

Matrix Coefficients of the Principal P_J -series and the Middle Discrete Series of $SU(2, 2)$

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§0. Motivation and introduction

Recently several people including the authors investigate various kinds of generalized spherical functions on relatively small Lie groups. The typical situation of the problem is as follows.

Let G be a semisimple Lie group with finite center and R a closed subgroup. Let π be an admissible (irreducible) representation of G , and η an irreducible admissible or unitary representation of R . Form the smooth induction of $\text{Ind}_R^G(\eta)$ from R to G . Then we consider the intertwining space $\text{Hom}_G(\pi, \text{Ind}_R^G(\eta))$.

Our concern is the case where R is “large enough” so that the above intertwining space becomes of finite dimension. This kind of problem is standard in the local theory of automorphic forms and representations.

If we take a nonzero element $T \in \text{Hom}_G(\pi, \text{Ind}_R^G(\eta))$, then the image $\text{Im}(T)$ of T is a realization (or a model) of π in $\text{Ind}_R^G(\eta)$.

A most popular case is when R is the maximal unipotent subgroup and η a unitary non-degenerate character of R . Then the intertwiners are called Whittaker functionals. These are investigated mainly to define automorphic L -functions in GL_n case (Jacquet, Piatetski-Shapiro and Shalika [8, 9]) not only for the real field but also for the p -adic fields. Fundamental results on Whittaker models over the real field are found in the works of Kostant [11], Vogan [24] and Matumoto [12, 13].

More generally, there is a notion of Gel’fand-Graev representation, where R is a “good” subgroup of some parabolic subgroup of G . An attempt to classify representation of G using this kind of realization in $\text{Ind}_R^G(\eta)$ is worked out by Yamashita [26, 27] for $G = SU(2, 2)$.

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If one wants to consider Fourier expansion of an automorphic form on G with respect to a discrete subgroup Γ , then depending on the choice of standard parabolic subgroups compatible with the rational structure of Γ , we have various different kinds of Fourier expansions: one is the classical expansion with respect to the Siegel parabolic subgroup (say, in $Sp(2, \mathbb{R})$ or $SU(2, 2)$), one is called the Fourier-Jacobi expansion, a Fourier expansion with respect to certain maximal parabolic subgroup ([1]).

However we know little about these expansions except for the holomorphic automorphic forms when G/K , K being a maximal compact subgroup of G , is hermitian symmetric. Beyond this classical objects, we did not yet have any concrete results even for the most basic problem to investigate the special functions which appear in Fourier expansions of automorphic forms. More serious problem is that without some kind of uniqueness of such special functions (local multiplicity one theorem), it is impossible even to define the notion of Fourier coefficients of automorphic forms.

But now, when π is a discrete series representation, and R is a stabilizer of a unitary character $\xi: N_S \rightarrow \mathbb{C}^{(1)}$ of the unipotent radical of the Siegel parabolic subgroup P_S of $G = Sp(2; \mathbb{R})$ or $SU(2, 2)$ inside P_S , we know certain explicit formulae for the realization of π in $\text{Ind}_R^G(\eta)$ and multiplicity one theorem as corollary ([2, 14]). Among others, Gon [2] gave a very interesting result for the discrete series representation of Gel'fand-Kirillov dimension 5: roughly speaking, such a discrete series has a non-trivial model only when ξ is indefinite. In some sense, it is a reverse of the well-known Köcher principle for holomorphic discrete series. Also one of the authors worked the case of Whittaker model for the group $SU(2, 2)$ ([3, 4, 5]).

To push forward in this direction, in this paper we return to the most classical problem, *i.e.*, to investigate the matrix coefficients of certain standard representations of $SU(2, 2)$. This is the case R is equal to the maximal compact subgroup K .

More specifically we have a few direct reasons to investigate this:

1. The functions in the Whittaker models investigated by Miyazaki and Gon, are considered as the confluence version of spherical functions;
2. The matrix coefficients of the discrete series are analogue of the Bergmann kernel in the case of holomorphic discrete series. The matrix coefficients of principal P_J -series may be useful to define certain Green functions which should have some relation with arithmetic intersection theory.

Apart from number theoretical problem, our results give examples of integrable system and special functions. Although there is a large literature on spherical functions, they mostly discuss only zonal spherical functions if the rank of $G \geq 2$. Our cases have non-trivial K -types.

The contents of this article are partly a short review on former results discussed above, and an announcement on recent results on the matrix coefficient as the title indicates.

The contents read as follows:

- §1. Notations, principal P_J -series and discrete series of $SU(2, 2)$.
- §2. Whittaker functions.
- §3. Generalized Whittaker functions (Siegel case).
- §4. Matrix coefficients: P_J -series
- §5. Matrix coefficients: the middle discrete series
- §6. Further problems and some comments

Let us discuss the part of the new results in more detail (§§4, 5).

Our concern in these sections is to have an explicit formula of the radial part of the matrix coefficients with some K -type of π ; π is either certain generalized principal series representation, *i.e.*, principal P_J -series, or the middle discrete series representation of the real unitary group $SU(2, 2)$ of signature $(2+, 2-)$.

The non-compact Lie group $G = SU(2, 2)$ of dimension 15 has two types of the standard maximal parabolic subgroups. One, which corresponds to the long root in the restricted root system (C_2 -type), that has non-abelian unipotent radical, is denote by $P_J = M_J A_J N_J$.

Given a discrete series representation σ of M_J and a complex linear form ν on the Lie algebra $\mathfrak{a}_J = \text{Lie}(A_J)$, the parabolic induction $\pi_{P_J} = \text{Ind}_{P_J}^G(\sigma \otimes e^{\nu+\rho_J} \otimes 1_{N_J})$ is a principal P_J -series representation of G . Here ρ_J denotes the half-sum of roots in the unipotent part N_J , $e^{\nu+\rho_J}: A_J \rightarrow \mathbb{C}^*$ the quasi-character with derivation $\nu + \rho_J: \mathfrak{a}_J \rightarrow \mathbb{C}$, and $\sigma \otimes e^{\nu+\rho_J}$ is the exterior tensor representation of the Levi part $M_J A_J$ of P_J .

As for the discrete series representations of $SU(2, 2)$, there exist six inequivalent discrete series representations having the same infinitesimal character according to Harish-Chandra. Their Gel'fand-Kirillov dimension (Dim) is between 4 and 6. We say π_{MD} the *middle* discrete series representation if it attains $\text{Dim } \pi_{\text{MD}} = 5$.

Consider the subset consisting of the highest weights of the K -types occurring in $\pi_{P_J}|_K$ in the weight lattice of the maximal compact subgroup K of G . Then we can find that it is a "translation" of the similar set of the highest weights of the K -types of a large discrete series representation of G , here "large" is in the sense of Kostant-Vogan. *i.e.*, $\text{Dim } \pi = 6$ in our situation. Therefore corresponding to the minimal

K -type of a large discrete series, we can consider the “corner” K -type of our π ([4, §4.3]).

Let τ be the irreducible finite-dimensional representation of K such that its dual τ^* is the corner (resp. the minimal) K -type of π_{P_J} (resp. π_{MD}).

Fix a basis $\{v_i\}_{0 \leq i \leq d}$ of the representation space W_τ of τ , and let $\{v_j^*\}$ be its dual basis in W_{τ^*} . Then we can form $(d+1) \times (d+1)$ -matrix

$$\Phi(g) := (\pi(g)v_i^*, v_j^*)_{0 \leq i, j \leq d} = (c_{i,j}(a))_{0 \leq i, j \leq d}.$$

Here $(*, *)$ is the given inner product on the representation space H_π of a Hilbert space representation π . The value of Φ is determined by its restriction $\Phi|_A$ to the radical part $A \cong (\mathbb{R}_{>0})^2$, because of the Cartan decomposition $G = KAK$.

Our main result is to show that 1) if $\pi = \pi_{P_J}$ the radial part of each entry of the matrix Φ satisfies a holonomic system which is equivalent to Appell’s hypergeometric system of type F_2 , in fact, it is the modified F_2 system ([23, p. 211, (2.1)]) and to have an integral representation of it in terms of Gaussian hypergeometric series (Theorem 4.4); 2) if $\pi = \pi_{\text{MD}}$ the radial part is expressed by the sum of the Gaussian hypergeometric functions ${}_2F_1$ of one variable with Harish-Chandra transformation of the variables (Theorem 5.4).

Let us explain the outline of the method of proof. Set $M = Z_A(K)$ the centralizer of A in K . Then $\Phi(mam^{-1}) = \Phi(a)$ for any $a \in A$ and $m \in M$. This M -compatibility implies that many entries of the matrix $\Phi(a)$ vanishes (cf. (4.15)). Let Ω be the Casimir operator in $Z(\mathfrak{g})$. Then, since π is quasi-simple, the A -radial part $\rho_A(\Omega)$ of Ω acts on $\Phi|_A$ as a scalar multiple of $\chi_\pi(\Omega)$, the value of the infinitesimal character χ_π of π at Ω :

$$\rho_A(\Omega)(\Phi|_A) = \chi_\pi(\Omega)(\Phi|_A).$$

Moreover we consider a gradient-type operator called the Schmid operator ∇_{η, τ^*} (§1.9). Using a non-compact root δ such that the shift of the corner K -type τ by δ is no longer a K -type of π , we can construct

$$P^{\delta^*} \circ \rho_A(\nabla_{\eta, \tau^*})(\Phi|_A) = 0. \quad (\text{cf. (1.7)})$$

Those differential operators produce a number of difference-differential equations satisfied by $c_{i,j}(a)$, whose straightforward solutions will give the results.

We make a few remarks. Our result of the case π_{P_J} is quite similar to that of Iida [7]. There should be an explanation by the fact that the restricted root system of $Sp(2; \mathbb{R})$ and $SU(2, 2)$ are both of C_2 -type.

As seen from the results, the complexity of the solution reflects the size of the representation of G . So the matrix coefficients of the “large” discrete series also satisfy the similar formula as that of π_{P_3} , which will be discussed elsewhere.

This article is based on the talks in the symposium held at Hayashibara Institute on July, 1997 and in the subsequent symposium held at Yukawa Institute in Kyoto University on August, 1997.

