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Modular Transformation of Twisted Characters of Admissible Representations and Fusion Algebras Associated to Non-Symmetric Transformation Matrices

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Dedicated to Professor Kiyosato Okamoto on his 60th Birthday

Abstract.

In this note we extend the Kac and Todorov's orbifold theory to the principal admissible representations. This gives rise to non-symmetric transformation matrices.

§1. Introduction

Recently Kac and Todorov [4] has shown a very beautiful theory on the modular transformations of twisted characters and on the connection of the fusion algebra of integrable representations of affine Lie algebras with that of finite groups. The aim of this note is to extend their theory to the characters of *principal admissible* representations of affine Lie algebras. As is known well, the character of an admissible representation given by a similar formula as the famous Weyl-Kac character formula is not a real character exactly but coincides with it via a kind of renormalization ([6] and [7]). Also the corresponding Lie group G itself does not always act on the actual space of admissible representation, since the integrable condition is no longer satisfied. So in order to extend the theory, we will have to be free from the geometric picture. In this note, the relation between an affine algebra and a finite group is only algebraic, and is not supposed to have a geometric meaning of orbifolds. And by paying its cost, one obtains a nice family of twisted characters even in the case for admissible representations. It may also be expected and now under investigation that our theory will lead us to the study of

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twisted characters of W-algebras as was done for non-twisted admissible characters via quantized Drinfeld-Sokolov reduction in [1].

We observe, by the simplest example of $\widehat{sl}(2,\mathbb{C})$, that non-symmetric transformation matrices and associated fusion algebras take place in the modular transformation of twisted characters of admissible representations. To take care of them, we need to extend the Lusztig's theory [9] on a "fusion datum" with a non-symmetric transformation matrix to a more general situation.

In this note, first we reconstruct the theory of Kac and Todorov for integrable representations in our *non-geometric* way, and then proceed to the admissible case.

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§2. Modular transformation of twisted theta functions

Let $A = (a_{ij})_{i,j=0,\cdots,\ell}$ be an affine Cartan matrix, and $\mathfrak{g}(A)$ be the corresponding affine Lie algebra with the Cartan subalgebra $\mathfrak{h}$. Let $h^\vee$ be its dual Coxeter number, and r (resp. $r^\vee$) denote the tier number of $\mathfrak{g}(A)$ (resp. of $\mathfrak{g}({}^tA)$). Throughout this paper, we follow the notation from [2], in particular from Chapter 13. The space $\mathfrak{h}$ is identified with its dual space $\mathfrak{h}^*$ by the standard inner product. Following [2] we introduce the coordinates in $\mathfrak{h}^* \cong \mathfrak{h} : h = (\tau, z, t) = 2\pi i(-\tau\Lambda_0 + z + t\delta)$, and put $H := \{\tau \in \mathbb{C};\ \mathrm{Im}\tau > 0\}$ and $Y := \{h \in \mathfrak{h}^*;\ \mathrm{Im}\tau > 0\}$.

Let $N := \bar{\mathfrak{h}}_{\mathbb{R}} \times \bar{\mathfrak{h}}_{\mathbb{R}} \times i\mathbb{R}$ be the Heisenberg group with multiplication

(1) $$(\alpha,\beta,u)\cdot(\alpha',\beta',u') = (\alpha+\alpha',\beta+\beta',u+u'+\pi i((\alpha|\beta')-(\alpha'|\beta))).$$

Consider the action of the Heisenberg group N and the group $SL_2(\mathbb{Z})$ on $\mathfrak{h}^*$:

(2)
$$(\alpha,\beta,u)\cdot(\tau,z,t) := \left(\tau, z+\alpha-\tau\beta, t+\frac{u}{2\pi i}-\frac{(\alpha|\beta)}{2}+\frac{\tau}{2}|\beta|^2-(\beta|z)\right)$$

and

(3) $$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\cdot(\tau,z,t) := \left(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}, t-\frac{c|z|^2}{2(c\tau+d)}\right).$$

Using the translation operator

(4) $$t_\beta h := h + (h|\delta)\beta - \left(\frac{|\beta|^2}{2}(h|\delta)+(h|\beta)\right)\delta.$$

introduced in Chapter 6 of [2], (2) is written as

$$(\alpha,\beta,u)\cdot h = t_\beta h + 2\pi i\alpha + (u - \pi i(\alpha|\beta))\delta. \tag{5}$$

The action of these groups $SL_2(\mathbb{Z})$ and N on the space $\mathfrak{h}^*$ is compatible in the sense that we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (\alpha,\beta,u)\cdot h := (a\alpha+b\beta, c\alpha+d\beta, u)\begin{pmatrix} a & b \\ c & d \end{pmatrix}\cdot h. \tag{6}$$

Now we consider the right action of the metaplectic group

$$Mp_2(\mathbb{Z}) := \{(A,j);\ A\in SL_2(\mathbb{Z}),\ \ j \text{ is a holomorphic function in } \tau\in H,\ \text{ such that } j(\tau)^2 = c\tau+d\}$$

and the Heisenberg group N on the space of holomorphic functions F on Y :

$$F|_{(A,j)}(\tau,z,t) := j(\tau)^{-\ell}F(A\cdot(\tau,z,t)),$$
$$F|_{(\alpha,\beta,u)}(\tau,z,t) := F((\alpha,\beta,u)\cdot(\tau,z,t)).$$

The very important function is $F^{\alpha,\beta}$ defined, for $\alpha,\beta\in\overline{\mathfrak{h}}^*$, by

$$F^{\alpha,\beta}(\tau,z,t) := F((\alpha,\beta,0)\cdot(\tau,z,t)), \tag{7}$$

namely

$$\begin{aligned} F^{\alpha,\beta}(\tau,z,t) &= F(t_\beta h + 2\pi i\alpha - \pi i(\alpha|\beta)\delta) \\ &= F\left(\tau, z+\alpha-\tau\beta, t-\frac{(\alpha|\beta)}{2}-(\beta|z)+\frac{\tau}{2}|\beta|^2\right). \end{aligned} \tag{8}$$

We note that functions $F^{\alpha,\beta}(\tau,0,0)$ were introduced in section 13.6 of [2], but here as above they are defined for *full* variables (τ,z,t). The modular transformation of these functions is easily calculated by using (6) as follows:

$$F^{\alpha,\beta}|_{(A,j)} = (F|_{(A,j)})^{a'\alpha+b'\beta,c'\alpha+d'\beta} \tag{9}$$

if $A\in SL_2(\mathbb{Z})$ and $A^{-1} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$. In particular for $S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, one has

$$F^{\alpha,\beta}|_{(S,j)} = (F|_{(S,j)})^{\beta,-\alpha}, \tag{10}$$

$$F^{\alpha,\beta}|_{(T,j)} = (F|_{(T,j)})^{\alpha-\beta,\beta}. \tag{11}$$

In the following sections, we study characters of integrable or admissible representations of affine Lie algebras *twisted* by a finite group Γ. Here, as its preliminaries, we recall some of usual notations for a finite group Γ:

(i) Let $E(\Gamma)$ denote the set of irreducible characters of Γ, or equivalently the set of all equivalence classes of irreducible representations of Γ.

(ii) For each $a \in \Gamma$, let $Z_\Gamma(a)$ (or simply $Z(a)$) denote the centralizer of a in Γ.

(iii) For $g, a \in \Gamma$ and $\sigma \in E(\Gamma)$, we put $a^g := gag^{-1}$ and σ^g is the character defined by $\sigma^g(h) := \sigma(g^{-1}hg)$ for all $h \in \Gamma$.

(iv) For $\sigma \in E(\Gamma)$, we put $\overline{\sigma}(g) := \overline{\sigma(g)}$. Since $\overline{\sigma}$ is the character of the contragredient representation of σ, we sometimes write σ^* in place of $\overline{\sigma}$.

Given a finite group Γ, we put

$$\tilde{X}(\Gamma) := \{(a, \sigma);\ a \in \Gamma \text{ and } \sigma \in E(Z(a))\},$$

and

$$X(\Gamma) := \tilde{X}(\Gamma)/\Gamma,$$

where the action of the group Γ on $\tilde{X}(\Gamma)$ is $(a, \sigma)^g := (a^g, \sigma^g)$.

Let $[a, \sigma]$ denote the equivalence class of (a, σ), namely the Γ-orbit through (a, σ). Then the associated transformation matrix is given in [9] as follows:

$$(12)\quad S_{[a,\sigma],[a',\sigma']} := \sum_{\substack{g \in \Gamma \text{ such that} \\ aga'g^{-1} = ga'^{-1}g^{-1}a}} \frac{\sigma'(g^{-1}a^{-1}g)\sigma(ga'^{-1}g^{-1})}{|Z(a)| \cdot |Z(a')|}$$

$$= \frac{1}{|Z(a)|} \sum_{\substack{(a'', \sigma'') \in [a', \sigma'] \\ \text{such that } a'' \in Z(a)}} \sigma(a''^{-1})\sigma''(a^{-1}).$$

§3. Twisted characters of integrable representations.

We recall the representation theory of affine Lie algebras from [2]. Given a non-negative integer m, let P_m^+ (or more exactly $P_m^+(A)$ if necessary) denote the set of all dominant integral forms (modulo $\mathbb{C}\delta$) of level m. For $\Lambda \in P_m^+$, let $ch(\Lambda)$ be the normalized character given by the formula

$$(13)\qquad ch(\Lambda) := \frac{A_{\Lambda+\rho}}{A_\rho},$$

where A_λ, for a strictly positive integral form λ, is a holomorphic function defined by

$$A_\lambda(h) := e^{-\frac{|\lambda|^2}{2n}(\delta|h)} \sum_{w\in W} \epsilon(w)e^{(w\lambda|h)} \quad (h \in Y),$$

and $n := (\lambda|\delta)$ is a positive integer. By its definition, it is easy to see the following:

$$A_\lambda(h+u\delta) = e^{u(\lambda|\delta)}A_\lambda(h),$$
$$A_\lambda(t_\alpha(h)) = A_\lambda(h) \text{ for all } \alpha \in M,$$

where M is the lattice in $\overline{\mathfrak{h}}^*$ defined as follows in Chapter 6 of [2]:

$$M := \begin{cases} \overline{Q} & \text{if } A \text{ is symmetric or is twisted not of type } A_{2\ell}^{(2)}, \\ \overline{Q}^\vee & \text{otherwise,} \end{cases}$$

namely

$$M := \begin{cases} \overline{Q}^\vee & \text{if } r=1 \text{ or } A = A_{2\ell}^{(2)}, \\ \overline{Q} & \text{otherwise,} \end{cases}$$

where $\overline{Q} := \sum_{i=1}^{\ell} \mathbb{Z}\alpha$ and $\overline{Q}^\vee := \sum_{i=1}^{\ell} \mathbb{Z}\alpha^\vee$. Note that one always has

$$M \subset \overline{Q}^\vee \subset \overline{Q}^*. \tag{14}$$

Lemma 3.1. *Let $M_1 \subset R_1$ be a pair of lattices in $\overline{\mathfrak{h}}^*$ such that*

$$M_1 \subset M. \tag{15}$$

Then, for $\Lambda \in P_m^+$ and $\alpha, \beta \in R_1$ and $\xi, \eta \in M_1$, the following formulas hold:

1) $e^{\pi i(\alpha+\xi|\beta+\eta)(\Lambda+\rho|\delta)}A_{\Lambda+\rho}^{\alpha+\xi,\beta+\eta}$
$= e^{2\pi i(\Lambda+\rho|\xi)}e^{2\pi i(\alpha|\eta)(\Lambda+\rho|\delta)}e^{\pi i(\alpha|\beta)(\Lambda+\rho|\delta)}A_{\Lambda+\rho}^{\alpha,\beta}$,

2) $e^{\pi i m(\alpha+\xi|\beta+\eta)}ch_\Lambda^{\alpha+\xi,\beta+\eta} = e^{2\pi i(\Lambda|\xi)}e^{2\pi i m(\alpha|\eta)}e^{\pi i m(\alpha|\beta)}ch_\Lambda^{\alpha,\beta}$.

