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Selberg's Eigenvalue Conjecture and the Siegel Zeros for Hecke L-series

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The present paper is an improved and corrected version of our previous paper [B-V]. The result deals with odd eigenfunctions for all the Hecke groups $\Gamma_0(N)$. This case does not seem to be accessible to methods based on the functional equation only. An important modification is, that we have to restrict ourselves to proving the absence of eigenvalues in an interval $(\frac{1}{4} - C(N), \frac{1}{4})$, where $C(N)$ is an efficient constant, depending on N, instead of $(0, \frac{1}{4})$. In the s-plane this corresponds to the interval $(\frac{1}{2}, \frac{1}{2} + \delta(N))$, where $\delta(N) = \frac{C}{\log N}$ and C is an explicit constant independent of N. It is crucial for the proof that the interval $(1 - \delta(N), 1)$ is free from zeros of the Dirichlet series associated with the cusp form eigenfunction with eigenvalue in $(\frac{1}{2}, \frac{1}{2} + \delta(N))$. Here we apply recent remarkable results by Hoffstein and Ramakrishnan [H-R] on the absence of Siegel zeros, see also [G-H-L]. Any improvement of these results will lead to an improvement of our results on the Selberg conjecture. It is noteworthy that the method of singular perturbations as opposed to previous methods leads to absence of eigenvalues in an interval near the continuous spectrum.

We indicate the ideas of the basic steps of the proof, giving first a few definitions.

Let $\Gamma(1) = PSL(2, \mathbb{Z})$ be the modular group and

$$\Gamma(N) = \{\gamma \in \Gamma(1) \mid \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N, N \in \mathbb{Z}_+\}$$

the principal congruence subgroup of $\Gamma(1)$ of level N. By definition Γ is a congruence subgroup of $\Gamma(1)$ if there exists N with the property $\Gamma(N) \subseteq \Gamma \subseteq \Gamma(1)$. The group Γ is a cofinite, discrete subgroup of $G = PSL(2, \mathbb{R})$. G acts on the hyperbolic plane (the upper half-plane) $H = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}, z = x + iy, y > 0$, by linear-fractional transformations, which are isometric relative to the Poincaré metric

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$ds^2 = y^{-2}(dx^2 + dy^2)$. Let $F = \Gamma \setminus H$ be the fundamental domain for Γ in H with the associated Hilbert space $\mathcal{H}(\Gamma) = L^2(F; d\mu)$, where $d\mu$ is the Poincaré measure $d\mu(z) = \frac{dxdy}{y^2}$. The automorphic Laplacian $A(\Gamma)$ is the self-adjoint operator in $\mathcal{H}(\Gamma)$, defined as the closure of $\tilde{A}(\Gamma)$, acting on smooth, Γ-automorphic functions f by

$$\tilde{A}(\Gamma)f = - \triangle f = -y^2\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)$$

The μ-measure of F is finite, $\mu(F) = [\Gamma(1) : \Gamma]\frac{\pi}{3}$, but F is not compact; here $[\Gamma(1) : \Gamma]$ is the index of Γ in $\Gamma(1)$. Then $A(\Gamma)$ has the continuous spectrum $[\frac{1}{4}, \infty)$ of multiplicity equal to the number of inequivalent cusps of F, an infinite discrete spectrum of eigenvalues $\lambda_i \in [\frac{1}{4}, \infty)$, an at most finite point spectrum in $(0, \frac{1}{4})$ and the eigenvalue $\lambda_0 = 0$.

In his fundamental paper [Se1] Selberg conjectured that there are no eigenvalues of $A(\Gamma)$ in the inverval $(0, \frac{1}{4})$.

In principle there are two possibilities for an eigenvalue $\lambda \in (0, \frac{1}{4})$: 1) λ corresponds to cusp forms 2) λ is a pole of the Eisenstein series. It is relatively simple to prove that there are no poles of the Eisenstein series in $(0, \frac{1}{4})$ for the congruence subgroups considered. This follows from the explicit expression for the determinant of the automorphic scattering matrix for $\Gamma(N)$. (See [He1], [Hu3]).

