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Characters of Non-Linear Groups

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§1. Introduction

Two of the primary methods of constructing automorphic forms are the Langlands program and Howe's theory of dual pairs.

The Langlands program concerns a reductive *linear* group G defined over a number field. Associated to G is its dual group ${}^LG$. The conjectural principle of functoriality says that a homomorphism ${}^LH \to {}^LG$ should provide a "transfer" of automorphic representations from H to those of G.

On the other hand Howe's theory of dual pairs, the theta correspondence, starts with the oscillator representation of the *non-linear* metaplectic group $Mp(2n)$, the two-fold cover of $Sp(2n)$. Restricting this automorphic representation to a commuting pair of subgroups (G, G') of $Mp(2n)$ gives a relationship between the automorphic representations of G and G'.

This suggests a natural question: is the theta-correspondence in some sense "functorial". As Langlands points out [16]: "the connection between theta series and functoriality is quite delicate, and therefore quite fascinating ... ". Now G and G' may be non-linear groups, and so even to define the notion of functoriality requires some work. In particular the L-groups of G and G' are not defined. Nevertheless it is reasonable to ask that theta-lifting be given by some sort of data on the "dual" side. This can be done in some cases in which the non-linearity of G and G' do not play an essential role. Nevertheless a proper understanding of the relationship between theta-lifting (and its generalizations) and functoriality requires bringing the representation theory of non-linear groups into the Langlands program.

Some discussion of the relation of the theta-correspondence to functoriality may be found in [15], [21], and [2]. The case of $U(3)$ has been

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discussed in great detail in [8].

Let $G = \mathbb{G}(\mathbb{F})$ be the $\mathbb{F}$ points of an algebraic group $\mathbb{G}$ defined over a local field $\mathbb{F}$. Suppose $\tilde{G}$ is a non-linear covering group of G. A representation of $\tilde{G}$ is *genuine* if it does not factor to any proper quotient. Our approach is to relate:

$$\widehat{\tilde{G}}_{gen} = \{\text{irreducible genuine representations of } \tilde{G}\}$$

to the representations of some linear group H (which may or may not be G). This would reduce questions about genuine representations of $\tilde{G}$ to those of H.

The approach we shall take here is through character theory, analogous to endoscopic transfer and base change. The first example of this is $GL(2)$ [6], which is related to the Shimura correspondence. We follow the roadmap, and some of the notation, of "the crux of the matter" ([12], §4), which gives more detail in the case of $GL(n)$. What is known (to this author) in general is discussed in section 5.

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§2. Non-Linear Groups

Let $\mathbb{F}$ be a local field of characteristic 0. Assume $\mathbb{G}$ is a simple, simply connected algebraic group defined over $\mathbb{F}$, and let $G = \mathbb{G}(\mathbb{F})$. Let A be an abelian, locally compact group with trivial G-action. Then $H^2(G, A)$ as defined by Moore [19] classifies the topological central extensions $\tilde{G}$ of G by A up to equivalence, i.e.

$$1 \to A \xrightarrow{\iota} \tilde{G} \xrightarrow{p} G \to 1 \tag{2.1}$$

where ι, p are continuous, ι is closed and p is open, i.e. induces an isomorphism $\tilde{G}/A \simeq G$.

The following Theorem of Prasad and Rapinchuk is the culmination of work which has a number of contributors. It originates in the fundamental paper of Matsumoto [18], and uses results of Deligne [4].

Theorem 2.2 [20]. *Suppose $\mathbb{F}$ is non-archimedean and G is isotropic, i.e. contains a one-dimensional split torus. Then*

$$H^2(G, A) \simeq Hom(\mu(\mathbb{F}), A) \tag{2.3}$$

Here $\mu(\mathbb{F})$ is the set of roots of unity in $\mathbb{F}$. Let $\mu_N(\mathbb{F})$ be the N^{th} roots of unity in $\mathbb{F}$.