Proof.

$$\begin{aligned} &A_{\Lambda+\rho}^{\alpha+\xi,\beta+\eta}(h) \\ &= A_{\Lambda+\rho}(t_{\beta+\eta}h + 2\pi i(\alpha+\xi) - \pi i(\alpha+\xi|\beta+\eta)\delta) \\ &= e^{2\pi i(\Lambda+\rho|\xi)}e^{-\pi i(\alpha+\xi|\beta+\eta)(\Lambda+\rho|\delta)}A_{\Lambda+\rho}(t_{\beta+\eta}h + 2\pi i\alpha) \qquad \text{by (15)} \end{aligned}$$

$$= e^{-\pi i(\alpha+\xi|\beta+\eta)(\Lambda+\rho|\delta)} A_{\Lambda+\rho}(t_{\beta+\eta}h + 2\pi i\alpha) \qquad \text{by (14) and (15)}$$
$$= e^{-\pi i(\alpha+\xi|\beta+\eta)(\Lambda+\rho|\delta)} e^{2\pi i(\alpha|\eta)(\Lambda+\rho|\delta)} A_{\Lambda+\rho}(t_\eta(t_\beta h + 2\pi i\alpha))$$
$$= e^{-\pi i(\alpha+\xi|\beta+\eta)(\Lambda+\rho|\delta)} e^{2\pi i(\alpha|\eta)(\Lambda+\rho|\delta)} A_{\Lambda+\rho}(t_\beta h + 2\pi i\alpha) \qquad \text{by (15)}$$
$$= e^{-\pi i(\alpha+\xi|\beta+\eta)(\Lambda+\rho|\delta)} e^{2\pi i(\alpha|\eta)(\Lambda+\rho|\delta)} e^{\pi i(\alpha|\beta)(\Lambda+\rho|\delta)}$$
$$A_{\Lambda+\rho}(t_\beta h + 2\pi i\alpha - \pi i(\alpha|\beta)\delta)$$
$$= e^{2\pi i(\alpha|\eta)(\Lambda+\rho|\delta)} e^{-\pi i(\alpha+\xi|\beta+\eta)(\Lambda+\rho|\delta)} e^{\pi i(\alpha|\beta)(\Lambda+\rho|\delta)} A^{\alpha,\beta}_{\Lambda+\rho}(h).$$

Thus 1) is proved. The formula 2) follows from 1) and (13). Q.E.D.

By Lemma 3.1, the *twisted character*

$$\chi^{\alpha,\beta}_\Lambda(\tau, z, t) := e^{\pi i m(\alpha|\beta)} ch^{\alpha,\beta}_\Lambda(\tau, z, t) \tag{16}$$

is well defined for $\Lambda \in P^+_m$ and $\alpha, \beta \in R_1/M_1$, if the conditions (15) and $m(M_1|R_1) \subset \mathbb{Z}$ are satisfied.

We now consider the situation where a datum $(m, \Gamma, M_\Gamma, R_\Gamma, \gamma)$ is given such that

(F1) m is a non-negative integer,

(F2) Γ is a finite group,

(F3) $M_\Gamma \subset R_\Gamma$ is a pair of lattices in $\overline{\mathfrak{h}}^*$ such that

$$M_\Gamma \subset M, \tag{17}$$
$$m(M_\Gamma|R_\Gamma) \subset \mathbb{Z}, \tag{18}$$

(F4) $\gamma : \Gamma \to R_\Gamma/M_\Gamma$ is a map such that

$$\gamma(a^g) = \gamma(a) \qquad \text{for all } a, g \in \Gamma, \tag{19}$$
$$\gamma(ab) = \gamma(a) + \gamma(b) \qquad \text{if } ab = ba. \tag{20}$$

We note that the conditions (19) and (20) imply that

$$\gamma(e) = 0, \tag{21}$$
$$\gamma(a^{-1}) = -\gamma(a), \tag{22}$$
$$\gamma(ab^{-1}) = \gamma(a) - \gamma(b) \qquad \text{if } ab = ba. \tag{23}$$

Associated to this finite group Γ, we consider the tranformation matrix on the set $X_\Gamma := \tilde{X}_\Gamma/\Gamma$:

$$S_{[g,\sigma],[g',\sigma']} = \frac{1}{|Z(g)|} \sum_{\substack{(g'',\sigma'')\in[g',\sigma'] \\ \text{such that } g''\in Z(g)}} \sigma(g''^{-1})\sigma''(g^{-1}). \tag{24}$$

For $\Lambda \in P_m^+$ and $(a,\sigma) \in \tilde{X}_\Gamma$, we put

$$(25) \qquad \chi_{(a,\sigma),\Lambda}(\tau,z,t) := \frac{1}{|Z(a)|} \sum_{b \in Z(a)} \sigma(b) \chi_\Lambda^{\gamma(b),\gamma(a)}(\tau,z,t).$$

Since $\chi_{(a,\sigma),\Lambda} = \chi_{(a,\sigma)^g,\Lambda}$, this formula (25) defines the function $\chi_{[a,\sigma],\Lambda}$.

We now turn to twisted affine algebras. It is known in [3] (see also [2]) that the modular transformation of characters of integrable representations in the case when $\mathfrak{g}(A)$ is a twisted affine algebra not of type $A_{2\ell}^{(2)}$ is given in terms of the *adjacent* root system. For a twisted affine Cartan matrix A not of type $A_{2\ell}^{(2)}$, its adjacent Cartan matrix A' $:= (\langle \alpha_i'^\vee, \alpha_j' \rangle)_{i,j=0,\cdots,\ell}$ is defined to be an affine Cartan matrix, of type $D_{\ell+1}^{(2)}$, $A_{2\ell-1}^{(2)}$, $E_6^{(2)}$ or $D_4^{(3)}$ according as the type of A is $A_{2\ell-1}^{(2)}$, $D_{\ell+1}^{(2)}$, $E_6^{(2)}$ or $D_4^{(3)}$, satisfying the conditions:

(C1)

$$\alpha_i' := \begin{cases} \dfrac{a_i}{a_i^\vee}\alpha_i & \text{if } A = A_{2\ell-1}^{(2)} \text{ or } D_{\ell+1}^{(2)}, \\[2ex] \dfrac{a_{\ell+1-i}}{a_{\ell+1-i}^\vee}\alpha_{\ell+1-i} & \text{if } A = E_6^{(2)} \text{ or } D_4^{(3)}, \end{cases}$$

$$\alpha_i'^\vee := \begin{cases} \dfrac{a_i^\vee}{a_i}\alpha_i^\vee & \text{if } A = A_{2\ell-1}^{(2)} \text{ or } D_{\ell+1}^{(2)}, \\[2ex] \dfrac{a_{\ell+1-i}^\vee}{a_{\ell+1-i}}\alpha_{\ell+1-i}^\vee & \text{if } A = E_6^{(2)} \text{ or } D_4^{(3)}, \end{cases}$$

for $1 \leq i \leq \ell$,

$$\text{(C2)} \qquad \delta' = \frac{1}{r}\delta \qquad \text{and} \qquad c' = c,$$

$$\text{(C3)} \qquad \Lambda_0' = \Lambda_0,$$

where $(a_0, \cdots, a_\ell)$ (resp. $(a_0^\vee, \cdots, a_\ell^\vee)$) is the label (resp. co-label) of A, and $c := \sum_{i=0}^{\ell} a_i^\vee \alpha_i^\vee$ (resp. c') is the canonical central element of $\mathfrak{g}(A)$ (resp. $\mathfrak{g}(A')$) (see Chapter 6 of [2]).

The Cartan subalgebra $\mathfrak{h}'$ of the adjacent affine Lie algebra $\mathfrak{g}(A')$ is identified with its dual space $\mathfrak{h}'^*$ by the standard bilinear form $(\ |\ )'$ on $\mathfrak{g}(A')$. But one can also identify the linear space $\mathfrak{h}'$ (resp. $\mathfrak{h}'^*$) with $\mathfrak{h}$ (resp. $\mathfrak{h}^*$) by conditions (C1), (C2) and (C3), and so the standard bilinear form $(\ |\ )$ on $\mathfrak{h}$ or $\mathfrak{h}^*$ is, in itself, looked as a bilinear form on $\mathfrak{h}'$ or $\mathfrak{h}'^*$. These two natural bilinear forms are related to each other as follows:

$$(26)\qquad (\ |\ )\text{ on }\mathfrak{h} = r(\ |\ )'\text{ on }\mathfrak{h}',$$

$$(27)\qquad (\ |\ )\text{ on }\mathfrak{h}^* = \frac{1}{r}(\ |\ )'\text{ on }\mathfrak{h}'^*.$$

Since $\overline{\mathfrak{h}}^* \cong \overline{\mathfrak{h}}'^*$, lattices M_Γ and R_Γ can be looked as lattices in $\mathfrak{h}'^*$. Then, by a similar calculation, one sees that

$$(28)\qquad \chi'_\Lambda\left(\frac{\tau}{r},\frac{z}{r},t\right)^{\alpha,\beta} := e^{\pi i m(\alpha|\beta)} ch_\Lambda\left(\frac{\tau}{r},\frac{z}{r},t\right)^{\alpha,\beta}$$

is well defined for $\Lambda \in P_m^+(A')$ and $\alpha,\beta \in R_\Gamma/M_\Gamma$, where the function $f(\frac{\tau}{r},\frac{z}{r},t)^{\alpha,\beta}$ is defined by

$$f\left(\frac{\tau}{r},\frac{z}{r},t\right)^{\alpha,\beta} = f\left(\frac{\tau}{r},\frac{z+\alpha-\tau\beta}{r},t-\frac{(\alpha|\beta)}{2}-(\beta|z)+\frac{\tau}{2}|\beta|^2\right).$$

We put

$$(29)\quad \chi'_{(a,\sigma),\Lambda}\left(\frac{\tau}{r},\frac{z}{r},t\right) := \frac{1}{|Z(a)|}\sum_{b\in Z(a)}\sigma(b)\chi'_\Lambda(\tau,z,t)^{\gamma(b),\gamma(a)},$$

for $\Lambda \in P_m^+(A')$ and $(a,\sigma)\in\tilde{X}_\Gamma$.

The modular transformation of integrable characters are given as follows in [2], [3] and [5] :

Lemma 3.2.

1) *If* $r=1$ or $A = A_{2\ell}^{(2)}$, *then*

$$ch_\Lambda|_S(\tau,z,t) = \sum_{\Lambda'\in P_m^+} a(\Lambda,\Lambda')ch_{\Lambda'}(\tau,z,t),$$

and if A is twisted not of type $A = A_{2\ell}^{(2)}$, *then*

$$ch_\Lambda|_S(\tau,z,t) = \sum_{\Lambda'\in P_m^+(A')} a(\Lambda,\Lambda')ch_{\Lambda'}\left(\frac{\tau}{r},\frac{z}{r},t\right),$$

where

(30)

$$a(\Lambda,\Lambda') := \frac{1}{(m+h^\vee)^{\ell/2}}|M^*/M'|^{-1/2}\sum_{w\in\overline{W}}\epsilon(w)e^{-\frac{2\pi i}{m+h^\vee}(w(\Lambda+\rho)|\Lambda'+\rho)}.$$

2) *If A is of type $X_N^{(r)}$ $(X = A, \cdots, G)$ and $A \neq A_{2\ell}^{(2)}$, then*

$$ch_\Lambda|_T(\tau, z, t) = e^{2\pi i s_\Lambda} ch_\Lambda(\tau, z, t),$$

where

$$\begin{aligned} s_\Lambda &:= h_\Lambda - \frac{1}{24} z_m, \qquad h_\Lambda := \frac{(\Lambda|\Lambda + 2\rho)}{2(m + h^\vee)}, \\ z_m &:= \frac{m}{m + h^\vee} \dim(\mathfrak{g}(X_N)). \end{aligned}$$

Under these setting, by a similar calculation as in [4], one obtains the following:

Theorem 3.1.