§1. Notations, principal P_J -series and discrete series of $SU(2, 2)$

1.1. Lie groups and Lie algebras

Our Lie group is the special unitary group of real dimension 15

$$G = SU(2, 2) = \{g \in SL_4(\mathbb{C}) \mid {}^t\bar{g}I_{2,2}g = I_{2,2}\}; \quad I_{2,2} = \text{diag}(1, 1, -1, -1)$$

with Lie algebra

$$\mathfrak{g} = \text{Lie}(G) = \left\{ \begin{pmatrix} X & Z \\ {}^t\bar{Z} & Y \end{pmatrix} \in \mathfrak{sl}_4(\mathbb{C}) \mid {}^t\bar{X} + X = {}^t\bar{Y} + Y = 0 \right\}.$$

We fix a maximal compact subgroup $K \cong S(U(2) \times U(2))$ of G with its Lie algebra

$$\mathfrak{k} = \text{Lie}(K) = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathfrak{sl}_4(\mathbb{C}) \mid {}^t\bar{A} + A = {}^t\bar{B} + B = 0 \right\}.$$

It is 7-dimensional. We fix a compact Cartan subgroup T of G consisting of the diagonal matrices in K . Complexification of \mathfrak{g} , \mathfrak{k} and $\mathfrak{t} = \text{Lie}(T)$ are denoted by $\mathfrak{g}_{\mathbb{C}}$, $\mathfrak{k}_{\mathbb{C}}$, and $\mathfrak{t}_{\mathbb{C}}$, respectively.

We denote by X_{ij} the matrix unit with 1 at (i, j) -component. The complement space of the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is given by

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & Z \\ {}^t\bar{Z} & 0 \end{pmatrix} \mid Z \in M_2(\mathbb{C}) \right\}.$$

A maximal abelian subspace in \mathfrak{p} is given by

$$\mathfrak{a} = \mathbb{R}H_1 \oplus \mathbb{R}H_2$$

with $H_1 = X_{13} + X_{31}$, $H_2 = X_{24} + X_{42}$. Moreover we set

$$A = \exp(\mathfrak{a}), \quad M = Z_K(A), \quad \mathfrak{m} = \mathfrak{z}_{\mathfrak{t}}(\mathfrak{a}).$$

Then

$$M = \{\exp(\theta\sqrt{-1}I_0)\gamma^j \mid \theta \in \mathbb{R}, j = 1, 2\}, \quad \mathfrak{m} = \mathbb{R}\sqrt{-1}I_0$$

with matrices

$$\gamma = I_0 = \text{diag}(1, -1, 1, -1).$$

Define matrices $h_i, e_{i,\pm}$ ($i = 1, 2$) by

$$\begin{aligned} h_1 &= \text{diag}(1, -1, 0, 0), & e_{1,+} &= X_{12}, & e_{1,-} &= X_{21}, \\ h_2 &= \text{diag}(0, 0, 1, -1), & e_{2,+} &= X_{34}, & e_{2,-} &= X_{43}. \end{aligned}$$

Let $\mathfrak{z}(\mathfrak{k}_{\mathbb{C}}) = \mathbb{C}I_{2,2}$ be the center of $\mathfrak{k}_{\mathbb{C}}$. Then $\{h_i, e_{i,\pm} \ (i = 1, 2)\}$ is a basis of the commutator algebra $[\mathfrak{k}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}}]$, and $\mathfrak{k}_{\mathbb{C}} = \mathfrak{z}(\mathfrak{k}_{\mathbb{C}}) \oplus [\mathfrak{k}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}}]$. The set $\{h_1, h_2, I_{2,2}\}$ is a basis of the Cartan subalgebra $\mathfrak{t}_{\mathbb{C}}$.

It is convenient to set $Z_{ij} = X_{ii} - X_{jj}$ for $(i, j) = (1, 3)$ or $(2, 4)$.

Then

$$Z_{13} = \frac{1}{2}(I_{2,2} + h_1 - h_2), \quad Z_{24} = \frac{1}{2}(I_{2,2} - h_1 + h_2).$$

1.2. The restricted root system

Define two linear forms $\lambda_i \in \mathfrak{a}^*$ ($i = 1, 2$) on \mathfrak{a} by $\lambda_i(H_j) = \delta_{ij}$. Then the restricted root system of \mathfrak{g} with respect to \mathfrak{a} , which is of type C_2 , is given by

$$\Delta = \Delta(\mathfrak{g}, \mathfrak{a}) = \{\pm\lambda_1 \pm \lambda_2, \pm 2\lambda_1, \pm 2\lambda_2\}.$$

For a fixed positive system $\Delta_+ = \{\lambda_1 \pm \lambda_2, 2\lambda_1, 2\lambda_2\}$, the associated root space decomposition is given as

$$\mathfrak{g} = \mathfrak{a} + \mathfrak{m} + \sum_{\lambda \in \Delta} \mathfrak{g}_{\lambda};$$

with $\mathfrak{g}_{2\lambda_1} = \mathbb{R}E_1$, $\mathfrak{g}_{2\lambda_2} = \mathbb{R}E_2$, $\mathfrak{g}_{\lambda_1 + \lambda_2} = \mathbb{R}E_3 + \mathbb{R}E_4$, $\mathfrak{g}_{\lambda_1 - \lambda_2} = \mathbb{R}E_5 + \mathbb{R}E_6$ and

$$\mathfrak{g}_{-\mu} = {}^t\mathfrak{g}_{\mu} = \{{}^tX \mid X \in \mathfrak{g}_{\mu}\}.$$

Here

$$\begin{aligned} E_1 &= \frac{\sqrt{-1}}{2} \begin{pmatrix} 1 & -1 & & \\ & 0 & -1 & 0 \\ 1 & & & \\ & 0 & & 0 \end{pmatrix}, & E_2 &= \frac{\sqrt{-1}}{2} \begin{pmatrix} 0 & & 0 & \\ & 1 & & -1 \\ 0 & & 0 & \\ & 1 & & -1 \end{pmatrix}, \\ E_3 &= \frac{1}{2} \begin{pmatrix} & 1 & & -1 \\ -1 & & 1 & \\ & 1 & & -1 \\ -1 & & 1 & \end{pmatrix}, & E_4 &= \frac{\sqrt{-1}}{2} \begin{pmatrix} & 1 & & -1 \\ 1 & & -1 & \\ & 1 & & -1 \\ 1 & & -1 & \end{pmatrix}, \end{aligned}$$

$$E_5 = \frac{1}{2} \begin{pmatrix} & 1 & & 1 \\ -1 & & 1 & \\ & 1 & & 1 \\ 1 & & -1 & \end{pmatrix}, \quad E_6 = \frac{\sqrt{-1}}{2} \begin{pmatrix} & 1 & & 1 \\ 1 & & -1 & \\ & 1 & & 1 \\ -1 & & 1 & \end{pmatrix},$$

regarding blank entries as zero. Put also $\mathfrak{n}_m = \sum_{\lambda \in \Delta^+} \mathfrak{g}_\lambda$.

1.3. The Casimir operator

By definition, the Casimir operator is given by

$$\Omega = H_1^2 + H_2^2 + \frac{1}{2}I_0^2 + 2 \sum_{j=1}^2 (E_j {}^t \bar{E}_j + {}^t \bar{E}_j E_j) + \sum_{j=3}^6 (E_j {}^t \bar{E}_j + {}^t \bar{E}_j E_j)$$

as an element of $U(\mathfrak{g})$. It is rewritten as

$$\Omega = H_1^2 + H_2^2 - 6H_1 - 2H_2 + \frac{1}{2}I_0^2 + 4 \sum_{j=1}^2 E_j {}^t \bar{E}_j + 2 \sum_{j=3}^6 E_j {}^t \bar{E}_j,$$

by using the commutation relation $[E_j, {}^t \bar{E}_j] = H_j$ ($j = 1, 2$), $[E_j, {}^t \bar{E}_j] = H_1 + H_2$ ($j = 3, 4$), and $[E_j, {}^t \bar{E}_j] = H_1 - H_2$ ($j = 5, 6$).

1.4. Hyperbolic trigonometric functions

We identify $(a_1, a_2) \in A$ with $(\mathbb{R}_+^\times)^2$ and set

$$\begin{aligned} \text{sh}(a_i) &= \frac{1}{2}(a_i - a_i^{-1}), & \text{ch}(a_i) &= \frac{1}{2}(a_i + a_i^{-1}), \\ \text{th}(a_i) &= \text{sh}(a_i) / \text{ch}(a_i), & \text{cth}(a_i) &= \text{ch}(a_i) / \text{sh}(a_i) \end{aligned}$$

and define two variable functions

$$\begin{aligned} D &= D(a_1, a_2) = \text{ch}(a_1)^2 - \text{ch}(a_2)^2 = \text{sh}(a_1)^2 - \text{sh}(a_2)^2 \\ &= \frac{1}{4} \left(\frac{a_1}{a_2} - \frac{a_2}{a_1} \right) \left(a_1 a_2 - \frac{1}{a_1 a_2} \right). \end{aligned}$$

Added to the basic relations

$$\text{ch}(a)^2 - \text{sh}(a)^2 = 1, \quad \text{sh}(a^2) = 2 \text{sh}(a) \text{ch}(a), \quad \text{ch}(a^2) = \text{ch}(a)^2 + \text{sh}(a)^2,$$

in this paper we particularly use

$$\begin{aligned} (1.1) \quad \text{sh}(a_1/a_2)^2 + \text{sh}(a_1 a_2)^2 &= \text{ch}(a_1^2) \text{ch}(a_2^2) - 1 \\ &= 2(\text{sh}(a_1)^2 + \text{sh}(a_2)^2) + 2 \text{sh}(a_1)^2 \text{sh}(a_2)^2 \\ \text{ch}(a_1 a_2) \text{sh}(a_1/a_2)^2 + \text{ch}(a_1/a_2) \text{sh}(a_1 a_2)^2 \\ &= 2 \text{ch}(a_1) \text{ch}(a_2) (\text{sh}(a_1)^2 + \text{sh}(a_2)^2). \end{aligned}$$

Moreover for Euler operators $\partial_i = a_i \partial / \partial a_i$ ($i = 1, 2$), we have

$$(1.2) \quad \begin{aligned} \partial_i(\operatorname{sh}(a_i)) &= \operatorname{ch}(a_i), & \partial_i(\operatorname{ch}(a_i)) &= \operatorname{sh}(a_i), \\ \partial_i \cdot \frac{D}{\operatorname{ch}(a_i)} &= \frac{D}{\operatorname{ch}(a_i)} \cdot \left(\frac{(-1)^{i-1} \operatorname{sh}(a_i)^2}{D} - \operatorname{cth}(a_i^2) + \frac{1}{\operatorname{sh}(a_i^2)} \right). \end{aligned}$$

1.5. K -modules and projectors

Let $\beta \in \mathfrak{k}_{\mathbb{C}}^*$. We mean by $\beta = [r, s; u]$ that $r = \beta(h_1)$, $s = \beta(h_2)$ and $u = \beta(I_{2,2})$. Then the non-compact roots $\tilde{\Delta}_n$ with respect to $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{k}_{\mathbb{C}})$ is

$$\tilde{\Delta}_n = \{[\pm 1, \pm 1; \pm 2]\}$$

and the positive system is fixed as $\tilde{\Delta}_{n,+} = \{[\pm 1, \pm 1; 2]\}$.