The question about eigenvalues $\lambda \in (0, \frac{1}{4})$, which correspond to cusp eigenfunctions, is very difficult (see the survey paper of Sarnak [Sa] for results and references). We mention here only several important results. Let $\lambda_1(\Gamma)$ denote the first non-zero eigenvalue of $A(\Gamma)$. Selberg proved $\lambda_1(\Gamma) \geq \frac{3}{16}$, using a sharp bound on Kloosterman sums due to André Weil. Luo, Rudnick and Sarnak [L-R-Sa] improved this estimate for even functions to $\lambda_1(\Gamma) \geq \frac{171}{784}$, using twisted Hecke L-series. There are also combinatorial and geometrical approaches to this problem. These methods work only for small levels $N \leq 17$, but give the desired result ([Hu1]). The interesting situation is that these methods predict rather $\lambda_1(\Gamma) \to 0$ when the genus of Γ goes to ∞. So we have to use the arithmetical nature of congruence groups in order to prove the Selberg conjecture.

In this paper we are concerned with the Hecke groups $\Gamma_0(N) = \left\{\binom{ab}{cd} \in \Gamma(1) \mid N|c\right\}$ and, what is more important for our method, with skew-symmetric (odd) eigenfunctions

$$v_j(-\bar{z}) = -v_j(z) \tag{1}$$

This skew-symmetry is allowed here, because the map $z \to -\bar{z}$ commutes with the Laplacian and maps Γ-automorphic functions to Γ-automorphic functions. If we are interested now in the smallest positive eigenvalue λ, with eigenfunction v_1 satisfying (1), then in many cases one can prove $\lambda_1 \geq \frac{1}{4}$, using the very strong Cheeger's inequality

$$\lambda > \frac{1}{4}\left(1 + \frac{4\pi}{\mu(D)}\right)$$

where λ is the eigenvalue of the function v with nodal domain D. But we don't know if we can apply this method in the general situation of the group $\Gamma_0(N)$. Huxley ([Hu2]) mentioned several examples with genus 0 and 1, where this approach gives $\lambda_1 > \frac{1}{4}$, but he also considered an example of a skew-symmetric eigenfunction for some discrete group of genus zero with $\lambda_1 < \frac{1}{4}$. (In that case Γ is a non-congruence group).

In this paper we consider a different method of studying small eigenvalues corresponding to eigenfunctions satisfying (1), based on a singular perturbation of $A(\Gamma)$ by a group of characters. This kind of perturbation was introduced and studied by Wolpert, Phillips and Sarnak in order to investigate the embedded (discrete) eigenvalues of $A(\Gamma)$ (See [W1], [P-Sa]).

Our main result is the following:

Theorem. *The operators $A(\Gamma_0(N))$ have no odd eigenfunctions corresponding to eigenvalues $\lambda = s(1-s)$ for $s \in (\frac{1}{2}, \frac{1}{2} + \frac{C}{\log N})$, where C is an efficient constant independent of N.*

Proof. It suffices to prove the result for odd new forms for general $\Gamma_0(N)$, using the fact proved by Maass that it holds for $\Gamma(1)$.

We first indicate the main steps of the proof. Let $\Gamma = \Gamma_0(N)$. We consider a family of selfadjoint operators $A(\Gamma; \chi_\alpha)$ defined by the Laplacian acting on functions $f(z)$ satisfying

$$f(\gamma z) = \chi_\alpha(\gamma) f(z), \gamma \in \Gamma,$$

where the $\chi_\alpha(\gamma)$ are one-dimensional, unitary representations (characters) of Γ, depending on a real parameter $\alpha \in [0, \varepsilon), \varepsilon > 0$ small, and $\chi_0 \equiv 1$. Thus, $A(\Gamma; 1) = A(\Gamma)$. The domain $D(A(\Gamma; \chi_\alpha))$ is a dense subspace of $L^2(F)$, varying with α. Using the idea of Phillips and Sarnak [P-Sa], we can transform these operators to operators acting on functions which are purely Γ-automorphic. For such a function f, i.e. $f(z) = f(\gamma z), \gamma \in \Gamma$, we define