1) *If $r = 1$ or $A = A_{2\ell}^{(2)}$, then*

$$\begin{aligned} &\chi_{[g,\sigma],\Lambda}|_S \\ = &\sum_{[g',\sigma'] \in X_\Gamma} \sum_{\Lambda' \in P_m^+} e^{-2\pi i m(\gamma(g)|\gamma(g'))} a(\Lambda, \Lambda') S_{[g,\sigma],[g',\sigma']} \cdot \chi_{[g',\sigma'],\Lambda'}. \end{aligned}$$

2) *If A is twisted not of type $A_{2\ell}^{(2)}$, then*

$$\begin{aligned} &\chi_{[g,\sigma],\Lambda}|_S(\tau, z, t) \\ &= \sum_{[g',\sigma'] \in X_\Gamma} \sum_{\Lambda' \in P_m^+(A')} e^{-2\pi i m(\gamma(g)|\gamma(g'))} a(\Lambda, \Lambda') S_{[g,\sigma],[g',\sigma']} \cdot \chi_{[g',\sigma'],\Lambda'} \\ &\times \left(\frac{\tau}{r}, \frac{z}{r}, t\right). \end{aligned}$$

3) *If $A \neq A_{2\ell}^{(2)}$, then*

$$\chi_{[g,\sigma],\Lambda}|_T = e^{2\pi i s_\Lambda} e^{\pi i m|\gamma(g)|^2} \frac{\sigma(g)}{\sigma(1)} \chi_{[g,\sigma],\Lambda}.$$

Proof. 1) By (16), (25), (10) and Lemma 3.2, one has

$$\begin{aligned} \chi_{[g,\sigma],\Lambda}|_S &= \frac{1}{|Z(g)|} \sum_{h \in Z(g)} \sigma(h) e^{\pi i m(\gamma(h)|\gamma(g))} ch_\Lambda^{\gamma(h),\gamma(g)}|_S \\ &= \frac{1}{|Z(g)|} \sum_{h \in Z(g)} \sum_{\Lambda' \in P_m^+} a(\Lambda, \Lambda') \sigma(h) e^{\pi i m(\gamma(h)|\gamma(g))} ch_{\Lambda'}^{\gamma(g),-\gamma(h)} \\ &= \frac{1}{|Z(g)|} \sum_{h \in Z(g)} \sum_{\Lambda' \in P_m^+} a(\Lambda, \Lambda') \sigma(h) e^{\pi i m(\gamma(h)|\gamma(g))} ch_{\Lambda'}^{\gamma(g),\gamma(h^{-1})}. \end{aligned}$$

Using

$$(31)\quad ch_{\Lambda'}^{\gamma(g),\gamma(h^{-1})} = e^{\pi i m(\gamma(h)|\gamma(g))} \sum_{\sigma' \in E(Z(h))} \overline{\sigma}'(g)\chi_{(h^{-1},\sigma'),\Lambda'},$$

which follows from the orthogonality of characters of a finite group, and writing a in place of h^{-1}, this is rewritten as follows:

$$\begin{aligned}
&\chi_{[g,\sigma],\Lambda}|_S \\
&= \frac{1}{|Z(g)|} \sum_{\Lambda' \in P_m^+} a(\Lambda,\Lambda') \sum_{a \in Z(g)} \sum_{\sigma' \in E(Z(a))} e^{-2\pi i m(\gamma(a)|\gamma(g))} \\
&\times \quad \sigma(a^{-1})\sigma'(g^{-1})\chi_{(a,\sigma'),\Lambda'} \\
&= \frac{1}{|Z(g)|} \sum_{\Lambda' \in P_m^+} a(\Lambda,\Lambda') \sum_{a \in Z(g)} \sum_{[a'',\sigma''] \in X_\Gamma} e^{-2\pi i m(\gamma(a'')|\gamma(g))} \\
&\times \sum_{\substack{(a,\sigma') \in [a'',\sigma''] \\ \text{such that} \\ a \in Z(g),\ \sigma' \in E(Z(a))}} \sigma(a^{-1})\sigma'(g^{-1})\chi_{[a'',\sigma''],\Lambda'}.
\end{aligned}$$

Thus we have proved 1). A similar calculation proves 2).

3) One has

$$\begin{aligned}
\chi_{[g,\sigma],\Lambda}|_T &= \frac{1}{|Z(g)|} \sum_{h \in Z(g)} \sigma(h) e^{\pi i m(\gamma(h)|\gamma(g))} ch_\Lambda^{\gamma(h),\gamma(g)}|_T \\
&= e^{2\pi i s_\Lambda} \frac{1}{|Z(g)|} \sum_{h \in Z(g)} a(\Lambda,\Lambda')\sigma(h) e^{\pi i m(\gamma(h)|\gamma(g))} ch_\Lambda^{\gamma(hg^{-1}),\gamma(g)}.
\end{aligned}$$

Using

$$(32)\quad ch_\Lambda^{\gamma(hg^{-1}),\gamma(g)} = e^{-\pi i m(\gamma(hg^{-1})|\gamma(g))} \sum_{\sigma' \in E(Z(g))} \overline{\sigma}'(hg^{-1})\chi_{(g,\sigma'),\Lambda},$$

this is rewritten as follows:

$$\chi_{[g,\sigma],\Lambda}|_T = e^{2\pi i s_\Lambda} e^{\pi i m|\gamma(g)|^2} \frac{1}{|Z(g)|} \sum_{\sigma' \in E(Z(g))} \sum_{h \in Z(g)} \sigma(h)\overline{\sigma}'(hg^{-1})\chi_{(g,\sigma'),\Lambda}.$$

Then by the orthogonality of characters this gives

$$\chi_{[g,\sigma],\Lambda}|_T = e^{2\pi i s_\Lambda} e^{\pi i m|\gamma(g)|^2} \frac{\sigma(g)}{\sigma(1)} \chi_{[g,\sigma],\Lambda},$$

proving 3). Now the proof is complete. Q.E.D.

Example 3.1. We consider the simplest case $A = A_1^{(1)} = \widehat{sl}(2,\mathbb{C})$, where

$$M = \overline{Q} = \overline{Q}^\vee = \mathbb{Z}\alpha_1 = \mathbb{Z}\alpha_1^\vee \tag{33}$$

and

$$\overline{P} := (\overline{Q}^\vee)^* = \mathbb{Z}\overline{\Lambda}_1 = \frac{1}{2}\mathbb{Z}\alpha_1^\vee, \tag{34}$$

since $\overline{\Lambda}_1 = \frac{1}{2}\alpha_1$. Let $\Gamma := (\frac{1}{2}\mathbb{Z})/4\mathbb{Z}$ be a cyclic group, and consider two lattices $M_\Gamma := 2\overline{Q} = 2\mathbb{Z}\alpha_1^\vee$ and $R_\Gamma := \frac{1}{2}\overline{P} = \frac{1}{4}\mathbb{Z}\alpha_1^\vee$. Let γ be a group isomorphism of Γ to R_Γ/M_Γ given by $\gamma(j) := j\overline{\Lambda}_1$ for all $j \in \frac{1}{2}\mathbb{Z}$, and then we get a datum $(m, \Gamma, R_\Gamma, M_\Gamma, \gamma)$ satisfying conditions (17) $\sim$ (20), for arbitrarily fixed non-negative integer m.

For each $s \in (\frac{1}{2}\mathbb{Z})/4\mathbb{Z}$, let $\varepsilon_s : \Gamma \to U(1)$ denote the character defined by $\varepsilon_s(j) := e^{\pi i s j}$. Then the set $E(\Gamma)$ of all characters of the group Γ is

$$E(\Gamma) = E\left((\frac{1}{2}\mathbb{Z})/4\mathbb{Z}\right) = \left\{\varepsilon_s;\ s \in (\frac{1}{2}\mathbb{Z})/4\mathbb{Z}\right\}.$$

We make use of a simplified notation:

$$F^{j,k}(h) := F^{\frac{j}{2}\alpha_1, \frac{k}{2}\alpha_1}(h) = F^{j\overline{\Lambda}_1, k\overline{\Lambda}_1}(h) \tag{35}$$

for a holomolphic function F on Y and $j, k \in \frac{1}{2}\mathbb{Z}$. Also let φ denote the Dynkin diagram automorphism of $A_1^{(1)}$ such that $\varphi(\alpha_i) := \alpha_{1-i}$ for $i = 0, 1$. Then it naturally induces the action on the weight space: $\varphi(\Lambda_i) := \Lambda_{1-i}$ for $i = 0, 1$.

First we note the following:

Lemma 3.3. *Let $j, k \in \mathbb{Z}$ and $\Lambda \in P_m^+$. Then*

$$\begin{aligned}
ch_\Lambda^{j,k} &= e^{-\frac{\pi i m}{2} jk} e^{\pi i j \langle \Lambda, \alpha_1^\vee \rangle} ch_{\varphi^k \Lambda}, \\
ch_\Lambda^{j+\frac{1}{2},k} &= e^{-\frac{\pi i m}{2}(j-\frac{1}{2})k} e^{\pi i j \langle \Lambda, \alpha_1^\vee \rangle} ch_{\varphi^k \Lambda}^{\frac{1}{2},0}, \\
ch_\Lambda^{j,k+\frac{1}{2}} &= e^{-\frac{\pi i m}{2} j(k+\frac{1}{2})} e^{\pi i j \langle \Lambda, \alpha_1^\vee \rangle} ch_{\varphi^k \Lambda}^{0,\frac{1}{2}}, \\
ch_\Lambda^{j+\frac{1}{2},k+\frac{1}{2}} &= e^{-\frac{\pi i m}{2}(jk+\frac{j}{2}-\frac{k}{2})} e^{\pi i j \langle \Lambda, \alpha_1^\vee \rangle} ch_{\varphi^k \Lambda}^{\frac{1}{2},\frac{1}{2}}.
\end{aligned}$$

Since $(\alpha_1|\alpha_1) = 2$, this gives immediately the following:

Lemma 3.3$'$. *Let $j,k\in\mathbb{Z}$ and $\Lambda\in P_m^+$. Then*

$$\begin{aligned}
e^{\pi i m(j\cdot\frac{\alpha_1}{2}|k\cdot\frac{\alpha_1}{2})}ch_{\Lambda}^{j,k} &= e^{\pi i j\langle\Lambda,\alpha_1^\vee\rangle}ch_{\varphi^k\Lambda},\\
e^{\pi i m((j+\frac{1}{2})\frac{\alpha_1}{2}|k\cdot\frac{\alpha_1}{2})}ch_{\Lambda}^{j+\frac{1}{2},k} &= e^{\frac{\pi i m}{2}k}e^{\pi i j\langle\Lambda,\alpha_1^\vee\rangle}ch_{\varphi^k\Lambda}^{\frac{1}{2},0},\\
e^{\pi i m(j\cdot\frac{\alpha_1}{2}|(k+\frac{1}{2})\frac{\alpha_1}{2})}ch_{\Lambda}^{j,k+\frac{1}{2}} &= e^{\pi i j\langle\Lambda,\alpha_1^\vee\rangle}ch_{\varphi^k\Lambda}^{0,\frac{1}{2}},\\
e^{\pi i m((j+\frac{1}{2})\frac{\alpha_1}{2}|(k+\frac{1}{2})\frac{\alpha_1}{2})}ch_{\Lambda}^{j+\frac{1}{2},k+\frac{1}{2}} &= e^{\frac{\pi i m}{2}(k+\frac{1}{4})}e^{\pi i j\langle\Lambda,\alpha_1^\vee\rangle}ch_{\varphi^k\Lambda}^{\frac{1}{2},\frac{1}{2}}.
\end{aligned}$$

We now consider the functions $\chi_{(k,\varepsilon_s),\Lambda}$ as defined by (25), for $k\in\Gamma$, $s\in(\frac{1}{2}\mathbb{Z})/4\mathbb{Z}$ and $\Lambda\in P_m^+$. By a simple calculation using the orthogonality of characters, one checks the following:

Lemma 3.4. *Let $k\in\mathbb{Z}/4\mathbb{Z}$ and $\Lambda\in P_m^+$, then*

1)

$$\chi_{(k,\varepsilon_s),\Lambda}=\begin{cases}\frac{1}{2}\left(ch_{\varphi^k\Lambda}+e^{\frac{\pi i m}{2}k}e^{-\frac{\pi i}{2}\langle\Lambda,\alpha_1^\vee\rangle}ch_{\varphi^k\Lambda}^{\frac{1}{2},0}\right) & \text{if } s=-\langle\Lambda,\alpha_1^\vee\rangle,\\ \frac{1}{2}\left(ch_{\varphi^k\Lambda}-e^{\frac{\pi i m}{2}k}e^{-\frac{\pi i}{2}\langle\Lambda,\alpha_1^\vee\rangle}ch_{\varphi^k\Lambda}^{\frac{1}{2},0}\right) & \text{if } s=-\langle\Lambda,\alpha_1^\vee\rangle+2,\\ 0 & \text{otherwise.}\end{cases}$$

2)

$$\chi_{(k+\frac{1}{2},\varepsilon_s),\Lambda}=\begin{cases}\frac{1}{2}\left(ch_{\varphi^k\Lambda}^{0,\frac{1}{2}}+e^{\frac{\pi i m}{2}(k+\frac{1}{4})}e^{-\frac{\pi i}{2}\langle\Lambda,\alpha_1^\vee\rangle}ch_{\varphi^k\Lambda}^{\frac{1}{2},\frac{1}{2}}\right) & \text{if } s=-\langle\Lambda,\alpha_1^\vee\rangle,\\ \frac{1}{2}\left(ch_{\varphi^k\Lambda}^{0,\frac{1}{2}}-e^{\frac{\pi i m}{2}(k+\frac{1}{4})}e^{-\frac{\pi i}{2}\langle\Lambda,\alpha_1^\vee\rangle}ch_{\varphi^k\Lambda}^{\frac{1}{2},\frac{1}{2}}\right) & \text{if } s=-\langle\Lambda,\alpha_1^\vee\rangle+2,\\ 0 & \text{otherwise.}\end{cases}$$