Let $d_1, d_2 \in \mathbb{Z}_{\geq 0}$ and $d_3 \in \mathbb{Z}$. For $d = [d_1, d_2; d_3] \in \mathfrak{k}_{\mathbb{C}}^*$, define $\tau_d \in \hat{K}$ and its representation space $V_d = \{f_{k_1 k_2}^{(d)} \mid 0 \leq k_j \leq d_j\}_{\mathbb{C}}$ by the following rule ($j = 1, 2$):

$$\begin{aligned} \tau_d(h_j) f_{k_1, k_2}^{(d)} &= (2k_j - d_j) f_{k_1, k_2}^{(d)}, & \tau_d(I_{2,2}) f_{k_1, k_2}^{(d)} &= d_3 f_{k_1, k_2}^{(d)}, \\ \tau_d(e_{j,+}) f_{k_1, k_2}^{(d)} &= (d_j - k_j) f_{k_1 + \delta_{1j}, k_2 + \delta_{2j}}^{(d)}, & \tau_d(e_{j,-}) f_{k_1, k_2}^{(d)} &= k_j f_{k_1 - \delta_{1j}, k_2 - \delta_{2j}}^{(d)}. \end{aligned}$$

We refer to $\{f_{k_1, k_2}^{(d)}\}$ as the standard basis. \hat{K} is exhausted by

$$\{(\tau_d, V_d) \mid d = [d_1, d_2; d_3], d_1 + d_2 + d_3 \in 2\mathbb{Z}\} \quad ([3, \text{Prop. 3.1}]).$$

The adjoint representation $\operatorname{Ad} = \operatorname{Ad}_{\mathfrak{p}_{\mathbb{C}}}$ of K on $\mathfrak{p}_{\mathbb{C}}$ is decomposed into a direct sum of two irreducible subrepresentations: $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ + \mathfrak{p}_-$, where

$$\mathfrak{p}_+ = \mathbb{C}X_{13} + \mathbb{C}X_{14} + \mathbb{C}X_{23} + \mathbb{C}X_{24}, \quad \mathfrak{p}_- = {}^t \mathfrak{p}_+.$$

In fact, $\operatorname{Ad}_{\pm} = \operatorname{Ad}|_{\mathfrak{p}_{\pm}}$ are isomorphic to $\tau_{[1,1;\pm 2]}$, respectively.

$$\begin{aligned} \iota_+ : (X_{23}, X_{13}, X_{24}, X_{14}) &\mapsto (f_{00}, f_{10}, -f_{01}, -f_{11}), \\ \iota_- : (X_{41}, X_{31}, X_{42}, X_{32}) &\mapsto (f_{00}, f_{01}, -f_{10}, -f_{11}), \quad ([3, \text{Prop. 3.10}].) \end{aligned}$$

The irreducible decomposition of $\mathfrak{k}_{\mathbb{C}}$ -module $V_d \otimes \mathfrak{p}_{\mathbb{C}}$ is given by

$$(1.3) \quad V_d \otimes \mathfrak{p}_{\mathbb{C}} = V_d \otimes \mathfrak{p}_- \oplus V_d \otimes \mathfrak{p}_-, \quad V_d \otimes \mathfrak{p}_{\pm} \simeq \bigoplus_{\delta \in \tilde{\Delta}_{n,\pm}} V_{d+\delta}$$

Therefore the projectors

$$(1.4) \quad P_d^{\pm\delta} : V_d \otimes \mathfrak{p}_{\mathbb{C}} \rightarrow V_{d\pm\delta} \quad (\delta \in \tilde{\Delta}_{n,+})$$

are uniquely determined up to constant and the explicit formula is found in [3, Lemma 3.12].

1.6. Principal P_J -series

Put $\mathfrak{a}_J = \mathbb{R}H_1$, $\mathfrak{n}_J = \mathfrak{g}_{2\lambda_1} + \mathfrak{g}_{\lambda_1+\lambda_2} + \mathfrak{g}_{\lambda_1-\lambda_2}$, $A_J = \exp \mathfrak{a}_J$, $N_J = \exp \mathfrak{n}_J$ and $\mathfrak{m}_J = \mathbb{R}H_2 + \mathbb{R}E_2 + \mathbb{R}\sqrt{-1}Z_{24} + \mathfrak{m}$. Then by definition, the Jacobi parabolic subgroup P_J of G is given by $P_J = M_J A_J N_J$ with $M_J = Z_K(\mathfrak{a}_J) \exp(\mathfrak{m}_J)$.

Let D_{p_0} be the discrete series of $SU(1, 1)$ with Blattner parameter p_0 ($|p_0| \geq 2$). We identify $M_J \simeq \mathbb{C}^{(1)} \times SU(1, 1)$. By definition, the principal P_J -series $\pi_{P_J}(m, p_0; \nu) = \text{Ind}_{P_J}^G (e^{\sqrt{-1}m\theta} D_{p_0} \otimes e^{\nu+\rho_J} \otimes 1)$ consists of the smooth functions on G satisfying

$$f(tm_0ang) = t^m a^{\nu+\rho_J} D_{p_0}(m_0) f(g) \quad ((t, m_0) \in M_J, a \in A_J, n \in N_J)$$

for quasi-character $a^{\nu+\rho_J}: A \rightarrow \mathbb{C}^\times$ of A ($\nu \in \mathbb{C}$, $\rho_J = 3\lambda_1$).

We consider as a typical multiplicity-one K -type of π_{P_J} , a corner K -type τ_d with property: 1) it is of minimum dimension in $\pi_{P_J}|_K$; 2) $\tau_{d+\delta}$ is not a K -type for a non-compact root δ such that $\tau_{d+\delta} \in \tilde{K}$. For example, if $\pi_{P_J} = \pi_{P_J}(m, p_0; \nu)$ ($m, p_0 > 0$), then the corner K -type is given by $d = [0, m; 2p_0 - m]$ ([4, p. 541]).

1.7. Middle discrete series

We fix the compact positive roots as $\tilde{\Delta}_c = \{[2, 0; 0], [0, 2; 0]\}$. There exist 6 positive systems containing $\tilde{\Delta}_c$. According to Harish-Chandra's result, a regular, dominant integral weight Λ with respect to the positive system (the Harish-Chandra parameter) defines a discrete series representation. If $\{[1, \pm 1; \pm 2]\}$ (resp. $\{[\pm 1, 1; \pm 2]\}$) is chosen as a (non-compact) positive system, the set of Harish-Chandra parameters is denoted by Ξ_{III} (resp. Ξ_{IV}). Precisely, for $\Lambda = [\Lambda_1, \Lambda_2; \Lambda_3]$, Λ is contained in Ξ_{III} (resp. Ξ_{IV}) when $\Lambda_1 - \Lambda_2 > |\Lambda_3|$ (resp. $\Lambda_2 - \Lambda_1 > |\Lambda_3|$). Then the resulting representation is of Gel'fand-Kirillov dimension 5, to which we refer as *the middle discrete series representation*.

It contains a unique minimal K -type τ_d ($d = [r, s; u]$). Its parameter d is obtained from the Harish-Chandra parameter by $d = \Lambda + \rho_n - \rho_c$ using a half sum ρ_n, ρ_c ([10, IX, §7]). In the middle discrete series case, one has $d = [\Lambda_1 \pm 1, \Lambda_2 \mp 1; \Lambda_3]$ for given Harish-Chandra $\Lambda \in \Xi_{\text{III}}$ or Ξ_{IV} .

1.8. R -model

Let R be a closed subgroup of G with double coset decomposition

$$(1.5) \quad G = RAK, \quad \mathfrak{g} = \text{Ad}(a^{-1})\mathfrak{r} + \mathfrak{a} + \mathfrak{k}.$$

Here a is a regular element in some vector subgroup A of G and $\mathfrak{r} = \text{Lie}(R)$. Let (η, V_η) an irreducible unitary (or Frechét) representation of

R and define a smooth induction $\text{Ind}_R^G(\eta)$ and its representation space $C_\eta^\infty(R \backslash G)$ consisting of V_η -valued smooth functions satisfying

$$f(\tau g) = \eta(r)f(g) \quad (r \in R),$$

on which G acts by right translation. For an irreducible admissible representation (π, H_π) of G , choose an element $\Phi_{\pi, \eta}$ of intertwining space:

$$\mathcal{I}(\pi, \eta) = \text{Hom}_{(\mathfrak{g}, K)}(H_{\pi, K}, C_\eta^\infty(R \backslash G)).$$

We call the image of $\Phi_{\pi, \eta}$ a R -model of π .