If $\mathbb{F}=\mathbb{C}$ then $H^2(G,A)=1$. If $\mathbb{F}=\mathbb{R}$ let K be a maximal compact subgroup of G, and $\pi_1(G)\simeq\pi_1(K)$ the ordinary topological fundamental group. This is isomorphic to $1,\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}$, and $H^2(G,A)\simeq Hom(\pi_1(G),A)$.

Definition 2.4. Assume $|\mu_N(\mathbb{F})|=N$. Let $G[N]$ be the central extension

$$1\to\mu_N(\mathbb{F})\to G[N]\to G\to 1$$

corresponding via (2.3) to the homomorphism $\zeta\to\zeta^{|\mu(F)|/N}$ taking $\mu(\mathbb{F})$ to $\mu_N(\mathbb{F})$.

If $\mathbb{F}=\mathbb{R}$ we define $G[N]$ similarly if $N=2$ and $\pi_1(G)\neq 1$.

Definition 2.5. Let $\tilde{G}$ be a central extension of G by A as in (2.1). Let π be an irreducible admissible representation of $\tilde{G}$. Then π is *genuine* if the central character χ_π of π is an injection when restricted to A.

The representation theory of extensions of G reduces to that of the groups $G[N]$. First of all we may assume A is cyclic, and secondly that the homomorphism on the right hand side of (2.3) is surjective. Henceforth we assume $|\mu_N(\mathbb{F})|=N$, and write $\tilde{G}=G[N]$.

Let T be a Cartan subgroup of G, and let $\tilde{T}=p^{-1}(T)$. A key point is that $\tilde{T}$ is not necessarily abelian.

Definition 2.6. An element t of $\tilde{T}$ is said to be *regular* if $p(g)$ is regular in T. Suppose $g\in\tilde{T}$ is a regular. Then

$$\boxed{g \text{ is } \textit{relevant} \text{ if } g\in Z(\tilde{T})}$$

where $Z(\tilde{T})$ denotes the center of $\tilde{T}$.

Let $\widehat{\tilde{G}}_{gen}$ be the equivalence classes of irreducible genuine admissible representations of $\tilde{G}$. For $\pi\in\widehat{\tilde{G}}_{gen}$ let Θ_π be its global character. For π a genuine virtual representation, i.e. a finite integral linear combination of irreducible representations $\pi=\sum_i a_i\pi_i$ we define $\Theta_\pi=\sum a_i\Theta_{\pi_i}$ as usual. We identify Θ_π with the function on the regular semisimple elements which represents it.

For $GL(n)$ the next Proposition may be found in [6] and [11].

Proposition 2.7. *Suppose $\pi\in\widehat{\tilde{G}}_{gen}$. If g is not relevant then* $\Theta_\pi(g)=0$.

Proof. If $g \in \tilde{T}$ is not in the center of $\tilde{T}$, then there exists $h \in \tilde{T}$ such that

$$\text{(a)} \qquad hgh^{-1} \neq g.$$

Projecting the left hand side gives

$$\text{(b)} \qquad p(hgh^{-1}) = p(h)p(g)p(h^{-1}) = p(g)$$

since T is abelian. Therefore $hgh^{-1} = zg$ for some $z \in \mu_N(\mathbb{F})$ with $z \neq 1$ by (a). Now Θ_π is conjugation invariant, so it takes the same value at g and $hgh^{-1} = zg$. This gives

$$\begin{aligned}\Theta_\pi(g) &= \Theta_\pi(zg)\\ &= \chi_\pi(z)\Theta_\pi(g).\end{aligned}$$

Since π is genuine $\chi_\pi(z) \neq 1$, proving the Proposition.

Remark. The Proposition holds for any conjugation invariant function f which is genuine, i.e. $f(zg) \neq f(g)$ for all $z \in \mu_N(\mathbb{F}), z \neq 1$.

That this condition is highly non-trivial is illustrated by:

Proposition 2.8 [18]. *Suppose $\mathbb{F}$ is p-adic, G is split, and T is a split Cartan subgroup. Then $\tilde{T}$ is abelian if and only if $G = Sp(2n)$ and $N = 2$, i.e. $\tilde{G}$ is the metaplectic group.*

Remark. In some sense the the metaplectic group exists "because" of the oscillator representation. The proposition is an example of the fact that the oscillator representation is distinguished.