From this one immediately has

Lemma 3.5. *Let $k,s\in(\frac{1}{2}\mathbb{Z})/4\mathbb{Z}$, then*

1)

$$\chi_{(k+2,\varepsilon_s),\Lambda}=\begin{cases}\chi_{(k,\varepsilon_s),\Lambda} & \text{if } m \text{ is even,}\\ \chi_{(k,\varepsilon_{s+2}),\Lambda} & \text{if } m \text{ is odd.}\end{cases}$$

2)

$$\chi_{(k,\varepsilon_{-\langle\varphi\Lambda,\alpha_1^\vee\rangle}),\varphi\Lambda}=\begin{cases}\chi_{(k+1,\varepsilon_{-\langle\Lambda,\alpha_1^\vee\rangle}),\Lambda} & \text{if } \langle\Lambda,\alpha_0^\vee\rangle \text{ is even,}\\ \chi_{(k+1,\varepsilon_{-\langle\Lambda,\alpha_1^\vee\rangle+2}),\Lambda} & \text{if } \langle\Lambda,\alpha_0^\vee\rangle \text{ is odd,}\end{cases}$$

3)

$$\chi_{(k,\varepsilon_{-\langle\varphi\Lambda,\alpha_1^\vee\rangle+2}),\varphi\Lambda} = \begin{cases} \chi_{(k+1,\varepsilon_{-\langle\Lambda,\alpha_1^\vee\rangle+2}),\Lambda} & \text{if } \langle\Lambda,\alpha_0^\vee\rangle \text{ is even,} \\ \chi_{(k+1,\varepsilon_{-\langle\Lambda,\alpha_1^\vee\rangle}),\Lambda} & \text{if } \langle\Lambda,\alpha_0^\vee\rangle \text{ is odd.} \end{cases}$$

So putting

$$\chi^{+}_{k,\Lambda} := \chi_{(k,\varepsilon_{-\langle\Lambda,\alpha_1^\vee\rangle}),\Lambda} \qquad \text{and} \qquad \chi^{-}_{k,\Lambda} := \chi_{(k,\varepsilon_{-\langle\Lambda,\alpha_1^\vee\rangle+2}),\Lambda},$$

for $k \in (\frac{1}{2}\mathbb{Z})/4\mathbb{Z}$ and $\Lambda \in P^{+}_{m}$, one obtains a family of holomorphic functions $\chi^{\pm}_{0,\Lambda}$ and $\chi^{\pm}_{\frac{1}{2},\Lambda}$, whose linear span is invariant under modular transformations. The transformation of these functions under S is explicitly computed by the formula in Theorem 2.1, and the fusion coefficients due to the Verlinde's formula are obtained as follows in particular when m =1 or 2.

1) The case when m=1; We put

$$x_0 := \chi^{+}_{0,\Lambda_0}, \quad x_1 := \chi^{-}_{0,\Lambda_0}, \quad x_2 := \chi^{+}_{0,\Lambda_1}, \quad x_3 := \chi^{-}_{0,\Lambda_1},$$
$$x_4 := \chi^{+}_{\frac{1}{2},\Lambda_0}, \quad x_5 := \chi^{-}_{\frac{1}{2},\Lambda_0}, \quad x_6 := \chi^{+}_{\frac{1}{2},\Lambda_1}, \quad x_7 := \chi^{-}_{\frac{1}{2},\Lambda_1}.$$

Then the transformation matrix is obtained as follows:

	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7
x_0	$\frac{1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$
x_1	$\frac{1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$	$\frac{-1}{2\sqrt{2}}$	$\frac{-1}{2\sqrt{2}}$	$\frac{-1}{2\sqrt{2}}$	$\frac{-1}{2\sqrt{2}}$
x_2	$\frac{1}{2\sqrt{2}}$	$\frac{-1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$	$\frac{-1}{2\sqrt{2}}$	$\frac{i}{2\sqrt{2}}$	$\frac{i}{2\sqrt{2}}$	$\frac{-i}{2\sqrt{2}}$	$\frac{-i}{2\sqrt{2}}$
x_3	$\frac{1}{2\sqrt{2}}$	$\frac{-1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$	$\frac{-1}{2\sqrt{2}}$	$\frac{-i}{2\sqrt{2}}$	$\frac{-i}{2\sqrt{2}}$	$\frac{i}{2\sqrt{2}}$	$\frac{i}{2\sqrt{2}}$
x_4	$\frac{1}{2\sqrt{2}}$	$\frac{-1}{2\sqrt{2}}$	$\frac{i}{2\sqrt{2}}$	$\frac{-i}{2\sqrt{2}}$	$\frac{\omega^7}{2\sqrt{2}}$	$\frac{\omega^3}{2\sqrt{2}}$	$\frac{\omega}{2\sqrt{2}}$	$\frac{\omega^5}{2\sqrt{2}}$
x_5	$\frac{1}{2\sqrt{2}}$	$\frac{-1}{2\sqrt{2}}$	$\frac{i}{2\sqrt{2}}$	$\frac{-i}{2\sqrt{2}}$	$\frac{\omega^3}{2\sqrt{2}}$	$\frac{\omega^7}{2\sqrt{2}}$	$\frac{\omega^5}{2\sqrt{2}}$	$\frac{\omega}{2\sqrt{2}}$
x_6	$\frac{1}{2\sqrt{2}}$	$\frac{-1}{2\sqrt{2}}$	$\frac{-i}{2\sqrt{2}}$	$\frac{i}{2\sqrt{2}}$	$\frac{\omega}{2\sqrt{2}}$	$\frac{\omega^5}{2\sqrt{2}}$	$\frac{\omega^7}{2\sqrt{2}}$	$\frac{\omega^3}{2\sqrt{2}}$
x_7	$\frac{1}{2\sqrt{2}}$	$\frac{-1}{2\sqrt{2}}$	$\frac{-i}{2\sqrt{2}}$	$\frac{i}{2\sqrt{2}}$	$\frac{\omega^5}{2\sqrt{2}}$	$\frac{\omega}{2\sqrt{2}}$	$\frac{\omega^3}{2\sqrt{2}}$	$\frac{\omega^7}{2\sqrt{2}}$

where

$$\omega := e^{\frac{\pi i}{4}} = \frac{1+i}{\sqrt{2}}.$$

And the fusion coefficient N_{x_i,x_j,x_k} is $= 1$ if (i,j,k) is a permutation of (0,0,0), (0,1,1), (0,2,3), (0,4,6), (0,5,7), (1,2,2), (1,3,3), (1,4,7), (1,5,6), (2,4,4), (2,5,5), (2,6,7), (3,4,5), (3,6,6), (3,7,7), and $= 0$ otherwise, and so the fusion algebra is

$$y_i \cdot y_j = y_{i+j \bmod 8},$$

by putting

$$y_0{:=}x_0,\ y_1{:=}x_6,\ y_2{:=}x_2,\ y_3{:=}x_5,\ y_4{:=}x_1,\ y_5{:=}x_7,\ y_6{:=}x_3,\ y_7{:=}x_4;$$

namely the group algebra over the cyclic group of order 8.

2) The case when $m = 2$; We put

$$x_0 := \chi^+_{0,2\Lambda_0},\quad x_1 := \chi^-_{0,2\Lambda_0},\quad x_2 := \chi^+_{0,\Lambda_0+\Lambda_1},\quad x_3 := \chi^-_{0,\Lambda_0+\Lambda_1},$$

$$x_4 := \chi^+_{0,2\Lambda_1},\quad x_5 := \chi^-_{0,2\Lambda_2},\quad x_6 := \chi^+_{\frac{1}{2},2\Lambda_0},\quad x_7 := \chi^-_{\frac{1}{2},2\Lambda_0},$$

$$x_8 := \chi^+_{\frac{1}{2},\Lambda_0+\Lambda_1},\quad x_9 := \chi^-_{\frac{1}{2},\Lambda_0+\Lambda_1},\quad x_{10} := \chi^+_{\frac{1}{2},2\Lambda_1},\quad x_{11} := \chi^-_{\frac{1}{2},2\Lambda_1}.$$

Then the transformation matrix is obtained as follows:

	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}
x_0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$	$\frac{1}{4}$	$\frac{1}{4}$
x_1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{-1}{4}$	$\frac{-1}{4}$	$\frac{-1}{2\sqrt{2}}$	$\frac{-1}{2\sqrt{2}}$	$\frac{-1}{4}$	$\frac{-1}{4}$
x_2	$\frac{1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$	0	0	$\frac{-1}{2\sqrt{2}}$	$\frac{-1}{2\sqrt{2}}$	$\frac{i}{2\sqrt{2}}$	$\frac{i}{2\sqrt{2}}$	0	0	$\frac{-i}{2\sqrt{2}}$	$\frac{-i}{2\sqrt{2}}$
x_3	$\frac{1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$	0	0	$\frac{-1}{2\sqrt{2}}$	$\frac{-1}{2\sqrt{2}}$	$\frac{-i}{2\sqrt{2}}$	$\frac{-i}{2\sqrt{2}}$	0	0	$\frac{i}{2\sqrt{2}}$	$\frac{i}{2\sqrt{2}}$
x_4	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{-1}{2\sqrt{2}}$	$\frac{-1}{2\sqrt{2}}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{-1}{4}$	$\frac{-1}{4}$	$\frac{1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$	$\frac{-1}{4}$	$\frac{-1}{4}$
x_5	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{-1}{2\sqrt{2}}$	$\frac{-1}{2\sqrt{2}}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{-1}{2\sqrt{2}}$	$\frac{-1}{2\sqrt{2}}$	$\frac{1}{4}$	$\frac{1}{4}$
x_6	$\frac{1}{4}$	$\frac{-1}{4}$	$\frac{i}{2\sqrt{2}}$	$\frac{-i}{2\sqrt{2}}$	$\frac{-1}{4}$	$\frac{1}{4}$	$\frac{-i}{4}$	$\frac{i}{4}$	$\frac{1}{2\sqrt{2}}$	$\frac{-1}{2\sqrt{2}}$	$\frac{i}{4}$	$\frac{-i}{4}$
x_7	$\frac{1}{4}$	$\frac{-1}{4}$	$\frac{i}{2\sqrt{2}}$	$\frac{-i}{2\sqrt{2}}$	$\frac{-1}{4}$	$\frac{1}{4}$	$\frac{i}{4}$	$\frac{-i}{4}$	$\frac{-1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$	$\frac{-i}{4}$	$\frac{i}{4}$
x_8	$\frac{1}{2\sqrt{2}}$	$\frac{-1}{2\sqrt{2}}$	0	0	$\frac{1}{2\sqrt{2}}$	$\frac{-1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$	$\frac{-1}{2\sqrt{2}}$	0	0	$\frac{1}{2\sqrt{2}}$	$\frac{-1}{2\sqrt{2}}$
x_9	$\frac{1}{2\sqrt{2}}$	$\frac{-1}{2\sqrt{2}}$	0	0	$\frac{1}{2\sqrt{2}}$	$\frac{-1}{2\sqrt{2}}$	$\frac{-1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$	0	0	$\frac{-1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$
x_{10}	$\frac{1}{4}$	$\frac{-1}{4}$	$\frac{-i}{2\sqrt{2}}$	$\frac{i}{2\sqrt{2}}$	$\frac{-1}{4}$	$\frac{1}{4}$	$\frac{i}{4}$	$\frac{-i}{4}$	$\frac{1}{2\sqrt{2}}$	$\frac{-1}{2\sqrt{2}}$	$\frac{-i}{4}$	$\frac{i}{4}$
x_{11}	$\frac{1}{4}$	$\frac{-1}{4}$	$\frac{-i}{2\sqrt{2}}$	$\frac{i}{2\sqrt{2}}$	$\frac{-1}{4}$	$\frac{1}{4}$	$\frac{-i}{4}$	$\frac{i}{4}$	$\frac{-1}{2\sqrt{2}}$	$\frac{1}{2\sqrt{2}}$	$\frac{i}{4}$	$\frac{-i}{4}$

And the fusion coefficient N_{x_i,x_j,x_k} is $= 1$ if (i, j, k) is a permutation of (0,0,0), (0,1,1), (0,2,3), (0,4,4), (0,5,5), (0,6,10), (0,7,11), (0,8,8), (0,9,9), (1,2,2), (1,3,3), (1,4,5), (1,6,11), (1,7,10), (1,8,9), (2,2,4), (2,3,5), (2,6,8), (2,7,9), (2,8,11), (2,9,10), (3,3,4), (3,6,9), (3,7,8), (3,8,10), (3,9,11), (4,6,6), (4,7,7), (4,8,9), (4,10,10), (4,11,11), (5,6,7), (5,8,8), (5,9,9), (5,10,11), and $= 0$ otherwise. From this one obtains the following product table:

	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}
x_0	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}
x_1	x_1	x_0	x_3	x_2	x_5	x_4	x_7	x_6	x_9	x_8	x_{11}	x_{10}
x_2	x_2	x_3	$x_1 + x_4$	$x_0 + x_5$	x_3	x_2	x_8	x_9	$x_7 + x_{10}$	$x_6 + x_{11}$	x_9	x_8
x_3	x_3	x_2	$x_0 + x_5$	$x_1 + x_4$	x_2	x_3	x_9	x_8	$x_6 + x_{11}$	$x_7 + x_{10}$	x_8	x_9
x_4	x_4	x_5	x_3	x_2	x_0	x_1	x_{10}	x_{11}	x_9	x_8	x_6	x_7
x_5	x_5	x_4	x_2	x_3	x_1	x_0	x_{11}	x_{10}	x_8	x_9	x_7	x_6
x_6	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_4	x_5	x_3	x_2	x_0	x_1
x_7	x_7	x_6	x_9	x_8	x_{11}	x_{10}	x_5	x_4	x_2	x_3	x_1	x_0
x_8	x_8	x_9	$x_7 + x_{10}$	$x_6 + x_{11}$	x_9	x_8	x_3	x_2	$x_0 + x_5$	$x_1 + x_4$	x_2	x_3
x_9	x_9	x_8	$x_6 + x_{11}$	$x_7 + x_{10}$	x_8	x_9	x_2	x_3	$x_1 + x_4$	$x_0 + x_5$	x_3	x_2
x_{10}	x_{10}	x_{11}	x_9	x_8	x_6	x_7	x_0	x_1	x_2	x_3	x_4	x_5
x_{11}	x_{11}	x_{10}	x_8	x_9	x_7	x_6	x_1	x_0	x_3	x_2	x_5	x_4

§4. Twisted characters of principal admissible representations.