To describe a spherical function, suppose that $(\tau, V_\tau) \in \widehat{K}$ is a multiplicity-one K -type of π . By restriction, $\Phi_{\pi, \eta}$ is regarded as an element of $\text{Hom}_K(V_\tau, C_\eta^\infty(R \backslash G))$. Define $V_\eta \otimes V_{\tau^*}$ -valued function Φ_{π, η, τ^*} by

$$\langle \Phi_{\pi, \eta, \tau^*}(g), v \rangle = \Phi_{\pi, \eta}(v)(g) \quad (v \in V_\tau)$$

for canonical pairing $\langle w \otimes v^*, v \rangle = v^*(v) \cdot w$ ($w \in V_\eta, v \in V_\tau, v^* \in V_{\tau^*}$). We say Φ_{π, η, τ^*} as a (η, τ^*) -spherical function of π , which belongs to

$$C_{\eta, \tau^*}^\infty(R \backslash G/K) = \{\phi: G \xrightarrow{C^\infty} V_\eta \otimes V_{\tau^*} \mid \phi(\tau g k) = \eta(r) \otimes \tau^*(k)^{-1} \phi(g)\}.$$

In view of (1.5), a function in $C_{\eta, \tau^*}^\infty(R \backslash G/K)$ is determined by its restriction to A , i.e., its A -radial part. We want to find differential equations of the A -radial part of Φ_{π, η, τ^*} .

Sometimes they define a holonomic system and the intertwining space $\mathcal{I}(\pi, \eta)$ is isomorphic to, or a subspace of the solution space of this system; when π belongs to the discrete series ([22, 26]) and R -model is the Whittaker model ([11]) is a typical case.

1.9. The Schmid operator

For $\phi \in C_{\eta, \tau^*}^\infty(R \backslash G/K)$, define ∇_{η, τ^*} by

$$(1.6) \quad \nabla_{\eta, \tau^*} \phi(g) = \sum_j R_{X_j} \phi(g) \otimes X_j$$

with orthonormal basis $\{X_k\}$ of \mathfrak{p} and R_X the differentiation of the right translation. ∇_{η, τ^*} is determined independently of the choice of such orthonormal bases.

Suppose that π has a special K -type τ with property that a constituent δ of $\tau \otimes \text{Ad}$ is not a K -type of π . Put the projector $P^\delta: \tau \otimes \text{Ad} \rightarrow \delta$. Then considering the composition

$$P^{\delta^*} \circ \nabla_{\eta, \tau^*}: C_{\eta, \tau^*}^\infty(R \backslash G/K) \rightarrow C_{\eta, \delta^*}^\infty(R \backslash G/K)$$

(δ^* is such that τ_{δ^*} could be contragredient of τ_{δ}), one has

$$(1.7) \quad P^{\delta^*} \circ \nabla_{\eta, \tau^*} \Phi_{\pi, \eta, \tau^*} = 0.$$

Such a special K -type exists if, 1) π : discrete series, then τ is its minimal K -type, 2) π : principal P_J -series, then τ is the corner K -type defined in §1.6.

1.10. A -radial part of the Schmid operator

Let $\phi \in C_{\eta, \tau^*}^{\infty}(R \backslash G/K)$ and $F = \phi|_A$. Then the A -radial part of ∇_{η, τ^*} is defined by

$$\rho_A(\nabla_{\eta, \tau^*})F = (\nabla_{\eta, \tau^*}\phi)|_A.$$

It is computed as follows.

Decompose $X \in \mathfrak{p}_{\mathbb{C}}$ into

$$X = \text{Ad}(a^{-1})X_{\tau} + X_{\mathfrak{a}} + X_{\mathfrak{t}}$$

along (1.5). Then

$$(1.8) \quad \begin{aligned} R_X \phi(a) \otimes X &= R_{\text{Ad}(a^{-1})X_{\tau}} \phi(a) \otimes X + R_{X_{\mathfrak{a}}} \phi(a) \otimes X + R_{X_{\mathfrak{t}}} \phi(a) \otimes X \\ &= (\eta(X_{\tau}) \otimes 1_{V_{\tau^*}})(\phi(a) \otimes X) + \partial(X_{\mathfrak{a}})\phi(a) \otimes X \\ &\quad - 1_{V_{\eta}} \otimes (\tau \otimes \text{Ad})(X_{\mathfrak{t}})(\phi(a) \otimes X) + \phi(a) \otimes \text{Ad}(X_{\mathfrak{t}})X \end{aligned}$$

where $\partial(X)\phi(a) = (d/dt)_{t=0}\phi(a \exp(tX))$. Thus, if we write

$$\Phi_{\pi, \eta, \tau^*}(g) = \sum_{w, v} c_{v, w}(g) w \otimes v$$

using bases $\{w\}$ of V_{η} and $\{v\}$ of V_{τ^*} , we can compute the A -radial part of (1.7). It is a system of difference-differential equations of the A -radial part of the coefficients $\{c_{v, w}(a)\}$.

In our late discussion, the group A is the same for various choice of R . It is the vector part of the maximally \mathbb{R} -split Cartan subgroup of G . For some ‘‘spherical’’ subgroup R , A might be 1-dimensional, but we do not discuss such cases in this paper.

§2. Whittaker functions

In this section, we select $R = N_{\mathfrak{m}}$ the maximal unipotent subgroup of G with Iwasawa decomposition $G = N_{\mathfrak{m}}AK$ and η a nondegenerate unitary character of N . We consider $\pi = \pi_{P_J}$ case, for only the *large* discrete series representations admit Whittaker models.

2.1. Differential equations of P_J -series Whittaker functions

Let η be a nondegenerate unitary character of N_m . We put

$$\eta_0 = \eta(E_5)^2 + \eta(E_6)^2, \quad \eta_{\frac{1}{2}} = \eta(E_5) - \sqrt{-1}\eta(E_6), \quad \eta_2 = \sqrt{-1}\eta(E_2),$$

then by definition, η_2 and η_0 are nonzero real numbers. Let π be a principal P_J -series representation and $\Phi = \Phi_{\pi, \eta, \tau^*}$ a spherical function in $C_{\eta, \tau^*}^\infty(N \backslash G/K)$ defined in §1.8, where $\tau \in \widehat{K}$ is a corner K -type of π .

For simplicity, we state the results in the case of $\pi = \pi_{P_J}(m, -p_0; \nu)$ with $m \geq 0$ and $p_0 > 0$. Then the corner K -type is given by $[m, 0; m - 2p_0]$. Let $d^* = [m, 0; 2p_0 - m]$ for later convenience.

Put $\Phi(g) = \sum_{j=0}^m c_j(g) f_{j0}^{(d^*)}$ using the standard basis. Then $c_j(g) = \langle \Phi(g), (f_{j0}^{(d^*)})^* \rangle \in C_\eta^\infty(N \backslash G)$ and is called a Whittaker function of π .

Theorem ([4, Theorem 4.7]). *Put*

$$c_j(a_1, a_2) = \exp(-\eta_2 a_2^2 / 2) a_1^{m+p_0+2-j} a_2^{p_0-j} h_j(a_1, a_2).$$

Then, h_j satisfies

$$(2.9) \quad \begin{cases} (\partial_1 \partial_2 - (a_1/a_2)^2 \eta_0) h_j = 0, \\ (\partial_1^2 + \partial_2^2 + 2\gamma_{\pi, j} (\partial_1 + \partial_2) - 2\eta_2 a_2^2 \partial_2 + \gamma_{\pi, j}^2 - \nu^2) h_j = 0 \end{cases}$$

with $\gamma_{\pi, j} = m + p_0 - 2j - 1$.

2.2. Integral representation of Whittaker functions

Let $W_{\kappa, \mu}(z)$ be a classical Whittaker function ([25, 16.12]) with Whittaker's differential equation:

$$(2.10) \quad \frac{d^2 W}{dz^2} + \left(-\frac{1}{4} + \frac{\kappa}{z} + \frac{1/4 - \mu^2}{z^2} \right) W = 0.$$

Then one of the solutions of (2.9) can be represented by the following integral form under the condition $\eta_2 > 0$.

Theorem ([4, Theorem 5.1]). *Suppose that $\eta_2 > 0$. Then,*

$$(2.11) \quad c_j(a_1, a_2) = c_0 \cdot (-8\eta_2)^{m-j} \eta_{\frac{1}{2}}^j e^{-\eta_2 a_2^2 / 2} a_1^{m+p_0+2-j} a_2^{p_0-j} \cdot \int_0^\infty t^{2j-m-p_0+1/2} W_{0, \nu}(t) \exp\left(\frac{4\eta_2 \eta_0 a_1^2}{t^2} - \frac{t^2}{16\eta_2 a_2^2}\right) \frac{dt}{t}.$$

Here c_0 is a constant independent of j .

We remark that the Whittaker function for the large discrete series has the similar integral representation ([5, Theorem 4.5]).

§3. Generalized Whittaker functions (Siegel case)

We briefly review the recent results of Gon [2] about the middle discrete series (§3.2) and the principal P_J -series representation (§3.3).

3.1. A spherical subgroup attached to the unipotent radial of the Siegel parabolic subgroup

Let $\mathfrak{n}_S = \mathfrak{g}_{2\lambda_1} + \mathfrak{g}_{2\lambda_2} + \mathfrak{g}_{\lambda_1 - \lambda_2}$, $N_S = \exp(\mathfrak{n}_S)$ and $L_S = Z_G(N_S)$. Then $P_S = L_S N_S$ is called a Siegel parabolic subgroup. Identifying N_S with hermitian matrices:

$$\exp(aE_1 + bE_2 + cE_3 + dE_4) \mapsto \begin{pmatrix} -a & c\sqrt{-1} + d \\ -c\sqrt{-1} + d & -b \end{pmatrix},$$

one can parameterize its (nondegenerate) unitary characters ξ by the (nondegenerate) hermitian matrices H_ξ by

$$\xi(X) = \exp(2\pi\sqrt{-1} \operatorname{tr}(H_\xi X)) \quad (X: \text{hermitian}).$$

Without loss of generality, we assume that $H_\xi = \operatorname{diag}(\xi_1, \xi_2)$ with $\xi_1 > 0$.