$$g(z) = f(z) \exp 2\pi i \alpha \operatorname{Re} \int_{z_0}^{z} \omega(t) dt = f(z) \Omega(z, \alpha)$$

where $z_0 \in H$ is some fixed point in general position and $\omega(t)$ is a holomorphic Γ-automorphic form of weight 2. We define the character χ_α by the formula

$$\chi_\alpha(\gamma) = \exp 2\pi i \alpha \operatorname{Re} \int_{z_0}^{\gamma z_0} \omega(t) dt.$$

Then we have

$$g(\gamma z) = \chi_\alpha(\gamma) g(z), \gamma \in \Gamma.$$

Applying the negative Laplacian $-\triangle = -4y^2 \frac{\partial^2}{\partial z \partial \bar{z}}$ to the function $g(z)$ we obtain that the operator $A(\Gamma; \chi_\alpha)$ is unitarily equivalent to the operator

$$L(\alpha) = - \triangle + \alpha M + \alpha^2 N$$

where $M = -4y^2 \pi i (\omega_1 \frac{\partial}{\partial x} - \omega_2 \frac{\partial}{\partial y}), N = 4y^2 \pi^2 \mid \omega \mid^2$ and $\omega = \omega_1 + i\omega_2$, $\mid \omega \mid^2 = \omega_1^2 + \omega_2^2$. The domain $D(L(\alpha))$ equals $\Omega(z, \alpha)^{-1} D(A(\Gamma; \chi_\alpha))$ and $L(\alpha) = \Omega(\cdot, \alpha)^{-1} A(\Gamma; \chi_\alpha) \Omega(\cdot, \alpha)$. Note that M maps odd functions to even and even to odd.

Let us consider in more detail the characters χ_α and define the form ω.

Let χ be a one-dimensional unitary representation of Γ. We call χ singular in the cusp z_j of the canonical fundamental domain F of Γ if $\chi(S_j) \neq 1$ and S_j is the generator of a parabolic subgroup $\Gamma_j \subset \Gamma$, which fixes the cusp z_j. Otherwise $\chi(S_j) = 1$, and χ is non-singular in z_j. The total degree $k(\Gamma; \chi)$ of non-singularity of χ relative to Γ is equal to the number of all essential cusps of F (i.e. pairwise non-equivalent), in which χ is non-singular. If $k(\Gamma; \chi) < h$ then the representation is singular and the operator $A(\Gamma; \chi)$ has an absolutely continuous spectrum $[\frac{1}{4}, \infty)$ of multiplicity $k(\Gamma; \alpha)$. Here h is the number of pairwise inequivalent cusps of F. This definition agrees with the notion of singular perturbation, since a character χ, which is singular in the cusp z_j, removes the corresponding part of the continuous spectrum. A cusp, for which the representation χ is singular, is called a closed cusp, otherwise it is open.

We now construct explicitly the forms ω, which are important for our investigation. It is not difficult to see that if $\omega(z)$ is a cusp form, then the representation χ_α is non-singular for $\alpha \in (0, \epsilon)$. We consider three different types of group $\Gamma_0(n)$ and define for each type a form ω as follows. Let $P(z)$ be the classical holomorphic Eisenstein series of weight 2

$$P(z) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) e^{2\pi i n z}.$$

Case 1. $\Gamma = \Gamma_0(p_1 \ldots p_n)$, where $p_1, \ldots, p_n$ are different primes. Since $\Gamma \subset \Gamma_0(p_1 p_2)$. we can use the form

$$\omega_1(z) = P(z) - P(p_1 z) - p_2 P(p_2 x) + p_2 P(p_1 p_2 z) \tag{2}$$

associated with the group $\Gamma_0(p_1 p_2)$.