Given an affine Lie algebra $\mathfrak{g}(A)$, we introduce the following lattice:

$$\tilde{M} := \begin{cases} \overline{Q}^* & \text{if } r = 1, \\ (\overline{Q}^\vee)^* & \text{if } r \geq 2. \end{cases}$$

A rational number $m = \frac{p}{u} - h^\vee$ is called an *principal admissible number* (or shortly *admissible number*) if p and u are mutually co-prime positive integers such that $p \geq h^\vee$ and $\gcd(u, r^\vee) = 1$. For an admissible number m, u always stands for its denominator. Given an admissible number m, we choose $\overline{y} \in \overline{W}$ and $\beta \in \tilde{M}$ satisfying

$$y((u-1)c + \alpha_0^\vee) \in \Delta_+^\vee, \tag{36}$$

$$y(\alpha_i^\vee) \in \Delta_+^\vee \quad \text{for } i = 1, \cdots, \ell, \tag{37}$$

where $y := t_\beta \overline{y}$ and $c := \sum_{i=0}^{\ell} a_i^\vee \alpha_i^\vee$ is the canonical central element of $\mathfrak{g}(A)$.

Let $P_{m,y}^+$ denote the set of all

$$\lambda := y(\lambda^\circ - (u-1)(m+h^\vee)\Lambda_0 + \rho) - \rho, \tag{38}$$

where $\lambda^\circ \in P_{u(m+h^\vee)-h^\vee}^+$ are dominant integral forms. A linear form λ defined by (38) is called a *principal admissible weight* (or shortly *admissible weight* henceforward in this paper) of level m. The notion of principal admissible weights and principal admissible numbers was introduced in [8].

Given an admissible number m, the set of all principal admissible weights of level m is

$$P_m^+ = \bigcup_{\substack{y:\ \text{satisfying} \\ (36)\ \text{and}\ (37)}} P_{m,y}^+ .$$

The character of the irreducible highest weight module $L(\lambda)$, for an admissible weight $\lambda \in P_{m,y}^+$, is known by [6] and [7] as follows (see also [10]):

$$ch_\lambda(h) = \frac{B_\lambda(h)}{A_\rho(h)}, \tag{39}$$

where we put

$$B_\lambda(h) := A_{\lambda^\circ+\rho}(\widehat{h}), \tag{40}$$

and

$$\begin{aligned} \widehat{h} &:= u\overline{y}^{-1}t_{\frac{\beta}{u}}\left(\tau, \frac{z}{u}, \frac{t}{u^2}\right) \\ &= \left(u\tau, \overline{y}^{-1}(z+\tau\beta), \frac{1}{u}(t+(\beta|z)+\frac{\tau}{2}|\beta|^2)\right). \end{aligned} \tag{41}$$

So one has

(42)

$$\begin{aligned} & B_\lambda(\tau, z, t) \\ = \ & A_{\lambda^\circ+\rho}\left(u\tau, \overline{y}^{-1}(z+\tau\beta), \frac{1}{u}(t+(\beta|z)+\frac{\tau}{2}|\beta|^2)\right) \\ = \ & e^{-\frac{|\lambda^\circ+\rho|^2}{2u(m+h^\vee)}(\delta|-2\pi iu\tau\Lambda_0)} \\ & \quad \times \sum_{w\in W} \epsilon(w) e^{(w(\lambda^\circ+\rho)|(u\tau, \overline{y}^{-1}(z+\tau\beta), \frac{1}{u}(t+(\beta|z)+\frac{\tau}{2}|\beta|^2)))} \\ = \ & q^{\frac{|\lambda^\circ+\rho|^2}{2(m+h^\vee)}} \sum_{w\in W} \epsilon(w) e^{(w(\lambda^\circ+\rho)|(u\tau, \overline{y}^{-1}(z+\tau\beta), \frac{1}{u}(t+(\beta|z)+\frac{\tau}{2}|\beta|^2)))}, \end{aligned}$$

where $q := e^{2\pi i\tau}$ as usual.

The map $h \to \widehat{h}$ defined by (41) for given y satisfies the following:

Lemma 4.1. *Let $h = (\tau, z, t) \in \mathfrak{h}^*$ and $\alpha \in \overline{\mathfrak{h}}^*$; then*

1) $\widehat{t_\alpha h} = \left(u\tau, \overline{y}^{-1}(z+\tau(\beta-\alpha)), \frac{1}{u}(t+(\beta-\alpha|z)+\frac{\tau}{2}|\beta-\alpha|^2)\right),$

2) $\quad t_{\frac{1}{u}\overline{y}^{-1}\alpha}\widehat{h} = \widehat{t_\alpha h}.$

Proof. Since

$$\begin{aligned} t_\alpha h &= h - 2\pi i\tau\alpha - \left(\frac{|\alpha|^2}{2}(-2\pi i\tau) + 2\pi i(\alpha|z)\right)\delta \\ &= \left(\tau, z - \tau\alpha, t + \frac{\tau}{2}|\alpha|^2 - (\alpha|z)\right), \end{aligned}$$

one has

$$\begin{aligned} &\widehat{t_\alpha h} \\ =& \left(u\tau, \overline{y}^{-1}(z - \tau\alpha+\tau\beta), \frac{1}{u}(t+\frac{\tau}{2}|\alpha|^2-(\alpha|z)+(z-\tau\alpha|\beta)+\frac{\tau}{2}|\beta|^2)\right) \\ =& \left(u\tau, \overline{y}^{-1}(z+\tau(\beta-\alpha)), \frac{1}{u}(t+\frac{\tau}{2}|\beta-\alpha|^2+(\beta-\alpha|z))\right), \end{aligned}$$

proving 1). The formula 2) is shown as follows:

$$\begin{aligned} & t_{\frac{1}{u}\overline{y}^{-1}\alpha}\widehat{h} \\ =& \widehat{h} - 2\pi i u\tau\cdot\frac{\overline{y}^{-1}\alpha}{u} - \left(\frac{|\overline{y}^{-1}\alpha|^2}{2u^2}(-2\pi i u\tau) + (\hat{h}|\frac{\overline{y}^{-1}\alpha}{u})\right)\delta \\ =& \widehat{h} + \left(0, -\tau\overline{y}^{-1}\alpha, \frac{\tau}{2u}|\alpha|^2 - \frac{1}{u}(z+\tau\beta|\alpha)\right) \\ =& \left(u\tau, \overline{y}^{-1}(z+\tau(\beta-\alpha)), \frac{1}{u}(t+(z|\beta)+\frac{\tau}{2}|\beta|^2+\frac{\tau}{2}|\alpha|^2-(z+\tau\beta|\alpha))\right) \\ =& \left(u\tau, \overline{y}^{-1}(z+\tau(\beta-\alpha)), \frac{1}{u}(t+\frac{\tau}{2}|\beta-\alpha|^2+(\beta-\alpha|z))\right) \\ =& \widehat{t_\alpha h} \qquad\qquad\qquad \text{by 1)}, \end{aligned}$$

proving 2). Q.E.D.

Lemma 4.2. *Let $y = t_\beta\overline{y}$, $\lambda \in P^+_{m,y}$, $z' \in \overline{Q}^*$ and $t' \in \mathbb{C}$, then*

$$B_\lambda(\tau, z+z', t+t') = e^{2\pi i(\lambda^\circ+\rho|z')}e^{2\pi i(m+h^\vee)(z'|\beta)}e^{2\pi i(m+h^\vee)t'}B_\lambda(\tau, z, t).$$

Proof. By (42), one has

$$\begin{aligned} & B_\lambda(\tau, z+z', t+t') \\ =& q^{\frac{|\lambda^\circ+\rho|^2}{2(m+h^\vee)}}\sum_{w\in W}\epsilon(w)e^{(w(\lambda^\circ+\rho)|(u\tau,\overline{y}^{-1}(z+z'+\tau\beta),\frac{1}{u}(t+t'+(\beta|z+z')+\frac{\tau}{2}|\beta|^2)))} \\ =& q^{\frac{|\lambda^\circ+\rho|^2}{2(m+h^\vee)}}e^{2\pi i(\lambda^\circ+\rho|\frac{t'}{u}\delta)}e^{2\pi i(\lambda^\circ+\rho|\frac{1}{u}(z'|\beta)\delta)} \end{aligned}$$

$$\times \sum_{w\in W} \epsilon(w) e^{(w(\lambda^\circ+\rho)|(u\tau, \overline{y}^{-1}(z+z'+\tau\beta), \frac{1}{u}(t+(\beta|z)+\frac{\tau}{2}|\beta|^2)))}$$
$$= q^{\frac{|\lambda^\circ+\rho|^2}{2(m+h^\vee)}} e^{2\pi i(m+h^\vee)t'} e^{2\pi i(m+h^\vee)(z'|\beta)}$$
$$\times \sum_{w\in W} \epsilon(w) e^{(w(\lambda^\circ+\rho)|(u\tau, \overline{y}^{-1}(z+z'+\tau\beta), \frac{1}{u}(t+(\beta|z)+\frac{\tau}{2}|\beta|^2)))}.$$

Now the lemma follows since $e^{2\pi i(w(\lambda^\circ+\rho)|\overline{y}^{-1}z')} = e^{2\pi i(\lambda^\circ+\rho|z')}$ for all $w \in W$. Q.E.D.

Lemma 4.3. *Let m be an admissible number and (M_1, R_1) be a pair of lattices in $\overline{\mathfrak{h}}^*$ such that*

$$M_1 \subset uM. \tag{43}$$

Then, for $\lambda \in P^+_{m,y}$ and $\alpha, \alpha' \in R_1$ and $\xi, \eta \in M_1$, the following formulas hold:

1) $e^{\pi i(m+h^\vee)(\alpha+\xi|\alpha'+\eta)} B_\lambda^{\alpha+\xi, \alpha'+\eta}(h)$
$$= e^{2\pi i(m+h^\vee)(\alpha|\eta)} e^{2\pi i(m+h^\vee)(\xi|\beta)} e^{\pi i(m+h^\vee)(\alpha|\alpha')} B_\lambda^{\alpha,\alpha'}(h),$$

2) $e^{\pi i m(\alpha+\xi|\alpha'+\eta)} ch_\lambda^{\alpha+\xi, \alpha'+\eta}(h)$
$$= e^{2\pi i m(\alpha|\eta)} e^{2\pi i(m+h^\vee)(\xi|\beta)} e^{\pi i m(\alpha|\alpha')} ch_\lambda^{\alpha,\alpha'}(h).$$

Proof. First we note that the condition (43) implies

$$M_1 \subset \overline{Q}^\vee \subset \overline{Q}^*, \tag{44}$$

by (14). From (8) and Lemma 3.2 one has

$$\begin{aligned} B_\lambda^{\alpha,\alpha'}(h) &= B_\lambda(t_{\alpha'}h + 2\pi i\alpha - \pi i(\alpha|\alpha')\delta) \\ &= e^{-\pi i(m+h^\vee)(\alpha|\alpha')} B_\lambda(t_{\alpha'}h + 2\pi i\alpha). \end{aligned} \tag{45}$$

So

$e^{\pi i(m+h^\vee)(\alpha+\xi|\alpha'+\eta)} B_\lambda^{\alpha+\xi,\alpha'+\eta}(h)$
$= B_\lambda(t_{\alpha'+\eta}h + 2\pi i(\alpha+\xi))$
$= e^{2\pi i(\lambda^\circ+\rho|\xi)} e^{2\pi i(m+h^\vee)(\xi|\beta)} B_\lambda(t_{\alpha'+\eta}h + 2\pi i\alpha)$ by Lemma 4.2 and (44)
$= e^{2\pi i(m+h^\vee)(\xi|\beta)} B_\lambda(t_{\alpha'+\eta}h + 2\pi i\alpha)$ by (44)
$= e^{2\pi i(m+h^\vee)(\xi|\beta)} e^{2\pi i(m+h^\vee)(\alpha|\eta)} B_\lambda(t_\eta(t_{\alpha'}h + 2\pi i\alpha))$ by Lemma 4.2
$= e^{2\pi i(m+h^\vee)(\xi|\beta)} e^{2\pi i(m+h^\vee)(\alpha|\eta)} A_{\lambda^\circ+\rho}((t_\eta(t_{\alpha'}h + 2\pi i\alpha))^\frown)$

$$= e^{2\pi i(m+h^\vee)(\xi|\beta)} e^{2\pi i(m+h^\vee)(\alpha|\eta)} A_{\lambda^\circ+\rho}(t_{\frac{1}{u}\overline{y}^{-1}\eta}(t_{\alpha'}\widehat{h+2\pi i\alpha}))$$

by Lemma 4.1.