Let $\operatorname{Stab}_{L_S}(\xi)$ be the stabilizer of ξ in L_S . Define $SU(\xi)$ as the identity component of $\operatorname{Stab}_{L_S}(\xi)$, isomorphic to $SU(2)$ or $SU(1, 1)$ according to H_ξ being definite or indefinite. Take $R = SU(\xi) \ltimes N_S$ and $\eta = \chi \otimes \xi$ for an irreducible unitary representation χ of $SU(\xi)$. Then one can show that

$$\mathfrak{g} = \operatorname{Ad}(a^{-1})(\mathfrak{su}(\xi) + \mathfrak{n}_S) + \mathfrak{a} + \mathfrak{k}.$$

An \mathfrak{sl}_2 -triple of $\mathfrak{su}(\xi)_\mathbb{C}$ is given by $\{h_\xi = I_0, e_{\xi,+}, e_{\xi,-}\}$:

$$e_{\xi,\pm} = |\det(2H_\xi)|^{-1/2} (\xi_1 {}^t E_5 - \xi_2 E_5 \pm \sqrt{-1}(\xi_1 {}^t E_6 + \xi_2 E_6)).$$

In this section, we refer to the R -model with $R = SU(\xi) \ltimes N_S$ as the *generalized Whittaker model*.

3.2. Generalized Whittaker functions for the middle discrete series

Theorem ([2, Theorems 13.11, 14.14]). *Let $\Lambda = [r - 1, s + 1; u] \in \Xi_{\text{III}}$ and π_Λ a middle discrete series representation with Harish-Chandra parameter Λ having $\tau_{[r,s;u]}$ as its minimal K -type. Then the generalized Whittaker model for π_Λ exists uniquely if and only if H_ξ is indefinite and if χ is a (limit of) discrete series D_{p_0} of H_ξ with $p_0 \equiv r + s \pmod{2}$ and $||p_0| - r| \leq s$.*

Remarkably it contrasts to well-known ‘‘Köcher principle’’ for the holomorphic case. This theorem, so to say, a Köcher principle *in reverse*, says that Fourier coefficients of automorphic forms belonging to the middle discrete series appear only over indefinite characters.

The description of the generalized Whittaker functions is as follows. Denoting the representation space of D_{p_0} as $V_{p_0} = \bigoplus_{j \in p_0 + 2\mathbb{Z}_{\geq 0}} \mathbb{C}v_j$ (we assume now $p_0 > 0$), we put

$$\Phi_{\pi, \eta, \tau^*}(g) = \sum_{h \in \mathbb{Z}_{\geq 0}} \sum_{k, w} b_{p_0+2h, k, w}(g) v_{p_0+2h} \otimes f_{k, w}^{(d)}.$$

Then $b_{j, k, w} = 0$ unless $j + 2k + 2w = r + s$ ([2, Lemma 14.8]) and the coefficient function $b_{p_0+2h, w} := b_{p_0+2h, (r+s-p_0)/2-w-h, w}$ is given as follows:

Theorem ([2, Theorem 14.14]). *For $a = (a_1, a_2) \in A$, one has*

$$(3.12) \quad b_{p_0+2h, w}(\xi_1^{-1/2} a_1, (-\xi_2)^{-1/2} a_2) = c \cdot \alpha_{w, h}^{(p_0, -u)}(a_1 a_2)^{r+2-p_0/2} \\ \cdot \mathcal{R}_h^{(p_0, -u)}(a_2/a_1) \cdot \mathcal{F}_w^{(p_0)}(-4\pi(a_1^2 - a_2^2)) \cdot e^{2\pi(a_1^2 - a_2^2)}$$

with constant c and

$$\alpha_{w, h}^{(\mu', \mu)} = \binom{r}{\frac{r+s-p_0}{2} - w - h} \binom{s}{w} \binom{\frac{\mu' - \mu}{2} - 1 + h}{h}, \\ \mathcal{R}_h^{(\mu', \mu)}(t) = t^{-h+\mu/2} {}_2F_1((\mu + \mu')/2, -h; 1 - h + (\mu - \mu')/2; t), \\ \mathcal{F}_w^{(\mu')}(t) = t^{-1+(\mu' - r - s)/2} {}_1F_1(-w; -s; t)$$

where ${}_2F_1$ and ${}_1F_1$ is the Gaussian hypergeometric function and Kummer’s confluent hypergeometric function, respectively.

Note that $\mathcal{R}_h^{(-u, p_0)}(t)$ and $\mathcal{F}_w^{(p_0)}(t)$ are well-defined Laurent polynomials under the condition of the above Theorem. See [2, Theorem 14.14 (b-3)] for $p_0 < 0$.

3.3. Generalized Whittaker functions of the principal P_J -series

In this subsection, we assume that H_ξ is definite. Fix an $(n + 1)$ -dimensional irreducible finite-dimensional representation χ_n of H_ξ and a standard basis $\{v_j\}_{j=0}^n$ of χ_n .

We consider the generalized Whittaker model for a principal P_J -series $\pi = \pi_{P_J}(m, -p_0; \nu)$, assuming $m \geq 0$, $p_0 > 0$ for simplicity. Then the corner K -type of π is given by $[m, 0; m - 2p_0]$ and as usual, set $d^* = [m, 0; 2p_0 - m]$.

Theorem ([2, Theorem 8.14]). *If the generalized Whittaker model exists for π , then it satisfies $n \geq m$ and $n + m \in 2\mathbb{Z}$.*

Put $n' = (n - m)/2 \in \mathbb{Z}$ and

$$\Phi_{\pi, \chi_n \otimes \xi, \tau_d^*}(g) = \sum_{j=0}^n \sum_{k=0}^m c_{j,k}(g) v_j \otimes f_{k,0}^{(d^*)}.$$

Then $c_{j,k} = 0$ unless $j = n' + m - k$ ([2, Lemma 6.2]); we simply write c_k instead of $c_{n'+m-k,k}$.

Theorem ([2, Theorem 8.15]). *For $(a_1, a_2) \in A$, one has*

$$(3.13) \quad c_k(\xi_1^{-1/2} a_1, \xi_2^{-1/2} a_2) = c \cdot (-1)^k \binom{m}{k} \binom{n}{n'+k} \cdot a_1^{m-k+p_0+2} a_2^{k+p_0+2} (a_1^2 - a_2^2)^{n'} \cdot e^{-2\pi(a_1^2+a_2^2)} \cdot \int_0^1 F(2\pi(a_1^2 t + a_2^2(1-t))) t^{n'+m-k} (1-t)^{n'+k} dt,$$

for constant c , where

$$F(z) = e^z z^{-(n+p_0+2)/2} W_{(p_0-n-2)/2, \nu/2}(2z)$$

and Whittaker's confluent hypergeometric function $W_{\kappa, \mu}$ (cf. (2.10)).

The similar results for $Sp(2; \mathbb{R})$ were obtained by Miyazaki [14]. In fact, the Whittaker function of P_J -series of $Sp(2; \mathbb{R})$ satisfies almost the same differential equations as those of $SU(2, 2)$.

§4. Matrix coefficients: P_J -series

4.1. (τ, τ^*) -matrix coefficients

From now on, we concentrate on the problem of matrix coefficients.

Let (π, H_π) be an irreducible admissible representation of G and (τ, V) its multiplicity-one K -type. We identify the representation spaces V, V^* with their unique images in H_π, H_π^* respectively.

The matrix coefficient of π is defined by

$$\langle \pi(g)v, w^* \rangle$$

for $v \in V \subset H_\pi, w^* \in V^* \subset H_\pi^*$. Put

$$\Phi_\pi(g) = \sum_{v,w} \langle \pi(g)v, w^* \rangle w \otimes v^* \in C_{\tau, \tau^*}^\infty(K \backslash G / K),$$

where $\{v\}$ and $\{w\}$ run bases of V and $\{v^*\}$ and $\{w^*\}$ are their dual bases, respectively.

We take $R = K$ and $\eta = \tau$ in the notation of §1.8, then Φ_π coincides with the spherical function Φ_{π, τ, τ^*} . In fact, defining $\Phi_{\pi, \tau}$ by

$$(4.14) \quad \Phi_{\pi, \tau}(v)(g) = \sum_{v, w} \langle \pi(g)v, w^* \rangle w,$$

then we see that $\Phi_{\pi, \tau}$ is in $\mathcal{I}(\pi, \tau)$.

We remark that because of $\langle \pi(g)v, w^* \rangle = \langle v, \pi^*(g^{-1})w^* \rangle$, we can regard $\Phi_\pi(g^{-1})$ as a matrix coefficients of π^* by changing the role of right and left actions of K .

In the following, we choose as such a basis the standard one introduced in §1.5. Namely, letting $\tau = \tau_d$ with $d = [r, s; u]$, then $\tau_d^* \simeq \tau_{d^*}$ ($d^* = [r, s; -u]$) so that $V = \bigoplus_{k_1, l_1} \mathbb{C} f_{k_1, l_1}^{(d)}$, $V^* = \bigoplus_{k_2, l_2} \mathbb{C} f_{k_2, l_2}^{(d^*)}$. For simplicity we write the index $M = (k_1, l_1; k_2, l_2)$. Then, taking $w = f_{k_1, l_1}^{(d)}$ and $v^* = f_{k_2, l_2}^{(d^*)}$, define

$$c_M(g) = \langle \pi(g)(f_{k_2, l_2}^{(d^*)})^*, (f_{k_1, l_1}^{(d)})^* \rangle.$$

Note that the isomorphism $(\tau_d)^* \simeq \tau_{d^*}$ is given by

$$(f_{k, l}^{(d)})^* \mapsto (-1)^{k+l} \binom{r}{k} \binom{s}{l} f_{r-k, s-l}^{(d^*)}.$$

The component $c_M(g)$ is determined uniquely by its restriction to A considering the Cartan decomposition $G = KAK$. For any element m of the centralizer $Z_K(A)$ of A in K , $\phi \in C_{\tau, \tau^*}^\infty(K \backslash G / K)$ satisfies

$$\phi(mam^{-1}) = \phi(a) \quad (a \in A).$$

Accordingly, if $c_M (M = (k_1, l_1; k_2, l_2))$ is not zero then it should satisfy

$$(4.15) \quad k_1 + l_1 + k_2 + l_2 = r + s.$$

4.2. Matrix coefficients of P_J -series

For simplicity, we consider only $\pi_{P_J} = \pi_{P_J}(m, -p_0; \nu)$ with $m, p_0 > 0$. Then the corner K -type of π_{P_J} is given by $[m, 0; -2p_0 + m]$. We put $d^* = [m, 0; 2p_0 - m]$ as in §4.1. Since $[m - 1, 1; m - 2p_0 - 2]$ does not give a K -type of π_{P_J} , it follows

$$(4.16) \quad P^\delta \circ \rho_A(\nabla_{\tau, \tau^*}) \Phi_{\pi_{P_J}} = 0 \quad (\delta = [-1, 1; -2]).$$

Obviously $\Phi_{\pi_{P_j}}$ satisfies the following equation for the Casimir element Ω ,

$$(4.17) \quad \rho_A(\Omega)\Phi = \chi(\pi_{P_j})\Phi, \quad \chi(\pi_{P_j}) = \nu^2 + (p_0 - 1)^2 - 10 + m^2/2.$$

We simplify each of them along §1.10. The consequence is shown in succeeding subsections.