Case 2. $\Gamma = \Gamma_0(p_0^2 N)$, where p_0 is a prime and N an integer. Since $\Gamma \subset \Gamma_0(p_0^2)$, we can use the form

$$\omega_2(z) = P(z) - (p_0 + 1)P(p_0 z) + p_0 P(P_0^2 z) \tag{3}$$

Case 3. $\Gamma = \Gamma_0(p_0)$, where p_0 is a prime. Using that $\Gamma_0(p_0) \supset \Gamma_0(p_0^2)$, we can apply the method of case 2 with the same form ω_2.

The representations χ_α generated by the forms ω defined in (2) and (3) are regular in the cusp ∞ and singular at least in two other cusps, $\alpha \in (0, \varepsilon)$. Note that in case 3 we actually work with the group $\Gamma_0(p_0^2)$.

We assume now, that there exist small eigenvalues of $A(\Gamma)$, $0 < \lambda_1 < \cdots < \lambda_n < \frac{1}{4}$, and let $d_i = \dim \mathcal{N}(A(\Gamma) - \lambda_i)$.

Lemma 1. *The resolvent $R(\Gamma, \alpha, \lambda) = (A(\Gamma, \alpha) - \lambda)^{-1}$ converges strongly to $R(\Gamma, \lambda) = (A(\Gamma) - \lambda)^{-1}$ for all λ in the resolvent set of $A(\Gamma)$, uniformly on compact sets.*

For $\Im\lambda \neq 0$ this follows from the fact that $E(\Gamma, \alpha, \mu) \to E(\Gamma, \mu)$ strongly as $\alpha \to 0$ for $\mu \in \mathbb{R}$, where $E(\mu)$ denotes the spectral projections (cf. [K]). Then, using the following important Lemma, we obtain the result also for λ real.

Lemma 2. *The number of eigenvalues $\lambda_j(\alpha)$, counting multiplicity, near the small eigenvalue λ of $A(\Gamma)$, is bounded for α small by the multiplicity of λ.*

The proof of Lemma 2 is based on the Selberg trace formula. A similar result is known in the case of pinching geodesics, see [He2] and [W2].

This follows from the explicit expression for the Fourier decomposition of the eigenfunction in a closed cusp.

Lemma 3. *A cusp form of $A(\Gamma)$ corresponding to a small eigenvalue can not be the limit as $\alpha \to 0$ of residues of Eisenstein series of the operators $A(\Gamma; \chi_\alpha)$.*

This Lemma is very important. The proof is based on the Maass-Selberg relation, which we state for several cusps as follows: We denote by $\tilde{E}_i(z;\chi_\alpha;s)$ the Eisenstein series in the cusp z_i with the constant term truncated at $y = Y$ and by k the number of open cusps. The entries of the scattering matrix of $A(\Gamma;\chi_\alpha)$ at s are denoted $\varphi_{in}(s;\chi_\alpha)$, $i, n = 1, \dots, k$.

Then

$$\int_F \tilde{E}_i(z;\chi_\alpha;s)\tilde{E}_i(z;\overline{\chi_\alpha};s')d\mu(z)$$
$$= \frac{Y^{s+s'-1} - \sum_{n=1}^k \varphi_{in}(s;\chi_\alpha)\varphi_{ni}(s';\overline{\chi_\alpha})Y^{1-s-s'}}{s+s'-1} + \frac{\varphi_{ii}(s;\chi_\alpha)Y^{s'-s} - \varphi_{ii}(s';\overline{\chi_\alpha})Y^{s-s'}}{s'-s}$$

Remark 1. *This result does not follow for embedded eigenvalues in the same way, because the first term on the r.h.s. in that case plays a role.*

Lemma 4. *Let λ be a small eigenvalue of multiplicity k with eigenspace consisting of odd eigenfunctions. Then there is a basis $\{f_1, \dots, f_k\}$ of the corresponding eigenspace and eigenfunctions $f_i(\alpha)$ with eigenvalues $\lambda_i(\alpha)$ of $A(\Gamma, \alpha)$ such that for $\alpha \to 0$.*

$$\begin{aligned} f_i(\alpha) &= f_i + \alpha f_{i1} + o(\alpha) \\ \lambda_i(\alpha) &= \lambda + o(\alpha). \end{aligned}$$

Moreover, $f_{i1} = f_i'(0)$ is exponentially decreasing in open cusps and at least decreasing like y^{1-s} in closed cusps. Also for $\alpha \neq 0$ the eigenfunctions of $A(\Gamma, \chi_\alpha)$ corresponding to small eigenvalues are exponentially decreasing in all cusps.