Since $\frac{1}{u}M_1 \subset M$ by (43), this becomes

$$= e^{2\pi i(m+h^\vee)(\xi|\beta)} e^{2\pi i(m+h^\vee)(\alpha|\eta)} A_{\lambda^\circ+\rho}(t_{\alpha'}\widehat{h+2\pi i\alpha})$$
$$= e^{2\pi i(m+h^\vee)(\xi|\beta)} e^{2\pi i(m+h^\vee)(\alpha|\eta)} B_\lambda(t_{\alpha'}h+2\pi i\alpha)$$
$$= e^{2\pi i(m+h^\vee)(\xi|\beta)} e^{2\pi i(m+h^\vee)(\alpha|\eta)} e^{\pi i(m+h^\vee)(\alpha|\alpha')} B_\lambda^{\alpha,\alpha'}(h) \qquad \text{by (45)},$$

proving 1). The formula 2) follows from 1) and Lemma 3.1. Q.E.D.

By this lemma, a holomorphic function

$$\chi_\lambda^{\alpha,\alpha'}(\tau,z,t) := e^{\pi i m(\alpha|\alpha')} ch_\lambda^{\alpha,\alpha'}(\tau,z,t) \tag{46}$$

is well-defined for an admissible weight $\lambda \in P_m^+$, and $\alpha, \alpha' \in R_1/M_1$ if conditions (43) and $m(M_1|R_1) \subset \mathbb{Z}$ are satisfied, since one always has $(M|\tilde{M}) \subset \mathbb{Z}$.

We now consider a datum $(m, \Gamma, M_\Gamma, R_\Gamma, \gamma)$ satisfying the following conditions:

(A1) m is an admissible number,

(A2) Γ is a finite group,

(A3) $M_\Gamma \subset R_\Gamma$ is a pair of lattices in $\overline{\mathfrak{h}}^*$ such that

$$M_\Gamma \subset uM, \tag{47}$$
$$m(M_\Gamma|R_\Gamma) \subset \mathbb{Z}, \tag{48}$$

(A4) $\gamma : \Gamma \to R_\Gamma/M_\Gamma$ is a map such that

$$\gamma(a^g) = \gamma(a) \qquad \text{for all } a, g \in \Gamma, \tag{49}$$
$$\gamma(ab) = \gamma(a) + \gamma(b) \qquad \text{if } ab = ba. \tag{50}$$

Associated to this finite group Γ, we consider the tranformation matrix (24) on the set $X_\Gamma := \tilde{X}_\Gamma/\Gamma$. For $\lambda \in P_m^+$ and $(a,\sigma) \in \tilde{X}_\Gamma$, we put

$$\chi_{(a,\sigma),\lambda}(\tau,z,t) := \frac{1}{|Z(a)|} \sum_{b \in Z(a)} \sigma(b) \chi_\lambda^{\gamma(b),\gamma(a)}(\tau,z,t). \tag{51}$$

Since $\chi_{(a,\sigma),\lambda} = \chi_{(a,\sigma)^g,\lambda}$, this defines a function $\chi_{[a,\sigma],\lambda}$ for $[a,\sigma] \in X_\Gamma$ and $\lambda \in P_m^+$.

The modular transformation of admissible characters is given in [7] as follows:

Lemma 4.4.

1) *If $r=1$ or $A=A_{2\ell}^{(2)}$, then*

$$ch_\lambda|_S(\tau,z,t) = \sum_{\lambda'\in P_m^+} a(\lambda,\lambda')ch_{\lambda'}(\tau,z,t),$$

and if A is twisted not of type $A=A_{2\ell}^{(2)}$, then

$$ch_\lambda|_S(\tau,z,t) = \sum_{\lambda'\in P_m^+(A')} a(\lambda,\lambda')ch_{\lambda'}\left(\frac{\tau}{r},\frac{z}{r},t\right),$$

where

$$a(\lambda,\lambda') := i^{|\overline{\Delta}_+|}\frac{1}{u^\ell(m+h^\vee)^{\ell/2}}|M^*/M'|^{-1/2}\epsilon(\overline{y}\overline{y'})$$
$$\times\sum_{w\in\overline{W}}\epsilon(w)e^{-\frac{2\pi i}{m+h^\vee}(w(\lambda+\rho)|\lambda'+\rho)}e^{-2\pi i((\lambda^\circ+\rho|\beta')+(\lambda'^\circ+\rho'|\beta)+(m+h^\vee)(\beta|\beta'))},$$

for $\lambda\in P_{m,y}^+(A)$ and $\lambda'\in P_{m,y'}^+(A')$.

2) *If A is of type $X_N^{(r)}$ $(X=A,\cdots,G)$ and $A\neq A_{2\ell}^{(2)}$, then*

$$ch_\lambda|_T(\tau,z,t) = e^{2\pi i s_\lambda}ch_\lambda(\tau,z,t),$$

where

$$s_\lambda := h_\lambda - \frac{1}{24}z_m, \qquad h_\lambda := \frac{(\lambda|\lambda+2\rho)}{2(m+h^\vee)},$$
$$z_m := \frac{m}{m+h^\vee}\dim(\mathfrak{g}(X_N)).$$

Now the argument goes quite similarly with integrable cases and we obtain the transformation formula for $\chi_{[g,\sigma],\lambda}$:

Theorem 4.1.

1) *If $r=1$ or $A=A_{2\ell}^{(2)}$, then*

$$\chi_{[g,\sigma],\lambda}|_S = \sum_{[g',\sigma']\in X_\Gamma}\sum_{\lambda'\in P_m^+} e^{-2\pi i m(\gamma(g)|\gamma(g'))}a(\lambda,\lambda')S_{[g,\sigma],[g',\sigma']}\cdot\chi_{[g',\sigma'],\lambda'}.$$

2) *If A is twisted not of type $A_{2\ell}^{(2)}$, then*

$$\chi_{[g,\sigma],\lambda}|_S(\tau,z,t)$$
$$= \sum_{[g',\sigma']\in X_\Gamma}\sum_{\lambda'\in P_m^+(A')} e^{-2\pi i m(\gamma(g)|\gamma(g'))}a(\lambda,\lambda')S_{[g,\sigma],[g',\sigma']}\cdot\chi_{[g',\sigma'],\lambda'}\left(\frac{\tau}{r},\frac{z}{r},t\right).$$

3) *If* $A \neq A_{2\ell}^{(2)}$, *then*

$$\chi_{[g,\sigma],\lambda}|_T = e^{2\pi i s_\lambda} e^{\pi i m|\gamma(g)|^2} \frac{\sigma(g)}{\sigma(1)} \chi_{[g,\sigma],\lambda}.$$

§5. A fusion datum with non-symmetric transformation matrix

The theory of a fusion datum associated to a non-symmetric transformation matrix was discussed by Lusztig [9]. In order to treat the modular transformation of twisted admissible characters, we extend his definition to a more general situation, and consider a fusion datum defined as follows:

Definition. *Let X be a finite set with a given element x_0 and with given commuting transformations $*, \sigma : X \to X$. Let $S = (S_{x,y})_{x,y\in X}$ and $T = (t_x \delta_{x,y})_{x,y\in X}$ be unitary matrices. We call $(X, x_0, *, \sigma, S, T)$ a fusion datum if the following conditions (FD1) - (FD6) are satisfied for all $x, y, z \in X$:*

(FD1) $S_{x,x_0} > 0$,
(FD2) $*$ *is involutive and* $x_0^* = x_0$,
(FD3) $S_{\sigma(x),\sigma^{-1}(y)} = S_{y,x}$,
(FD4) $S_{x^*,y} = \overline{S_{x,y}}$,
(FD5)

$$N_{x,y,z} := \sum_{w\in X} \frac{S_{x,w}S_{y,w}S_{z,w}}{S_{x_0,w}} = \sum_{w\in X} \frac{S_{w,x}S_{w,y}S_{w,z}}{S_{w,x_0}} \in \mathbb{Z}_{\geq 0},$$

(FD6) $S^4 = (ST^{-1})^3 = (ST)^6 =$ *the identity matrix.*

The condition (FD3) implies that the matrix $(S_{\sigma(x),y})_{x,y\in X}$ is symmetric. It follows from (FD3) and (FD4) that

$$S_{x,y^*} = \overline{S_{x,y}}. \tag{52}$$

We note that (52) and the condition (FD4), together with the unitarity of S, imply

$$\sum_{w\in X} S_{x,w}S_{y,w} = \sum_{w\in X} S_{w,x}S_{w,y} = \delta_{x,y^*}. \tag{53}$$

Given a fusion datum $(X, x_0, *, \sigma, S, T)$, the $\mathbb{C}$-linear span R of X becomes a commutative and associative algebra by the multiplication

$x \cdot y := \sum_{z\in X} N_{x,y,z^*} z$ with the unity x_0, called the fusion algebra associated to $(X, x_0, *, \sigma, S, T)$. This multiplication satisfies $(x \cdot y)^* = x^* \cdot y^*$ by conditions (FD4) and (FD5), namely the map $*$ is an involutive automorphism of R.

Some examples of such fusion data are supplied by twisted admissible characters of affine Lie algebras.

Example 5.1. Let us consider an affine algebra $A = A_1^{(1)} = \widehat{sl}(2,\mathbb{C})$, and let $m = \dfrac{p}{u}$ be an admissible number. The set of all admissible weights of level m is

$$P_m^+ = \{\Lambda_{m;k,n}\ ;\ k,n \in \mathbb{Z}_{\geq 0},\ 0 \leq k \leq u-1,\ 0 \leq n \leq p-2\},$$

where

$$\begin{aligned} \Lambda_{m;k,n} &:= (m-n+k(m+2))\Lambda_0 + (n-k(m+2))\Lambda_1 \\ &= m\Lambda_0 + \frac{n-k(m+2)}{2}\alpha_1, \end{aligned} \tag{54}$$

and an explicit formula of the modular transformation of these characters is given in [6].

Let $\Gamma := (\frac{1}{2}\mathbb{Z})/4u\mathbb{Z}$ be a cyclic group, and consider two lattices $M_\Gamma := 2u\overline{Q} = 4u\mathbb{Z}\overline{\Lambda_1}$ and $R_\Gamma := \frac{1}{2}\overline{P} = \frac{1}{2}\mathbb{Z}\overline{\Lambda_1}$. Let γ be a group isomorphism of Γ to R_Γ/M_Γ given by $\gamma(j) := j\overline{\Lambda}_1$ for all $j \in \frac{1}{2}\mathbb{Z}$, and then we get a datum $(m, \Gamma, R_\Gamma, M_\Gamma, \gamma)$ satisfying conditions (47) $\sim$ (50).