4.3. Euler-Darboux equation

Firstly, consider the radial part of $\nabla = \nabla_{\tau, \tau^*}$ itself. Since $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ + \mathfrak{p}_-$, ∇ is decomposed into $\nabla_+ + \nabla_-$ canonically. Set $X_{ij}^{(+)} = X_{ij}$, $X_{ij}^{(-)} = X_{ji}$. Then in view of (1.8), the radial part of ∇_{\pm} is computed as follows:

Proposition 4.1. *Denote $\tau = \tau_d \otimes 1$ and $\tau_{\pm}^* = 1 \otimes \tau^* \otimes \text{Ad}_{\pm}$. Then*

$$(4.18) \quad \rho_A(\nabla_{\pm})\phi = \sum_{i=1,2, j=3,4} \mathcal{T}_{ij}^{\pm}(\phi \otimes X_{ij}^{\pm}),$$

where

$$\mathcal{T}_{13}^{\pm} = \frac{1}{2} \left(\partial_1 \mp \text{sh}(a_1^2)^{-1} \tau(Z_{13}) + \text{cth}(a_1^2) (2 \mp \tau_{\pm}^*(Z_{13})) + \frac{2}{D} \text{sh}(a_1^2) \right),$$

$$\mathcal{T}_{24}^{\pm} = \frac{1}{2} \left(\partial_2 \mp \text{sh}(a_2^2)^{-1} \tau(Z_{24}) + \text{cth}(a_2^2) (2 \mp \tau_{\pm}^*(Z_{24})) - \frac{2}{D} \text{sh}(a_2^2) \right),$$

$$\begin{aligned} \mathcal{T}_{14}^{\pm} = \pm \frac{1}{D} & \left(\text{ch}(a_1) \text{sh}(a_2) \tau(e_{\mp}^1) + \text{sh}(a_1) \text{ch}(a_2) \tau(e_{\mp}^2) \right. \\ & \left. + \text{sh}(a_2) \text{ch}(a_2) \tau_{\pm}^*(e_{\mp}^1) + \text{sh}(a_1) \text{ch}(a_1) \tau_{\pm}^*(e_{\mp}^2) \right), \end{aligned}$$

$$\begin{aligned} \mathcal{T}_{23}^{\pm} = \mp \frac{1}{D} & \left(\text{sh}(a_1) \text{ch}(a_2) \tau(e_{\pm}^1) + \text{ch}(a_1) \text{sh}(a_2) \tau(e_{\pm}^2) \right. \\ & \left. + \text{sh}(a_1) \text{ch}(a_1) \tau_{\pm}^*(e_{\pm}^1) + \text{sh}(a_2) \text{ch}(a_2) \tau_{\pm}^*(e_{\pm}^2) \right). \end{aligned}$$

Operating $P^{[-1,1;-2]}$ to (4.18), one has the system of difference-differential equations in terms of the components $c_{k_1; k_2} := c_{(k_1, 0; k_2, 0)}$ of $\Phi_{\pi_{P_j}}$. Eventually, the explicit formula of (4.16) reads

$$(4.19) \quad \begin{aligned} & 2(k_1 + 1)^2 \text{sh}(a_1) \text{ch}(a_2) D^{-1} c_{k_1+1; k_2-1} \\ & \quad = -k_2 (\partial_1 + \gamma_{1,+} \text{sh}(a_1^2)^{-1} - \gamma_{2,-} \text{cth}(a_1)^2 \\ & \quad \quad \quad + (k_1 + 1) \text{sh}(a_1^2) D^{-1}) c_{k_1; k_2}, \\ & 2(k_2 + 1)^2 \text{ch}(a_1) \text{sh}(a_2) D^{-1} c_{k_1-1; k_2+1} \\ & \quad = k_1 (\partial_2 - \gamma_{1,-} \text{sh}(a_2^2)^{-1} + \gamma_{2,+} \text{cth}(a_2)^2 \\ & \quad \quad \quad - (k_2 + 1) \text{sh}(a_2^2) D^{-1}) c_{k_1; k_2} \end{aligned}$$

with $\gamma_{i,\pm} = \gamma_{i,\pm,k_1,k_2} = (k_1 - k_2 \pm (-1)^i(2p_0 - m))/2$. Consulting (1.2), we see that (4.19) are equivalent to

$$(4.20) \quad \left((\partial_1 + \gamma_{1,+} \operatorname{sh}(a_1^2)^{-1} - \gamma_{2,-} \operatorname{cth}(a_1^2) + (k_1 + 1) \operatorname{sh}(a_1^2) D^{-1}) \right. \\ \left. \cdot (\partial_2 - \gamma_{1,-} \operatorname{sh}(a_2^2)^{-1} + \gamma_{2,+} \operatorname{cth}(a_2^2) - (k_2 + 1) \operatorname{sh}(a_2^2) D^{-1}) \right. \\ \left. + k_1(k_2 + 1) \operatorname{sh}(a_1^2) \operatorname{sh}(a_2^2) D^{-2} \right) c_{k_1;k_2} = 0.$$

Set $c_{k_1;k_2}(a) = \operatorname{ch}(a_1)^{A+} \operatorname{ch}(a_2)^{A-} h_{k_1}(a)$ with $A_{\pm} = (u \pm (k_1 - k_2))/2$. (Note that $k_1 + k_2 = r + s$.) Then it is rewritten in terms of h_{k_1} as

$$(4.21) \quad (\partial_1 \partial_2 - (k_2 + 1) \operatorname{sh}(a_2^2) D^{-1} \partial_1 + (k_1 + 1) \operatorname{sh}(a_1^2) D^{-1} \partial_2) h_{k_1} = 0.$$

The case of $m = 0$ is slightly different. Because of the lack of non-compact roots which make the corner K -type vanish, we use another equation generated by the composition of two Schmid operators. The resulting equation is the same as such mentioned above.

4.4. Poisson equation

Next consider (4.17). Put $F_i = E_i - {}^t \bar{E}_i$ and

$$D_{i,\pm} = \begin{cases} a_1 a_2 \pm a_1^{-1} a_2^{-1} & i = 3, 4, \\ a_1/a_2 \pm a_2/a_1 & i = 5, 6. \end{cases}$$

The radial part of the Casimir operator is obtained by a similar calculation:

Proposition 4.2. *It holds for $\phi \in C^\infty(A)$ that*

$$(4.22) \quad \rho_A(\Omega)\phi = \left(\sum_{i=1,2} \left(\partial_i^2 + 2(\operatorname{cth}(a_i^2) + (-1)^{i-1} \operatorname{sh}(a_i^2) D^{-1}) \partial_i \right. \right. \\ \left. \left. + \operatorname{sh}(a_i^2)^{-2} ((\tau(F_i)^2 + \tau^*(F_i)^2) + 2 \operatorname{ch}(a_i^2) (\tau \otimes \tau^*)(F_i)) \right) \right. \\ \left. + \sum_{i=3,\dots,6} 2D_{i,-}^{-2} (\tau(F_i)^2 + D_{i,+} (\tau \otimes \tau^*)(F_i) + \tau^*(F_i)^2) \right. \\ \left. + 2^{-1} \tau^*(h_1 + h_2)^2 \right) \phi.$$

Hence the equations satisfied by $c_{k_1;k_2}$ result from (4.22). Taken h_{k_1} in place of $c_{k_1;k_2}$, they finally become

$$(4.23) \quad (\partial_1^2 + \partial_2^2 + 2(-\gamma_{1,+} \operatorname{sh}(a_1^2)^{-1} + (\gamma_{2,-} + 1) \operatorname{cth}(a_1^2) + \operatorname{sh}(a_1^2) D^{-1}) \\ + k_2 \operatorname{cth}(a_1) (\operatorname{sh}(a_1)^2 + \operatorname{sh}(a_2)^2) D^{-1}) \partial_1$$

$$+ 2(\gamma_{1,-} \operatorname{sh}(a_1^2)^{-1} + (1 - \gamma_{2,+}) \operatorname{cth}(a_1^2) - \operatorname{sh}(a_2^2) D^{-1} - k_1 \operatorname{cth}(a_2)(\operatorname{sh}(a_1)^2 + \operatorname{sh}(a_2)^2) D^{-1}) \partial_2 + \chi'(\pi_{P_3}) h_{k_1} = 0$$

with $\chi'(\pi_{P_3}) = \chi(\pi_{P_3}) + 2m + (2p_0 - m)(2p_0 - m - 8)/2$.

4.5. Integral representation of solutions

Now we introduce new variables y_1, y_2 by $y_i = -\operatorname{sh}(a_i)^2$. Then the above equations (4.20), (4.23) for h_{k_1} are rewritten as follows.

Theorem 4.3. *Let $y_i = -\operatorname{sh}(a_i)^2$ for $(a_1, a_2) \in A$. Then,*

1. (Euler-Darboux equation)

$$\left(\frac{\partial^2}{\partial y_1 \partial y_2} - \frac{k_2 + 1}{y_1 - y_2} \frac{\partial}{\partial y_1} + \frac{k_1 + 1}{y_1 - y_2} \frac{\partial}{\partial y_2} \right) h_{k_1} = 0,$$

2. (Poisson equation)

$$\begin{aligned} & \left(4y_1(y_1 - 1) \frac{\partial^2}{\partial y_1^2} + 4y_2(y_2 - 1) \frac{\partial^2}{\partial y_2^2} + \chi(\pi_{P_3}) \right. \\ & \quad + 2 \left((\gamma_{2,-} + 2)(2y_1 - 1) + \gamma_{1,+} + 2\mathcal{S}_1(y_1, y_2) \right) \frac{\partial}{\partial y_1} \\ & \quad \left. + 2 \left((2 - \gamma_{2,+})(2y_2 - 1) + \gamma_{1,-} + 2\mathcal{S}_2(y_1, y_2) \right) \frac{\partial}{\partial y_2} \right) h_{k_1} = 0. \end{aligned}$$

Here we put $\mathcal{S}_i(y_1, y_2) = (2y_i + k_{i'}(y_1 + y_2))(y_i - 1)(y_i - y_{i'})^{-1}$ for $(i, i') = (1, 2), (2, 1)$.