Proof. The expansion to first order follows from Kato's perturbation theory [K], using Lemma 2. Here we use that Mf_i is small because f_i is a cusp form, so $f_{i1} = -R'_\lambda M f_i$ is in $L^2(F)$, where R'_λ is the reduced resolvent of $A(\Gamma)$ at λ.

It then follows from Fourier expansion that f_i decays exponentially and decay at least like y^{1-s} in closed cusps and from Lemma 3, that f_i and f_{i1} decay exponentially in open cusps.

Finally, $\lambda_{i1} = \lambda_i'(0) = (Mf_i, f_i) = 0$, because f_i is odd and Mf_i is even.

Lemma 5. *Let $\lambda \in (0, \frac{1}{4})$ be an eigenvalue of $A(\Gamma)$ with eigenfunction f, and let $E = E(z,s)$ be the Eisenstein series of $A(\Gamma)$ corresponding to the cusp at ∞ evaluated at $\lambda = s(1-s)$.*

Then

$$I(s) = \int_F (Mf)(z)E(z,s)d\mu(z) = 0 \tag{4}$$

Proof. From Lemma 4 follows for $i = 1, \dots, k$ the first order perturbation equation

$$A(\Gamma)f_1 + Mf = \lambda f_1 + \lambda_1 f \tag{5}$$

(4) then follows from (5) by Lemma 4.

The integral $I(s)$ was introduced by Phillips and Sarnak in connection with the study of embedded eigenvalues. The second part of the proof of Theorem 1 consists in proving that $I(s) \neq 0$ in the interval $(\frac{1}{2}, \frac{1}{2} + \delta(N))$ for new, odd cusp forms and $\Gamma = \Gamma_0(N)$, in contradiction to (4).

We first consider the groups $\Gamma = \Gamma_0(N)$ for general N. We recall some important properties of non-holomorphic new forms for Γ. We are interested here in cusp forms which are eigenfunctions for the operator $A(\Gamma)$. Let us denote the closed subspace of $\mathcal{H}(\Gamma)$ generated by all such eigenfunctions by $\mathcal{H}_0(\Gamma)$. The space $\mathcal{H}_0(\Gamma)$ decomposes as a direct sum of the subspaces of even and odd functions,

$$\mathcal{H}_0(\Gamma) = \mathcal{H}_0^+(\Gamma) \oplus \mathcal{H}_0^-(\Gamma).$$

A function $f \in \mathcal{H}_0^-(\Gamma)$ is defined by $f(-\bar{z}) = -f(z)$. This involution is well-defined in the space of Γ- automorphic functions; it commutes with the Laplacian and also with the Hecke operators T_p, which we define as follows for primes p, according to whether $p \nmid N$ or $p|N$

$$T_p f(z) = p^{-1}\left(\sum_{n=0}^{p-1} f(\frac{z+n}{p}) + f(pz)\right) \qquad \text{for } p \nmid N$$

$$T_p f(z) = p^{-1}\sum_{n=0}^{p-1} f(\frac{z+n}{p}) \qquad \text{for } p \mid N$$

