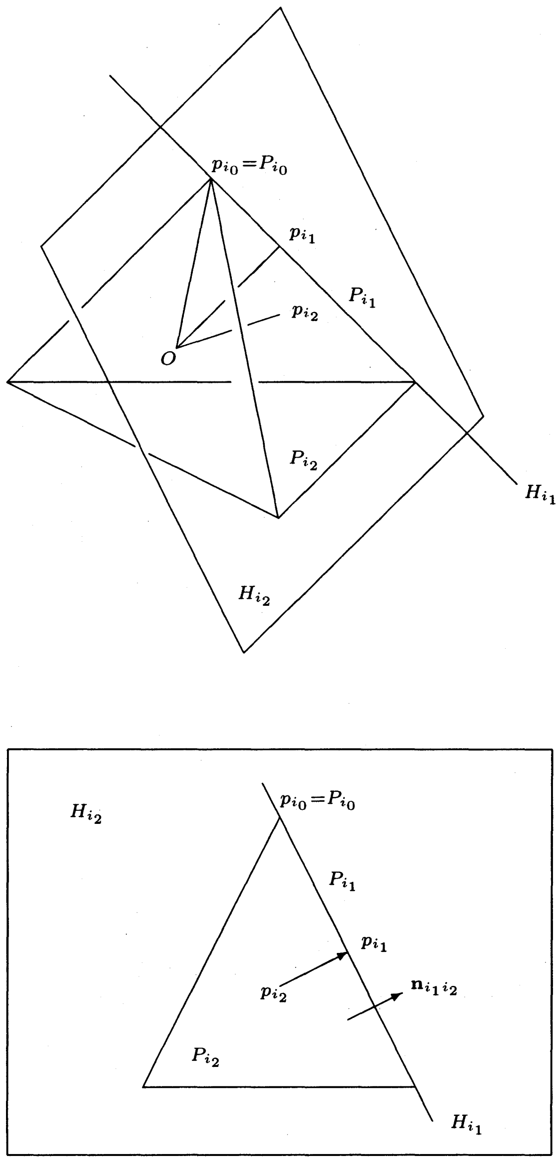

FIG 5. Combinatorial Structure of P

where $\tau_m^{(k)}(\xi)$ *is the homogeneous polynomial of degree* m *defined by*

$$\tau_m^{(k)}(\xi) = \sum_{i \in I(k)} [i] h_m^{(k+1)}(\langle p_{i_0}, \xi \rangle, \langle p_{i_1}, \xi \rangle, \ldots, \langle p_{i_k}, \xi \rangle).$$

Conversely, any weak solution of (6.1) *is real analytic, and belongs to* $\mathcal{H}_{P(k)}(\Omega)$.

The system (6.1) enjoys the following remarkable property.

Theorem 6.2 ([16]). *The system* (6.1) *is holonomic. In particular, the solution space of* (6.1) *is finite dimensional.*

The proof of Theorems 6.1 and 6.2 is based on geometry and combinatorics of the polytope P. These theorems play a central role in establishing Theorem 3.1.

§7. Open Problem

Let $\mathcal{H}_n$ be the linear space of all *harmonic polynomials* in n-variables. Note that $\mathcal{H}_n$ is infinite dimensional. By Theorem 3.1, if P is an n-dimensional polytope with ample symmetry (this means that the symmetry group of P is irreducible), then $\mathcal{H}_{P(k)}$ is a finite-dimensional linear subspace of $\mathcal{H}_n$. Now a natural question arises: *As the polytope* P *approximates the unit ball* B^n, *does the function space* $\mathcal{H}_{P(k)}$ *approximate* $\mathcal{H}_n$ *?* More precisely this problem is formulated as follows (see also Figure 6).

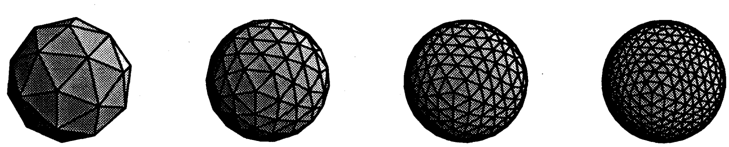

FIG 6. Geodesic Domes

Problem 7.1. Is there an infinite sequence $\{P_m\}_{m \in \mathbb{N}}$ of n-dimensional polytopes with ample symmetry such that the following conditions are satisfied ?

(1) $P_m \to B^n$ as $m \to \infty$ (Hausdorff convergence),

(2) $\cdots \subset \mathcal{H}_{P_{m-1}(k)} \subset \mathcal{H}_{P_m(k)} \subset \mathcal{H}_{P_{m+1}(k)} \subset \cdots,$
(3) $\bigcup_{m \in \mathbb{N}} \mathcal{H}_{P_m(k)} = \mathcal{H}_n.$

If $n = 2$, the answer to this question is *yes* for $k = 0, 1, 2$. Indeed, we can take P_m to be a regular convex m-gon with center at the origin. However, if $n \geq 3$, the problem becomes quite difficult. For the vertex problem ($k = 0$), we can also say that the answer is *yes*, but the proof of it is based on a very deep result from *spherical designs* (see [26]). For the remaining cases $n \geq 3$ and $k = 1, 2, \ldots, n$, the problem is completely open.

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