For each $s \in (\frac{1}{2}\mathbb{Z})/4u\mathbb{Z}$, let $\varepsilon_s : \Gamma \to U(1)$ denote the character defined by $\varepsilon_s(j) := e^{\frac{\pi i s j}{u}}$. Then the set $E(\Gamma)$ of all characters of the group Γ is

$$E(\Gamma) = E\left((\frac{1}{2}\mathbb{Z})/4u\mathbb{Z}\right) = \left\{\varepsilon_s;\ s \in (\frac{1}{2}\mathbb{Z})/4u\mathbb{Z}\right\}.$$

First we note the following formulas where the function $F^{j,j'}$, for a holomorphic function F on Y, is as defined by (35):

Lemma 5.1. *Let $j, j' \in \mathbb{Z}$ and $\Lambda_{m;k,n} \in P_m^+$. Then*

$$\begin{aligned} e^{\pi i m(j\cdot\frac{\alpha_1}{2}|j'\cdot\frac{\alpha_1}{2})} ch^{j,j'}_{\Lambda_{m;k,n}} &= \varepsilon_{nu-kp}(j) ch_{\Lambda_{m;k+j',n}}, \\ e^{\pi i m((j+\frac{1}{2})\frac{\alpha_1}{2}|j'\cdot\frac{\alpha_1}{2})} ch^{j+\frac{1}{2},j'}_{\Lambda_{m;k,n}} &= e^{\frac{\pi i m}{2}j'} \varepsilon_{nu-kp}(j) ch^{\frac{1}{2},0}_{\Lambda_{m;k+j',n}}, \\ e^{\pi i m(j\cdot\frac{\alpha_1}{2}|(j'+\frac{1}{2})\frac{\alpha_1}{2})} ch^{j,j'+\frac{1}{2}}_{\Lambda_{m;k,n}} &= \varepsilon_{nu-kp}(j) ch^{0,\frac{1}{2}}_{\Lambda_{m;k+j',n}}, \\ e^{\pi i m((j+\frac{1}{2})\frac{\alpha_1}{2}|(j'+\frac{1}{2})\frac{\alpha_1}{2})} ch^{j+\frac{1}{2},j'+\frac{1}{2}}_{\Lambda_{m;k,n}} &= e^{\frac{\pi i m}{2}(j'+\frac{1}{4})} \varepsilon_{nu-kp}(j) ch^{\frac{1}{2},\frac{1}{2}}_{\Lambda_{m;k+j',n}}, \end{aligned}$$

where

$$(55)\ ch_{\Lambda_{m;k+su,n}} := \begin{cases} ch_{\Lambda_{m;k,n}} & \textit{if } s \textit{ is even,} \\ -ch_{\Lambda_{m;k,p-2-n}} & \textit{if } s \textit{ is odd.} \end{cases}$$

We now consider the functions $\chi_{(j,\varepsilon_s),\Lambda_{m;k,n}}$ as defined by (51) for $j \in \Gamma$, $s \in (\frac{1}{2}\mathbb{Z})/4u\mathbb{Z}$ and $\Lambda_{m;k,n} \in P^+_m$. By a simple calculation using the orthogonality of characters, one obtains the following:

Lemma 5.2. *Let $j \in \mathbb{Z}/4u\mathbb{Z}$ and $\Lambda_{m;k,n} \in P^+_m$, then*

1)

$$\chi_{(j,\varepsilon_s),\Lambda_{m;k,n}} = \begin{cases} \frac{1}{2}\left(ch_{\Lambda_{m;j+k,n}} + e^{\frac{\pi i m}{2}(j+k)} e^{\pi i k} e^{-\frac{\pi i n}{2}} ch^{\frac{1}{2},0}_{\Lambda_{m;j+k,n}}\right) & \textit{if } s{=}kp{-}nu, \\ \frac{1}{2}\left(ch_{\Lambda_{m;j+k,n}} - e^{\frac{\pi i m}{2}(j+k)} e^{\pi i k} e^{-\frac{\pi i n}{2}} ch^{\frac{1}{2},0}_{\Lambda_{m;j+k,n}}\right) & \textit{if } s{=}kp{-}nu{+}2u, \\ 0 & \textit{otherwise.} \end{cases}$$

2)

$$\chi_{(j+\frac{1}{2},\varepsilon_s),\Lambda_{m;k,n}} = \begin{cases} \frac{1}{2}\left(ch^{0,\frac{1}{2}}_{\Lambda_{m;j+k,n}} + e^{\frac{\pi i m}{2}(j+k+\frac{1}{4})} e^{\pi i k} e^{-\frac{\pi i n}{2}} ch^{\frac{1}{2},\frac{1}{2}}_{\Lambda_{m;j+k,n}}\right) & \textit{if } s{=}kp{-}nu, \\ \frac{1}{2}\left(ch^{0,\frac{1}{2}}_{\Lambda_{m;j+k,n}} - e^{\frac{\pi i m}{2}(j+k+\frac{1}{4})} e^{\pi i k} e^{-\frac{\pi i n}{2}} ch^{\frac{1}{2},\frac{1}{2}}_{\Lambda_{m;j+k,n}}\right) & \textit{if } s{=}kp{-}nu{+}2u, \\ 0 & \textit{otherwise.} \end{cases}$$

From this one sees

$$(56)\quad \chi_{(j,\varepsilon_s),\Lambda_{m;k,n}} = \chi_{(j',\varepsilon_{s+2u(k+k')}),\Lambda_{m;k',n}} \qquad \text{if } j+k = j'+k',$$

for $j, j', s, s' \in (\frac{1}{2}\mathbb{Z})/4u\mathbb{Z}$, and obtains a family of holomorphic functions

$$(57)\qquad \chi^+_{j,(k,n)} := \chi_{(j,\varepsilon_{kp-nu}),\Lambda_{m;k,n}},$$

$$(58)\qquad \chi^-_{j,(k,n)} := \chi_{(j,\varepsilon_{kp-nu+2u}),\Lambda_{m;k,n}},$$

with $j = 0, \frac{1}{2}$ and $\Lambda_{m;k,n} \in P^+_m$, whose linear span is invariant under modular transformation. The transformation of these functions under S is given as follows:

Lemma 5.3.

1)

$$\chi^{\pm}_{0,(k,n)}|_S = \frac{1}{2}\sum_{k',n'} a(\Lambda_{m;k,n},\Lambda_{m;k',n'})\left(\chi^{+}_{0,(k',n')}+\chi^{-}_{0,(k',n')}\right)$$

$$\pm\frac{1}{2}e^{\frac{\pi ipk}{2u}}e^{-\frac{\pi in}{2}}\left\{\sum_{k'=0}^{u-2}\sum_{n'=0}^{p-2} a(\Lambda_{m;k,n},\Lambda_{m;k'+1,n'})\left(\chi^{+}_{\frac{1}{2},(k',n')}+\chi^{-}_{\frac{1}{2},(k',n')}\right)\right.$$

$$\left.-\sum_{n'=0}^{p-2} a(\Lambda_{m;k,n},\Lambda_{m;0,p-2-n'})\left(\chi^{+}_{\frac{1}{2},(u-1,n')}+\chi^{-}_{\frac{1}{2},(u-1,n')}\right)\right\},$$

2)

$$\chi^{\pm}_{\frac{1}{2},(k,n)}|_S = \frac{1}{2}\sum_{k',n'} a(\Lambda_{m;k,n},\Lambda_{m;k',n'})e^{-\frac{\pi ipk'}{2u}}e^{\frac{\pi in'}{2}}\left(\chi^{+}_{0,(k',n')}-\chi^{-}_{0,(k',n')}\right)$$

$$\pm\frac{1}{2}e^{-\frac{\pi im}{4}}e^{\frac{\pi ipk}{2u}}e^{-\frac{\pi in}{2}}$$

$$\times\left\{\sum_{k'=0}^{u-2}\sum_{n'=0}^{p-2} a(\Lambda_{m;k,n},\Lambda_{m;k'+1,n'})e^{-\frac{\pi ipk'}{2u}}e^{\frac{\pi in'}{2}}\left(\chi^{+}_{\frac{1}{2},(k',n')}-\chi^{-}_{\frac{1}{2},(k',n')}\right)\right.$$

$$\left.-\sum_{n'=0}^{p-2} a(\Lambda_{m;k,n},\Lambda_{m;0,p-2-n'})e^{-\frac{\pi ip(u-1)}{2u}}e^{\frac{\pi in'}{2}}\left(\chi^{+}_{\frac{1}{2},(u-1,n')}-\chi^{-}_{\frac{1}{2},(u-1,n')}\right)\right\}.$$

And the transformation under T is easily calculated from Theorem 4.1 :

Lemma 5.4.

1) $\chi^{\pm}_{0,(k,n)}|_T = e^{\frac{\pi i}{12}((2n-3k+2)^2-3)}\chi^{\pm}_{0,(k,n)}$,

2) $\chi^{\pm}_{\frac{1}{2},(k,n)}|_T = \pm e^{\frac{\pi i}{12}(2n-3k+\frac{1}{2})^2}\chi^{\pm}_{\frac{1}{2},(k,n)}$.

We note that the transformation matrix of S with respect to this basis is not symmetric in general; e.g.,

$$\chi^{+}_{0,(0,0)}|_S = \cdots + \frac{1}{2}a(\Lambda_{m;0,0},\Lambda_{m;1,0})\chi^{+}_{\frac{1}{2},(0,0)} + \cdots,$$

and

$$\chi^{+}_{\frac{1}{2},(0,0)}|_S = \frac{1}{2}a(\Lambda_{m;0,0},\Lambda_{m;0,0})\chi^{+}_{0,(0,0)} + \cdots.$$

We now look at the cases when $m = -\frac{1}{2}$ and $-\frac{4}{3}$ in more detail:
1) The case when $m = -\frac{1}{2}$;

The transformation matrix $a_{(k,n),(k',n')} := a(\Lambda_{-\frac{1}{2};k,n}, \Lambda_{-\frac{1}{2};k',n'})$ of level $m = -\frac{1}{2}$ is given by

$$\begin{pmatrix} a_{(00),(00)} & a_{(00),(01)} & a_{(00),(10)} & a_{(00),(11)} \\ a_{(01),(00)} & a_{(01),(01)} & a_{(01),(10)} & a_{(01),(11)} \\ a_{(10),(00)} & a_{(10),(01)} & a_{(10),(10)} & a_{(10),(11)} \\ a_{(11),(00)} & a_{(11),(01)} & a_{(11),(10)} & a_{(11),(11)} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{-1}{2} & \frac{i}{2} & \frac{i}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{i}{2} & \frac{i}{2} \end{pmatrix},$$

and Lemma 4.3 gives the following transformation formulas of $\chi^{\pm}_{j,(k,n)}$:

$$\begin{aligned} \chi^{\pm}_{0,(k,n)}|_S = & \frac{1}{2} \sum_{k',n'=0,1} a_{(k,n),(k',n')} \left(\chi^{+}_{0,(k',n')} + \chi^{-}_{0,(k',n')} \right) \\ & \pm \frac{1}{2} e^{\frac{3\pi i k}{4}} e^{-\frac{\pi i n}{2}} \left\{ \sum_{n'=0,1} a_{(k,n),(1,n')} \left(\chi^{+}_{\frac{1}{2},(0,n')} + \chi^{-}_{\frac{1}{2},(0,n')} \right) \right. \\ & \left. - \sum_{n'=0,1} a_{(k,n),(0,1-n')} \left(\chi^{+}_{\frac{1}{2},(1,n')} + \chi^{-}_{\frac{1}{2},(1,n')} \right) \right\}, \\ \chi^{\pm}_{\frac{1}{2},(k,n)}|_S = & \frac{1}{2} \sum_{k',n'=0,1} a_{(k,n),(k',n')} e^{-\frac{3\pi i k'}{4}} e^{\frac{\pi i n'}{2}} \left(\chi^{+}_{0,(k',n')} - \chi^{-}_{0,(k',n')} \right) \\ & \pm \frac{1}{2} e^{\frac{3\pi i k}{4}} e^{-\frac{\pi i n}{2}} \left\{ e^{\frac{\pi i}{8}} \sum_{n'=0,1} a_{(k,n),(1,n')} e^{\frac{\pi i n'}{2}} \left(\chi^{+}_{\frac{1}{2},(0,n')} - \chi^{-}_{\frac{1}{2},(0,n')} \right) \right. \\ & \left. + e^{\frac{3\pi i}{8}} \sum_{n'=0,1} a_{(k,n),(0,1-n')} e^{\frac{\pi i n'}{2}} \left(\chi^{+}_{\frac{1}{2},(1,n')} - \chi^{-}_{\frac{1}{2},(1,n')} \right) \right\}. \end{aligned}$$