For general n the operators T_n are defined using the T_p and their multiplicative properties. These operators T_n have important multiplicative properties and commute with each other and with $A(\Gamma)$. Also $T_n\mathcal{H}_0^-(\Gamma) \subset \mathcal{H}_0^-(\Gamma)$. We then consider a common basis of eigenfunctions for all the operators T_n and $A(\Gamma)$ in $\mathcal{H}_0^-(\Gamma)$. Let f be such an eigenfunction with eigenvalue λ; then we have the Fourier expansion

$$f(z) \equiv f(z,\bar{z}) = \sum_{n\neq 0} b_n \sqrt{y} K_{s-\frac{1}{2}}(2\pi \mid n \mid y) e^{2\pi i n x} \quad , \quad z = x + iy \quad ,$$

where $b_n = -b_{-n}$, $A(\Gamma)f = \lambda f$, $\lambda = s(1-s)$ and K is the modified Bessel function,

$$T_p f(z,\bar{z}) = \sum_{n\neq 0} \{b_{np} + p^{-1} b_{n/_p}\} \sqrt{y} K_{s-\frac{1}{2}}(2\pi \mid n \mid y) e^{2\pi i n x} \quad , p \nmid N$$

$$T_p f(z,\bar{z}) = \sum_{n\neq 0} b_{np} \sqrt{y} K_{s-\frac{1}{2}}(2\pi \mid n \mid y) e^{2\pi i n x} \quad , p \mid N$$

where $b_{n/_p} = 0$ if $p \nmid n$.

The structure of old and new Hecke forms plays an important role (see [A-L], [W1]). We consider the subspace of new odd Hecke cusp forms $\mathcal{H}_{0,N}^- \subset \mathcal{H}_0^-(\Gamma)$. The normalized eigenfunctions $f \in \mathcal{H}_{0,N}^-(\Gamma)$ have the following additional properties of the Fourier coefficients b_n:

$$\begin{aligned} b_1 &= 1 && \\ b_{np} - b_n b_p + b_{n/_p} &= 0 && \text{for } p \nmid N \\ b_{pn} &= 0 && \text{for } p, p^2 \mid N \\ b_{pn} \neq 0 \text{ and } b_{pn} &= b_p b_n && \text{for } p \mid N \text{ and } p^2 \nmid N \end{aligned} \tag{6}$$

Let us assume as before that $f(z,\bar{z})$ is a non-trivial Hecke eigenfunction in $\mathcal{H}_{0,N}^-$ corresponding to the eigenvalue $\lambda = s(1-s)$ of $A(\Gamma)$, $\frac{1}{2} < s < 1$,

$$f(z,\bar{z}) = \sum_{n\neq 0} b_n \sqrt{y} K_{s-\frac{1}{2}}(2\pi \mid n \mid y) e^{2\pi i n x}.$$

Let $\omega(z)$ be any of the forms ω_1 and $\omega_2$defined by (2) and (3),

$$\omega(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}.$$

The integral $I(s)$ is well defined and given by

$$I(s) = 4\pi i \int_F y^2 (\bar{\omega}(z) \frac{\partial f}{\partial z} + \omega(z) \frac{\partial f}{\partial \bar{z}}) E(z,s) d\mu(z).$$

We will now show that for $s \in (\frac{1}{2}, \frac{1}{2} + \delta(N))$

$$I(s) \neq 0 \tag{7}$$

We investigate this integral first for $Res > 1$ by unfolding the integral as Phillips and Sarnak do for $\Gamma_0(4)$ (see [P-Sa]). After some standard calculations we obtain

$$I(s) = -2(2\pi)^{-s+\frac{1}{2}} \int_0^\infty y^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(y) e^{-y} dy \cdot \sum_{n=1}^\infty \frac{a_n b_n}{n^{s+\frac{1}{2}}}.$$

The multiple to the Dirichlet series is never zero for $\frac{1}{2} < s < 1$, because the integrand is positive. Hence in order to prove (7) we have to study the Dirichlet series

$$L(s) = \sum_{n=1}^\infty \frac{a_n b_n}{n^{s+\frac{1}{2}}}.$$

We introduce the L-series

$$L_f(s) = \sum_{n=1}^\infty \frac{b_n}{n^s}$$

and the function

$$\Omega(s) = \pi^{-s} N^{s/2} \Gamma\left(\frac{s + s_j + \frac{1}{2}}{2}\right) \Gamma\left(\frac{s - s_j + \frac{3}{2}}{2}\right) f(s).$$