This system is quite similar to that of Iida ([7, Proposition 8.1]). We obtain an analogous integral formula.

Theorem 4.4.

$$h_k = c_0 \int_0^1 {}_2F_1(\mu_+, \mu_-; 2; -\operatorname{sh}(a_1)^2 t - \operatorname{sh}(a_2)^2 (1 - t)) t^k (1 - t)^{m-k} dt.$$

with $\mu_{\pm} = (m - p_0 + 3 \pm \nu)/2$ and a constant c_0 .

§5. Matrix coefficients: the middle discrete series

Let π_{Λ} ($\Lambda \in \Xi_{\text{III}} \cup \Xi_{\text{IV}}$) be a middle discrete series representation of G (§1.7). Then, the Blattner parameter of π_{Λ} in $\Lambda \in \Xi_{\text{III}}$ (resp. $\Lambda \in \Xi_{\text{IV}}$) is $\Lambda + [1, -1; 0]$ (resp. $\Lambda + [-1, 1; 0]$). We only treat the case $\Lambda = [r - 1, s + 1; u] \in \Xi_{\text{III}}$ as the other case ($\Lambda \in \Xi_{\text{IV}}$) is similar.

We take as η in §1.8 a minimal K -type τ of π_Λ and consider the spherical function $\Phi_\Lambda = \Phi_{\pi, \tau, \tau^*}$. As seen in §1.9, $\nabla_{\tau, \tau^*} \Phi_\Lambda$ vanishes when it is projected by P^δ ($\delta \in \{[1, 1; \pm 2], [1, -1; \pm 2]\}$). These differential equations characterize spherical functions Φ_Λ ([26, Theorem A]).

Write $\Phi_\Lambda(a) = \sum_M c_M(a) f_{k_1, l_1}^{(d)} \otimes f_{k_2, l_2}^{(d^*)}$ for $M = (k_1, l_1; k_2, l_2)$, $d^* = [r, s; -u]$ as in §4.1. As is done in §4.3, the radial parts

$$(5.24) \quad P^\delta \circ \rho_A(\nabla_{\tau, \tau^*}) \Phi_\Lambda = 0 \quad (\delta \in \{[1, 1; \pm 2], [1, -1; \pm 2]\})$$

are computed exactly using Proposition 4.1. Accordingly the system of difference-differential equations for c_M is found by an elementary but rather tediously long calculation to reduce the system into simpler forms step by step. Here we only write the final step.

For simplicity, put

$$c_M(a) = \binom{r}{k_1} \binom{r}{k_2} \binom{s}{l_1} \binom{s}{l_2} \frac{(\operatorname{sh}(a_1) \operatorname{sh}(a_2))^{|s-l_1-l_2|}}{(\operatorname{ch}(a_1) \operatorname{ch}(a_2))^{(r+s+2)/2}} \cdot \tilde{c}_M(a)$$

and use the following notation

$$p = \operatorname{ch}(a_1) \operatorname{ch}(a_2), \quad t = (\operatorname{ch}(a_1) / \operatorname{ch}(a_2))^2.$$

Then one has,

$$\begin{aligned} \partial_p &= (\operatorname{cth}(a_1) \partial_1 + \operatorname{cth}(a_2) \partial_2) / 2, \\ \partial_t &= (\operatorname{cth}(a_1) \partial_1 - \operatorname{cth}(a_2) \partial_2) / 4. \end{aligned}$$

We obtain the system of differential-difference equations of $c_M(a)$ (hence, of $\tilde{c}_M(a)$) from the radial part of (5.24). For instance, \tilde{c}_M satisfies

$$(\partial_p - l_1) \tilde{c}_M = l_2 p \tilde{c}_{(k_1, l_1; k_2+1, l_2-1)} + l_1 \tilde{c}_{(k_1, l_1-1; k_2+1, l_2)} \quad (r \neq k_2),$$

therefore, it can be written in the form

$$(5.25) \quad \tilde{c}_M(a) = \sum_{i=0}^{l_1+l_2} (-1)^{r-k_1-l_1} p^{l_1+l_2-i} s_{M,i}(t).$$

For simplicity we denote

$$\begin{aligned} \partial_t \{u; k\} &= \partial_t + u/4 + k(t+1)(t-1)^{-1}/2, \\ z_\pm(t) &= t^{1/2} \pm t^{-1/2}. \end{aligned}$$

Comparing the coefficients as a polynomial of p , one can derive the following from the system.

Lemma 5.1. 1. If $0 \leq k_2 < r$, then,

$$(5.26) \quad (l_2 - i)s_{M,i} = l_2 s_{(k_1, l_1; k_2+1, l_2-1), i} - l_1 s_{(k_1, l_1-1; k_2+1, l_2), i-1},$$

$$(5.27) \quad 2z_-(t)\partial_t\{u; k_2 + 1\}s_{M,i} = 2k_1 s_{(k_1-1, l_1; k_2+1, l_2), i} \\ + l_2(2s_{(k_1, l_1; k_2+1, l_2-1), i-1} - z_+(t)s_{(k_1, l_1; k_2+1, l_2-1), i}) \\ - l_1(z_+(t)s_{(k_1, l_1-1; k_2+1, l_2), i-1} - 2s_{(k_1, l_1-1; k_2+1, l_2), i}),$$

$$(5.28) \quad (s - l_2 + 1)s_{M, i+1} = (s - l_1)s_{(k_1, l_1+1; k_2, l_2-1), i} \\ + (s - l_2 - i)s_{(k_1, l_1; k_2+1, l_2-1), i+1} - (l_1 - i)z_+(t)s_{(k_1, l_1; k_2+1, l_2-1), i} \\ + (2l_1 + l_2 - s - i)s_{(k_1, l_1; k_2+1, l_2-1), i-1}.$$

2. If $0 \leq k_1 < r$, then,

$$(5.29) \quad (l_1 - i)s_{M,i} = l_1 s_{(k_1+1, l_1-1; k_2, l_2), i} - l_2 s_{(k_1+1, l_1; k_2, l_2-1), i-1},$$

$$(5.30) \quad 2z_-(t)\partial_t\{-u; k_1 + 1\}s_{M,i} = 2k_2 s_{(k_1+1, l_1; k_2-1, l_2), i} \\ + l_1(2s_{(k_1+1, l_1-1; k_2, l_2), i-1} - z_+(t)s_{(k_1+1, l_1-1; k_2, l_2), i}) \\ - l_2(z_+(t)s_{(k_1+1, l_1; k_2, l_2-1), i-1} - 2s_{(k_1+1, l_1; k_2, l_2-1), i}),$$

$$(5.31) \quad (s - l_1 + 1)s_{M, i+1} = (s - l_2)s_{(k_1, l_1-1; k_2, l_2+1), i} \\ + (s - l_1 - i)s_{(k_1+1, l_1-1; k_2, l_2), i+1} - (l_2 - i)z_+(t)s_{(k_1+1, l_1-1; k_2, l_2), i} \\ + (2l_2 + l_1 - s - i)s_{(k_1+1, l_1-1; k_2, l_2), i-1}.$$

5.1. Peripheral entries and Gaussian hypergeometric functions

First assume that $l_1 = l_2 = 0$. We simply write $s_{k_1, k_2} = s_{(k_1, 0; k_2, 0), 0}$. By (5.27) and (5.30), it follows

$$z_-(t)\partial_t\{u; k_2 + 1\}s_{k_1; k_2} - k_1 s_{k_1-1; k_2+1} = 0, \\ z_-(t)\partial_t\{-u; k_1 + 1\}s_{k_1; k_2} - k_2 s_{k_1+1; k_2-1} = 0.$$

Eliminating $s_{k_1-1; k_2+1}$, we have

$$(\partial_t\{-u; k_1 + 1\} \cdot \partial_t\{u; k_2 + 1\} - k_1(k_2 + 1)z_-(t)^{-2})s_{k_1; k_2} = 0.$$

Because $r + s = k_1 + k_2$, it becomes

$$\left(\partial_t^2 + \frac{r+s+2t+1}{2} \frac{1}{t-1} \partial_t + \frac{u(k_1 - k_2)}{8} \frac{t+1}{t-1} + \frac{\gamma_{d; k_1, k_2}}{16} \right) s_{k_1; k_2} = 0$$

with $\gamma_{d;k_1,k_2} = (r+s+2)^2 - (k_1 - k_2)^2 - u^2$. This is a Gaussian hypergeometric differential equation. Its Riemann's P -scheme is:

$$P \left[\begin{array}{ccc} 0 & 1 & \infty \\ \frac{r+s+2}{4} - \frac{k_1-k_2+u}{4} & 0 & \frac{r+s+2}{4} + \frac{k_1-k_2-u}{4} \\ \frac{r+s+2}{4} + \frac{k_1-k_2+u}{4} & -(r+s+1) & \frac{r+s+2}{4} - \frac{k_1-k_2-u}{4} \end{array} \right].$$

For later use, let $\Phi(m_1, m_2) = \Phi(m_1, m_2; u; t)$ be a regular function around 1 having the P -scheme

$$P \left[\begin{array}{ccc} 0 & 1 & \infty \\ \frac{m_1+m_2+2}{4} - \frac{m_1-m_2+u}{4} & 0 & \frac{m_1+m_2+2}{4} - \frac{m_1-m_2-u}{4} \\ \frac{m_1+m_2+2}{4} + \frac{m_1-m_2+u}{4} & -(m_1+m_2+1) & \frac{m_1+m_2+2}{4} + \frac{m_1-m_2-u}{4} \end{array} \right]$$

with condition

$$\Phi(m_1, m_2; u; 1) = \binom{m_1 + m_2}{m_1}^{-1}.$$

We also write $\Phi(m) = \Phi(m, r+s-m)$ for simplicity. Then it follows $s_{(k_1,0;k_2,0),0} = c_0 \Phi(k_1)$ (c_0 : a constant) since the matrix coefficients are regular at $1 \in G$.