§3. General Lifting

Our discussion of lifting of characters for non-linear groups is modeled on the theory of endscopy and twisted endoscopy, including base change. In broad outline this is defined as follows.

The ingredients are:

(L.1) Reductive groups G, H,

(L.2) a map

$$H_0/\sim_{st} \xrightarrow{t} G_0/\sim$$

where:

- G_0 is the strongly regular semisimple elements of G (recall g is strongly regular if the centralizer of g is a torus),

$\sim$ is conjugacy by G,
$\sim_{st}$ is stable conjugacy, i.e. conjugacy by $H(\overline{F})$,

(L.3) Transfer factors: a map

$$\Phi : H_0 \times G_0 \to \mathbb{C}$$

satisfying a number of conditions. In particular $\Phi(h,g) = 0$ unless $g = t(h)$, in which case

$$|\Phi(h,g)| = |\Delta(h)|/|\Delta(g)|$$

where Δ is the usual Weyl denominator (with well defined absolute value).

Suppose π_H is a stable virtual character of H, with character Θ_{π_H}.

Definition 3.1.

$$\text{(3.2)(a)} \qquad t_*(\Theta_{\pi_H})(g) = \sum_{\{h|t(h)=g\}/\sim_{st}} \Phi(h,g)\Theta_{\pi_H}(h) \quad (g \in G_0).$$

This is a finite sum, and defines a conjugation invariant function on G_0. This is conjectured to be the character of a virtual representation π_G of G: we say π_G is the *lift* of π_H, and write $t_*(\pi_H) = \pi_G$, if $t_*(\Theta_{\pi_H}) = \Theta_{\pi_G}$, i.e.

$$\text{(3.2)(b)} \qquad \Theta_{\pi_G}(g) = \sum_{\{h|t(h)=g\}/\sim} \Phi(h,g)\Theta_{\pi_H}(h).$$

In the case of twisted endoscopy (e.g. base change) conjugacy is replaced by twisted conjugacy, and characters by twisted characters.

In the standard theory lifting (transfer) of orbital integrals is defined first, and lifting of characters is defined to be dual to this. Formula (3.2)(a) is then a consequence of the Weyl integration formula. It is possible, and sometimes convenient, to take (3.2)(a) as the definition, and prove a result on characters directly without the use of orbital integrals. One advantage of this approach is that the representation theory may suggest what to do (for example see Remark 5.13). The corresponding result on orbital integrals should then follow by similar arguments.

The transfer factors are a critical and difficult part of the theory. In the case of endoscopy and twisted endoscopy they contain deep arithmetic information ([17], [13]). It is necessary to define them carefully to insure the right hand side of (3.2) (which is a priori only a conjugation invariant function) is in fact a virtual character.

§4. Lifting for Non-Linear Groups

We return to the setting of section 2: $\mathbb{F}$ is a local field of charactertistic zero, $\mathbb{G}$ is a simple simply connected reductive group defined over $\mathbb{F}$, $G = \mathbb{G}(\mathbb{F})$ and $\tilde{G} = G[N]$.

We seek the following data:

(L.1) a linear reductive group $\mathbb{H}$ defined over $\mathbb{F}$, with $\mathbb{F}$-points H,

(L.2) a notion of stable conjugacy of strongly regular semisimple elements of $\tilde{G}$, and a map

$$H_0/\sim_{st}\xrightarrow{t} G_{0,relevant}/\sim_{st}$$

The key point here is that the image should be the relevant semisimple elements of $\tilde{G}_0$, (cf. Proposition 2.7).

(L.3) Transfer factors

$$\Phi : H_0 \times \tilde{G}_0 \to \mathbb{C}$$

satisfying $\Phi(h, g) = 0$ unless $g = t(h)$, in which case

$$|\Phi(h, g)| = |\Delta(h)|/|\Delta(g)|$$

For π_H a stable virtual character of H *define* t_* by (3.1)(a):

$$t_*(\Theta_{\pi_H})(g) = \sum_{\{h|t(h)=g\}/\sim_{st}} \Phi(h, g)\Theta_{\pi_H}(h).$$

This is a stably conjugation invariant function on $\tilde{G}_0$. The hope is to define the data such that *it is a genuine stable virtual character or* 0, *and every such virtual character arises this way.*

Note that t_* conjecturally involves stable virtual characters on both H and $\tilde{G}$. This is analogous to transfer from the quasisplit inner form G_{qs} of an algebraic group G to G.