It is known that the function $\Omega(s)$ is entire and satisfies the functional equation

$$\Omega(1-s) = \pm\Omega(s).$$

In this series $b_{pn} = 0$ if $p^2 \mid N$. Hence we have

$$L(s) = \prod_{p\mid N, p^2\nmid N} \sum_{k=0}^\infty \frac{a_{p^k} b_{p^k}}{p^{k(s+\frac{1}{2})}} \cdot \prod_{p\nmid N} \sum_{k=0}^\infty \frac{a_{p^k} b_{p^k}}{p^{k(s+\frac{1}{2})}} \tag{8}$$

The second product in (8) is the same in all three cusps. We shall prove that it is different from 0. We have

$$\prod_{p\nmid N} \sum_{k=0}^\infty \frac{a_{p^k} b_{p^k}}{p^{k(s+\frac{1}{2})}} = \prod_{p\nmid N} \sum_{k=0}^\infty \frac{b_{p^k} \sigma_1(p^k)}{p^{k(s+\frac{1}{2})}} = \prod_{p\nmid N} \frac{1}{1-p} \sum_{k=0}^\infty \left(\frac{b_{p^k}}{p^{k(s+\frac{1}{2})}} - \frac{p b_{p^k}}{p^{k(s-\frac{1}{2})}} \right).$$

Now the calculation is well known and is similar to that of Phillips - Sarnak for $\Gamma_0(4)$. The last product equals

$$\prod_{p\nmid N} \frac{1}{1-p}\left[(1-b_p p^{-(s+\frac{1}{2})}+p^{-2s-1})^{-1}-p(1-b_p p^{-(s-\frac{1}{2})}+p^{-2s+1})^{-1}\right] =$$

$$= \prod_{p\nmid N} \frac{1-p^{-2s}}{(1-b_p p^{-(s+\frac{1}{2})}+p^{-2s-1})(1-b_p p^{-(s-\frac{1}{2})}+p^{-2s+1})}$$

We have

$$\prod_{p\nmid N}(1-p^{-2s}) = \frac{1}{\zeta(2s)}\prod_{p|N}(1-p^{-2s})^{-1} \neq 0 \quad , \quad \frac{1}{2} < s < 1.$$

The L-series corresponding to the Hecke form f is well known and is given by

$$L(f,s) = \prod_{p\nmid N} \frac{1}{1-b_p p^{-s}+p^{-2s}} \cdot \prod_{p|N} \frac{1}{1-b_p p^{-s}}$$

Hence we obtain the result

$$L(s) = A(s)\frac{1}{\zeta(2s)}L(f,s+\frac{1}{2})L(f,s-\frac{1}{2})$$

with

$$A(s) = \prod_{p|N}(1-p^{-2s})^{-1}(1-b_p p^{-s+\frac{1}{2}})(1-b_p p^{-s-\frac{1}{2}}).$$

We have $A(s)(\zeta(2s))^{-1} \neq 0 \quad , \quad \frac{1}{2} < s < 1.$

It is clear also that $L(f,s+\frac{1}{2}) \neq 0$ for $\frac{1}{2} < s < 1$; this follows from the convergence of the Euler product for $L(f,s)$. Then we apply Theorem C of Hoffstein and Ramakrishnan [H-R] and the functional equation for $L(f,s)$ to $L(f,s-\frac{1}{2})$, using the fact that f is a new Hecke form of weight zero.

We now consider the first product in the formula (8). We want to prove that this is not 0. Here we treat the three cases separately.

Case 1. $\Gamma_0(p_1p_2\dots p_n)$. We apply the form ω_1 given by (2) to the eigenfunction f.

This form is regular at ∞ and belongs to $\Gamma_0(p_1p_2)$ and therefore also to the subgroup $\Gamma_0(p_1p_2\dots p_n)$. For new forms f, which belong to $\Gamma_0(p_1p_2\dots p_n)$, the conditions (6) are satisfied.