5.2. Reduction of general coefficients $s_{M,i}$

To describe general solutions, we introduce the notion of height h and bias b . Write $M = (k_1, l_1; k_2, l_2)$ as before. Define $h = h(M, i) = \min(i, l_1, l_2, l_1 + l_2 - i)$, $l_{\min} = \min(l_1, l_2)$, $l_{\max} = \max(l_1, l_2)$ and

$$b = b(M, i) = \begin{cases} 0 & i \leq l_{\min}, \\ (\text{sgn}(l_2 - l_1)) \cdot (i - l_{\min}) & l_{\min} \leq i \leq l_{\max}, \\ l_2 - l_1 & l_{\max} \leq i. \end{cases}$$

Then one has the following:

Proposition 5.2. *It holds that*

$$(5.32) \quad s_{M,i} = \sum_{j=b-h}^{b+h} Q_j(-z_+) \Phi(k_1 + l_1 + j)$$

for a polynomial $Q_j(t) = Q_j(M, i; t)$ of degree $h - |j - b|$ satisfying

$$Q_j(-z_+) = (-1)^{\deg Q_j} Q_j(z_+).$$

In fact, $Q_j(t)$ depends only on l_1 and l_2 .

5.3. Polynomials $Q_j(z_+)$

Proposition 5.2 says that Q_j is in the form

$$Q_j(z_+) = \sum_{m \geq 0} \beta_m(M, i, j) z_+^{h-|j|-2m}.$$

By the difference equations of Q_j deduced from (5.28), those of $\beta_m(M, i, j)$ are obtained by comparing the coefficient of $z_+^{h-|j|-2m}$ as follows: If $i < l_{\min}$, $j \geq 0$, then

$$(5.33) \quad \begin{aligned} (s-i+1)i\beta_m(M, i, j) &= (s-l_1)l_2\beta_m(M+(0, 1; 0, -1), i-1, j-1) \\ &+ l_1(s-l_2-i+1)\beta_{m-1}(M+(0, -1; 1, 0), i-1, j+1) \\ &+ (l_1-i+1)l_2\beta_m(M+(0, 0; 1, -1), i-1, j) \\ &+ (2l_1+l_2-s-i+1)l_2\beta_{m-1}(M+(0, 0; 1, -1), i-2, j). \end{aligned}$$

If $j < 0$, then

$$(5.34) \quad \begin{aligned} (s-i+1)i\beta_m(M, i, j) &= (s-l_1)l_2\beta_{m-1}(M+(0, 1; 0, -1), i-1, j-1) \\ &+ l_1(s-l_2-i+1)\beta_m(M+(0, -1; 1, 0), i-1, j+1) \\ &+ (l_1-i+1)l_2\beta_m(M+(0, 0; 1, -1), i-1, j) \\ &+ (2l_1+l_2-s-i+1)l_2\beta_{m-1}(M+(0, 0; 1, -1), i-2, j). \end{aligned}$$

Its solution is in the following, which is proved by induction.

Proposition 5.3. *Assume that $0 \leq l_1 + l_2 \leq s$, $i < l_{\min}$. Then,*

$$\beta_m = \alpha_m(M; i, j) \sum_{n=0}^m \binom{s-l_1}{j_+ + m - n} \binom{s-l_2}{j_- + m - n} \binom{s-i+n}{n}$$

for

$$\alpha_m(M; i, j) = \binom{i-|j|-m}{m} \binom{i}{m}^{-1} \binom{s}{i}^{-1} \binom{l_1}{i-j_+-m} \binom{l_2}{i-j_--m},$$

where $j_{\pm} = \max\{\pm j, 0\}$.

The results for $s \leq l_1 + l_2$ is also obtained similarly.

One can check that each $\beta_m(M, i, j)$ are well defined and are nonzero if and only if $i-|j|-2m \geq 0$, $l_1-i+j_++m \geq 0$ and $l_2-i+j_--m \geq 0$. Consequently,

Theorem 5.4. *Let π_Λ be a middle discrete series representation with $\Lambda = [r - 1, s + 1; u] \in \Xi_{\text{III}}$, and $\tau = \tau_d$ the minimal K -type of π_Λ with $d = [r, s; u]$. For a (τ, τ^*) -matrix coefficient Φ_{π, τ, τ^*} , put*

$$\Phi_{\pi, \tau, \tau^*}(g) = \sum_{k_1, l_1; k_2, l_2} c_{(k_1, l_1; k_2, l_2)}(g) f_{k_1, l_1; k_2, l_2}.$$

Then, $c_M(a)$ ($M = (k_1, l_1; k_2, l_2)$) is zero unless $k_1 + l_1 + k_2 + l_2 = r + s$ and the explicit formula is given by the following:

1. Suppose that $l_1 + l_2 \leq s$. The matrix coefficients $c_{k_1, l_1; k_2, l_2}(a_1, a_2)$ can be expressed as follows:

$$\begin{aligned} c_M(a_1, a_2) &= c_0(-1)^{r-k_1-l_1} (\text{sh}(a_1) \text{sh}(a_2))^{s-l_1-l_2} \\ &\times \sum_{i=0}^{l_1+l_2} (\text{ch}(a_1) \text{ch}(a_2))^{-(r+s+2)/2+l_1+l_2-i} \binom{r}{k_1} \binom{r}{k_2} \binom{s}{l_1} \binom{s}{l_2} \\ &\times \sum_{j=-h}^h (-1)^{h-|j|} \sum_{m=0}^{\lfloor \frac{h-|j|}{2} \rfloor} \beta_m(M, i, j+b) \left(\frac{\text{ch}(a_1)}{\text{ch}(a_2)} + \frac{\text{ch}(a_2)}{\text{ch}(a_1)} \right)^{h-|j|-2m} \\ &\times \Phi \left(k_1 + l_1 + j + b, k_2 + l_2 - j - b; u; \left(\frac{\text{ch}(a_1)}{\text{ch}(a_2)} \right)^2 \right), \end{aligned}$$

for $b = b(M, i)$, $h = h(M, i)$.

2. Suppose that $s < l_1 + l_2 \leq 2s$. Let $b' = b(M', i)$ and $h' = h(M', i)$ for $M' = (r - k_1, s - l_1; r - k_2, s - l_2)$. Then,

$$\begin{aligned} c_M(a_1, a_2) &= c_0(-1)^{r-k_1-l_1} (\text{sh}(a_1) \text{sh}(a_2))^{-s+l_1+l_2} \\ &\times \sum_{i=0}^{l_1+l_2} (\text{ch}(a_1) \text{ch}(a_2))^{-(r+s+2)/2+2s-l_1-l_2-i} \binom{r}{k_1} \binom{r}{k_2} \binom{s}{l_1} \binom{s}{l_2} \\ &\times \sum_{j=-h'}^{h'} (-1)^{h'-|j|} \sum_{m=0}^{\lfloor \frac{h'-|j|}{2} \rfloor} \beta_m(M', i, j+b') \left(\frac{\text{ch}(a_1)}{\text{ch}(a_2)} + \frac{\text{ch}(a_2)}{\text{ch}(a_1)} \right)^{h'-|j|-2m} \\ &\times \Phi \left(k_2 + l_2 + j + b', k_1 + l_1 - j - b'; -u; \left(\frac{\text{ch}(a_1)}{\text{ch}(a_2)} \right)^2 \right). \end{aligned}$$

§6. Further problems and some comments

The spherical functions for $R = K$, $R = SO(\xi) \cdot N_S$ and $R = N_m$ are different realizations of the same π . We have two natural problems among these realizations:

1. The change of the holonomic systems of spherical functions with corner K -type with respect to the change of R ($K \rightarrow SO(\xi) \times N_S \rightarrow N_m$), seems the confluence process of the singularity. Is there any natural geometric formulation of the process in terms of the deformation of R and their representations?
2. Find the integral kernel to represent the intertwining operator from one realization of π in $\text{Ind}_{R_1}^G(\eta_1)$ to another in $\text{Ind}_{R_2}^G(\eta_2)$, for different (R_i, η_i) ($i = 1, 2$).

Because the (restricted) root system of $Sp(2; \mathbb{R})$ is the same as that of $SU(2, 2)$, the spherical functions of the both groups are very similar. So we give recent literature on the spherical functions on $Sp(2; \mathbb{R})$; it is mainly concerned about the principal P_J -series and the large discrete series:

$R = N_m, \eta$ nondegenerate character: [21] for the large discrete series, [16] for the principal P_J -series. Also in a different context, Niwa [20] gives an integral formula for the class-1 principal series.

The holonomic system for the principal series is obtained by [15].

$R = N_m, \dim(\eta) = \infty$: Narita [18] for the holomorphic discrete series.

$R \supset N_S, \eta$ definite character: Miyazaki [14]. Also Niwa [19] gives an integral formula for the class-1 principal series.

$R \supset N_J, \dim(\eta) = \infty$: Hirano [6].

$R = K$: Iida [7] for the principal P_J -series,

$R = SL(2) \times SL(2), \eta$ unitary: Moriyama [17].

The first named author also works on the case $G = SU(2, 2)$, $R = Sp(2; \mathbb{R})$ for the principal P_J -series (in preparation). The last named author find an explicit formula for the A -radial part of the matrix coefficients with minimal K -type of the large discrete series of $Sp(2; \mathbb{R})$ quite recently (in preparation). The similar result for the large discrete series of $SU(2, 2)$ is very probable.

We should also mention that in view of the isogenies $SU(2, 2) \sim SO(4, 2)$ and $Sp(2; \mathbb{R}) \sim SO(3, 2)$, some generalization of our results to the general orthogonal group $SO(n, 2)$ is probable. For example we believe the matrix coefficients of the principal P_J -series and the middle discrete series of $SO(2n, 2)$ resembles to those of $SU(2, 2)$.

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