Changing a basis $\varphi^{\pm}_{j,k,n} := (-1)^{k+n} \chi^{\pm}_{\frac{j}{2},(k,n)}$ and renaming them as

$$\begin{aligned} & x_0 := \varphi^{+}_{0,0,0}, \quad x_8 := \varphi^{-}_{0,0,0}, \quad x_4 := \varphi^{+}_{0,0,1}, \quad x_{12} := \varphi^{-}_{0,0,1}, \\ & x_{10} := \varphi^{+}_{0,1,0}, \quad x_2 := \varphi^{-}_{0,1,0}, \quad x_{14} := \varphi^{+}_{0,1,1}, \quad x_6 := \varphi^{-}_{0,1,1}, \\ & x_1 := \varphi^{+}_{1,0,0}, \quad x_9 := \varphi^{-}_{1,0,0}, \quad x_5 := \varphi^{+}_{1,0,1}, \quad x_{13} := \varphi^{-}_{1,0,1}, \\ & x_{11} := \varphi^{+}_{1,1,0}, \quad x_3 := \varphi^{-}_{1,1,0}, \quad x_{15} := \varphi^{+}_{1,1,1}, \quad x_7 := \varphi^{-}_{1,1,1}, \end{aligned}$$

the modular transformations $x_i|_T = t_i x_i$ and $x_i|_S = \sum_{0 \le j \le 15} S_{ij} x_j$ are explicitly given as follows :

$$\begin{aligned} & t_0 = t_8 = \zeta^4, \quad t_4 = t_{12} = -\zeta^4, \quad t_2 = t_6 = t_{10} = t_{14} = -\zeta^{40}, \\ & t_1 = t_{15} = \zeta, \quad t_7 = t_9 = -\zeta, \quad t_5 = t_{11} = \zeta^{25}, \quad t_3 = t_{13} = -\zeta^{25}, \end{aligned}$$

and

$$(S_{ij})_{i,j=0,\cdots,15} = \tfrac{1}{4}\times$$

$$\begin{pmatrix}
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & \theta^9 & \theta^2 & \theta^{11} & \theta^4 & \theta^{13} & \theta^6 & \theta^{15} & -1 & \theta & \theta^{10} & \theta^3 & \theta^{12} & \theta^5 & \theta^{14} & \theta^7 \\
1 & \theta^{10} & \theta^4 & \theta^{14} & -1 & \theta^2 & \theta^{12} & \theta^6 & 1 & \theta^{10} & \theta^4 & \theta^{14} & -1 & \theta^2 & \theta^{12} & \theta^6 \\
1 & \theta^{11} & \theta^6 & \theta & \theta^{12} & \theta^7 & \theta^2 & \theta^{13} & -1 & \theta^3 & \theta^{14} & \theta^9 & \theta^4 & \theta^{15} & \theta^{10} & \theta^5 \\
1 & \theta^{12} & -1 & \theta^4 & 1 & \theta^{12} & -1 & \theta^4 & 1 & \theta^{12} & -1 & \theta^4 & 1 & \theta^{12} & -1 & \theta^4 \\
1 & \theta^{13} & \theta^{10} & \theta^7 & \theta^4 & \theta & \theta^{14} & \theta^{11} & -1 & \theta^5 & \theta^2 & \theta^{15} & \theta^{12} & \theta^9 & \theta^6 & \theta^3 \\
1 & \theta^{14} & \theta^{12} & \theta^{10} & -1 & \theta^6 & \theta^4 & \theta^2 & 1 & \theta^{14} & \theta^{12} & \theta^{10} & -1 & \theta^6 & \theta^4 & \theta^2 \\
1 & \theta^{15} & \theta^{14} & \theta^{13} & \theta^{12} & \theta^{11} & \theta^{10} & \theta^9 & -1 & \theta^7 & \theta^6 & \theta^5 & \theta^4 & \theta^3 & \theta^2 & \theta \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & \theta & \theta^2 & \theta^3 & \theta^4 & \theta^5 & \theta^6 & \theta^7 & -1 & \theta^9 & \theta^{10} & \theta^{11} & \theta^{12} & \theta^{13} & \theta^{14} & \theta^{15} \\
1 & \theta^2 & \theta^4 & \theta^6 & -1 & \theta^{10} & \theta^{12} & \theta^{14} & 1 & \theta^2 & \theta^4 & \theta^6 & -1 & \theta^{10} & \theta^{12} & \theta^{14} \\
1 & \theta^3 & \theta^6 & \theta^9 & \theta^{12} & \theta^{15} & \theta^2 & \theta^5 & -1 & \theta^{11} & \theta^{14} & \theta & \theta^4 & \theta^7 & \theta^{10} & \theta^{13} \\
1 & \theta^4 & -1 & \theta^{12} & 1 & \theta^4 & -1 & \theta^{12} & 1 & \theta^4 & -1 & \theta^{12} & 1 & \theta^4 & -1 & \theta^{12} \\
1 & \theta^5 & \theta^{10} & \theta^{15} & \theta^4 & \theta^9 & \theta^{14} & \theta^3 & -1 & \theta^{13} & \theta^2 & \theta^7 & \theta^{12} & \theta & \theta^6 & \theta^{11} \\
1 & \theta^6 & \theta^{12} & \theta^2 & -1 & \theta^{14} & \theta^4 & \theta^{10} & 1 & \theta^6 & \theta^{12} & \theta^2 & -1 & \theta^{14} & \theta^4 & \theta^{10} \\
1 & \theta^7 & \theta^{14} & \theta^5 & \theta^{12} & \theta^3 & \theta^{10} & \theta & -1 & \theta^{15} & \theta^6 & \theta^{13} & \theta^4 & \theta^{11} & \theta^2 & \theta^9
\end{pmatrix},$$

where $\theta := e^{\frac{\pi i}{8}}$ and $\zeta := e^{\frac{\pi i}{48}}$.

These matrices S and T satisfy conditions (FD1)-(FD6) with the involutions

$$x_i^* := x_{16-i},$$

$$\sigma(x_i) := \begin{cases} x_{i+8 \mod 16} & \text{if } i \text{ is even,} \\ x_i & \text{if } i \text{ is odd.} \end{cases}$$

The associated fusion algebra is $x_i \cdot x_j = x_{i+j \mod 16}$, namely the group algebra over the cyclic group of order 16.

2) The case when $m = -\frac{4}{3}$;

The transformation matrix of level $m = -\frac{4}{3}$ is

$$(a_{k,k'})_{k,k'=0,1,2} := \left(a(\Lambda_{-\frac{4}{3};k,0}, \Lambda_{-\frac{4}{3};k',0})\right)_{k,k'=0,1,2} = \begin{pmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{\omega}{\sqrt{3}} & \frac{\omega^2}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} & \frac{\omega^2}{\sqrt{3}} & \frac{\omega}{\sqrt{3}} \end{pmatrix},$$

where $\omega := e^{\frac{\pi i}{3}}$, and Lemma 5.3 gives the following transformation formulas of $\chi^{\pm}_{j,(k,0)}$'s:

$$\begin{aligned}
\chi^{\pm}_{0,(k,0)}|_S &= \frac{1}{2}\sum_{k'=0}^{2} a_{k,k'}\left(\chi^{+}_{0,(k',0)} + \chi^{-}_{0,(k',0)}\right)\\
&\pm \frac{1}{2}e^{\frac{\pi i k}{3}}\Bigg\{\sum_{k'=0,1} a_{k,k'+1}\left(\chi^{+}_{\frac{1}{2},(k',0)}+\chi^{-}_{\frac{1}{2},(k',0)}\right)\\
&\qquad -a_{k,0}\left(\chi^{+}_{\frac{1}{2},(2,0)}+\chi^{-}_{\frac{1}{2},(2,0)}\right)\Bigg\},\\
\chi^{\pm}_{\frac{1}{2},(k,0)}|_S &= \frac{1}{2}\sum_{k'=0}^{2} a_{k,k'}e^{-\frac{\pi i k'}{3}}\left(\chi^{+}_{0,(k',0)} - \chi^{-}_{0,(k',0)}\right)\\
&\pm \frac{1}{2}e^{\frac{\pi i(k+1)}{3}}\Bigg\{\sum_{k'=0,1} a_{k,k'+1}e^{-\frac{\pi i k'}{3}}\left(\chi^{+}_{\frac{1}{2},(k',0)} - \chi^{-}_{\frac{1}{2},(k',0)}\right)\\
&\qquad -a_{k,0}e^{-\frac{2\pi i}{3}}\left(\chi^{+}_{\frac{1}{2},(2,0)} - \chi^{-}_{\frac{1}{2},(2,0)}\right)\Bigg\}.
\end{aligned}$$

Changing a basis $\varphi^{\pm}_{j,k} := i(-1)^k\chi^{\pm}_{\frac{j}{2},(k,0)}$ and renaming them as

$$\begin{array}{lll}
x_0 = x_{(000)} := \varphi^{+}_{0,0}, & x_5 = x_{(101)} := \varphi^{+}_{0,1}, & x_8 = x_{(200)} := \varphi^{+}_{0,2},\\
x_{11} = x_{(211)} := \varphi^{+}_{1,0}, & x_2 = x_{(010)} := \varphi^{+}_{1,1}, & x_7 = x_{(111)} := \varphi^{+}_{1,2},\\
x_1 = x_{(001)} := \varphi^{-}_{0,0}, & x_4 = x_{(100)} := \varphi^{-}_{0,1}, & x_9 = x_{(201)} := \varphi^{-}_{0,2},\\
x_{10} = x_{(210)} := \varphi^{-}_{1,0}, & x_3 = x_{(011)} := \varphi^{-}_{1,1}, & x_6 = x_{(110)} := \varphi^{-}_{1,2},
\end{array}$$

the modular transformation $x_i|_T = t_i x_i$ and $x_i|_S = \sum_{0\le j\le 11} S_{ij}x_j$ are explicitly given as follows:

$$\begin{array}{llll}
t_0 = t_1 = \zeta^4, & t_8 = t_9 = -\zeta^4, & t_4 = t_5 = -\zeta^{40}, & \\
t_2 = t_7 = \zeta^{25}, & t_3 = t_6 = -\zeta^{25}, & t_{11} = \zeta, & t_{10} = -\zeta,
\end{array}$$

and $\quad (S_{ij})_{i,j=0,\cdots,11} = \frac{1}{2\sqrt{3}} \times$

$$\begin{pmatrix}
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\
1 & 1 & -1 & -1 & \omega^4 & \omega^4 & \omega & \omega & \omega^2 & \omega^2 & \omega^5 & \omega^5 \\
1 & 1 & 1 & 1 & \omega^4 & \omega^4 & \omega^4 & \omega^4 & \omega^2 & \omega^2 & \omega^2 & \omega^2 \\
1 & -1 & 1 & -1 & \omega^4 & \omega & \omega^4 & \omega & \omega^2 & \omega^5 & \omega^2 & \omega^5 \\
1 & -1 & -1 & 1 & \omega^4 & \omega & \omega & \omega^4 & \omega^2 & \omega^5 & \omega^5 & \omega^2 \\
1 & 1 & -1 & -1 & \omega^2 & \omega^2 & \omega^5 & \omega^5 & \omega^4 & \omega^4 & \omega & \omega \\
1 & 1 & 1 & 1 & \omega^2 & \omega^2 & \omega^2 & \omega^2 & \omega^4 & \omega^4 & \omega^4 & \omega^4 \\
1 & -1 & 1 & -1 & \omega^2 & \omega^5 & \omega^2 & \omega^5 & \omega^4 & \omega & \omega^4 & \omega \\
1 & -1 & -1 & 1 & \omega^2 & \omega^5 & \omega^5 & \omega^2 & \omega^4 & \omega & \omega & \omega^4
\end{pmatrix},$$

where $\omega := e^{\frac{\pi i}{3}}$ and $\zeta := e^{\frac{\pi i}{48}}$.

We consider three transformations $*$ and σ_s $(s = 1, 2)$ of the set

$$X := \{x_i \ ; \ i = 0, \cdots, 11\} = \{x_{(i,j,k)} \ ; \ i \in \mathbb{Z}/3\mathbb{Z} \text{ and } j, k \in \mathbb{Z}/2\mathbb{Z}\}$$

defined by

$$x^*_{(i,j,k)} := x_{(3-i \bmod 3,\, j,\, k)} \quad \text{and} \quad \sigma_s(x_j) := x_{\tilde{\sigma}_s(j)},$$

where $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ are permutations:

$$\begin{aligned}
\tilde{\sigma}_1 &:= (0,1)(4,5)(8,9) \\
&= \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 1 & 0 & 2 & 3 & 5 & 4 & 6 & 7 & 9 & 8 & 10 & 11 \end{pmatrix}, \\
\tilde{\sigma}_2 &:= (0,1,2)(4,5,6)(8,9,10) \\
&= \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 1 & 2 & 0 & 3 & 5 & 6 & 4 & 7 & 9 & 10 & 8 & 11 \end{pmatrix}.
\end{aligned}$$

Then $(X, x_0, *, \sigma_s, S, T)$ $(s = 1, 2)$ is a fusion datum, whose associated fusion algebra is

$$x_{(i,j,k)} \cdot x_{(i',j',k')} = x_{(i+i' \bmod 3,\, j+j' \bmod 2,\, k+k' \bmod 2)},$$

namely the group algebra over $\mathbb{Z}/3\mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})^2$.

We remark that, in this example, σ_2 is not involutive and t_{x^*} is not necessarily equal to t_x.

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