The product in question contains besides a factor, which is clearly non-zero, the following two factors in which the form ω_1 depends essentially only on p_1 and p_2,

$$\sum_{k=0}^{\infty} \frac{a_{p_1^k} b_{p_1^k}}{p_1^{k(s+\frac{1}{2})}} \quad , \quad \sum_{k=0}^{\infty} \frac{a_{p_2^k} b_{p_2^k}}{p_2^{k(s+\frac{1}{2})}}$$

It is clear that for each k

$$a_{p_1^k} = -24 p_1^k$$

From that follows that the first sum is equal to

$$-24 \frac{1}{1 - \frac{b_{p_1}}{p_1^{s-\frac{1}{2}}}} \neq 0$$

It turns out that

$$a_{p_2^k} = -24 \quad \text{for all } k.$$

Hence the second sum equals

$$\frac{-24}{1 - \frac{b p_2}{p_2^{s+\frac{1}{2}}}} \neq 0$$

Case 2. We apply the form ω_2 given by (3) to the function f. The first product in (6) is of the form

$$\prod_{p \mid p_0^2 N, p^2 \nmid p_0^2 N} \quad \sum_{n=1}^{\infty} \frac{a_{p^k} b_{p^k}}{p^{k(s+\frac{1}{2})}}$$

because $b_{p_0 n} = 0$ and $b_{pn} = 0$ if $p^2 \mid N$.

From the definition of ω_2 follows

$$(-24)^{-1} a_{p^k} = \sigma_1(p^k) \text{ for } p \mid p_0^2 N, p^2 \nmid p_0^2 N.$$

Then

$$\sum_{k=0}^{\infty} \frac{a_{p^k} b_{p^k}}{p^{k(s+\frac{1}{2})}} = -24 \sum_{k=0}^{\infty} \frac{b_p^k \frac{1-p^{k+1}}{1-p}}{p^{k(s+\frac{1}{2})}} \neq 0$$

Case 3. For the new form f, which belongs to $\Gamma_0(p_0)$, we have

$$\begin{aligned} b_1 &= 1 \\ b_{np} - b_n b_p + b_{n/p} &= 0 \qquad\qquad p \neq p_0 \\ b_{pn} &= b_n b_p \neq 0 \qquad\qquad p = p_0 \end{aligned}$$

The form f is not new for the group $\Gamma_0(p_0^2)$, but that is not required in what follows. The function f will now be considered as belonging to the Hilbert space for $\Gamma_0(p_0^2)$. The first product in (6) is now

(9)
$$\sum_{k=0}^{\infty} \frac{a_{p_0^k} b_{p_0^k}}{p_0^{k(s+\frac{1}{2})}}.$$

We have

$$a_{p_0^k} = \begin{cases} (k=0) & a_1 = -24 \\ (k=1) & a_{p_0} = -24\sigma(p_0) - (p_0+1)(-24)\sigma(1) = 0 \\ (k>1) & -24\left(\frac{1-p_0^{k+1}}{1-p_0} - (p_0+1)\frac{1-p_0^k}{1-p_0} + p_0\frac{1-p_0^{k-1}}{1-p_0}\right) = 0 \end{cases}$$

From this we obtain that the series (8) equals $-24 \neq 0$.

From this follows the Theorem.

Remark 2. *One may ask why by this method we cannot prove non-existence of embedded eigenvalues λ corresponding to odd eigenfunctions, contradicting known results. The answer is no because in this case the proof of Lemma 5 breaks down, while the proof of (7) is still valid. Due to a result of Selberg* [Se2] *on the existence of resonances of the operators $A(\Gamma, \alpha)$ accumulating at every point of the continuous spectrum of A, however, it is not clear what (7) implies since asymptotic perturbation theory does not apply here.*

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Added in proof. The theory developed here can be established for Hecke groups with real primitive characters. In that case there exist regular perturbations given by holomorphic Eisenstein series of weight 2. The perturbation theory in that case is in fact simpler than for the singular perturbations considered in the present paper. For example, the proof of Lemma 4 is immediate.

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