§5. Examples

Character theory as in Section 4 or related results are known in the following cases, which will be discussed in more detail. In each case $G, \tilde{G}$ and H are as in §4.

(1) $G = H = GL(n, \mathbb{F})$

(a) For $n = 2$ this is due to Flicker [6]. For general n there is a series of papers by Flicker, Kazhdan and Patterson [10], [11], [12]. The most complete results are for tempered representations.

   (b) For $\mathbb{F} = \mathbb{C}$ this operation preserves unitarity [29] (there are no covers in this case).
   (c) For $\mathbb{F} = \mathbb{R}$ unitarity is preserved as well [3]. In (b) and (c) the correspondence of unitary representations is computed explicitly.

(2) Unramified Representations. Let $\tilde{G}$ be cover of a split, simply connected group over a p-adic field $\mathbb{F}$. Savin [25] constructs a split linear group H and an isomorphism of Iwahori Hecke algebras. This gives a candidate for H in this case. Huang proves that unitary is preserved [9].

(3) $G = Sp(2n, \mathbb{F})$ and $N = 2$. Take $H = SO(2n+1, \mathbb{F})$ (split). If $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ there is a bijection of genuine stable virtual characters between $\widetilde{Sp}(2n, \mathbb{F})$ and stable virtual characters of $SO(2n+1, \mathbb{F})$ [1]. For $\mathbb{F}$ p-adic and $n = 1$ the same result holds [26]. In the real case the corresponding result on orbital integrals is due to Renard [23].

(4) $G = SL(n, \mathbb{F})$ for $\mathbb{F}$ p-adic, and $N = n$. Take $H = PGL(n)$, as predicted by (2) [25]. The character relations hold in this case, as can be derived from (1a).

(1) <u>Kazhdan-Patterson Lifting for $GL(n, \mathbb{F})$</u>

Let $\mathbb{F}$ be a local field of characteristic 0, $G = GL(n, \mathbb{F})$. Fix N. Since G is not simple some extra work is required to define an N-fold cover $\tilde{G}$ [11]. There is an additional integral parameter c coming from the center, and we write $\tilde{G} = GL(n, \mathbb{F})[N, c]$. The restriction of $GL(n, \mathbb{F})[N, c]$ to $SL(n, \mathbb{F})$ is $SL(n, \mathbb{F})[N]$. The case n=2 is due to Flicker [6].

The center $Z(\tilde{G})$ of $\tilde{G}$ is a bit complicated ([11], Proposition 0.1.1): it is the inverse image of

$$\{xI \,|\, x^{n-1+2nc} \in \mathbb{F}^{*N}\}. \tag{5.1)(a}$$

Note that if $(n-1+2nc, N) = 1$ (greatest common divisor) this becomes

$$\{xI \,|\, x \in \mathbb{F}^{*N}\}. \tag{5.1)(b}$$

Let T be any Cartan subgroup of G.

Lemma 5.2 ([11], Proposition 0.1.4). *Suppose $g \in \tilde{T}$ is regular. Then g is relevant if and only if $g \in Z(\tilde{G})p^{-1}(T^N)$.*

If $(n-1+2nc, N) = 1$ by (5.1)(b) the relevant set is therefore $p^{-1}(T^N)$. This holds for example if $n = 2$ and $c = 0$. In general the center causes some some technical difficulties. Formally we may take $c = -\frac{1}{2}$; this is the approach of [24], provided $|\mu_{2N}(\mathbb{F})| = 2N$.

In any event the Lemma suggests taking $H = G$, and the N^{th} power map for the orbit correspondence. More precisely, let s be a map $s : G \to \tilde{G}$ satisfying $p \circ s = Id$. A simple observation [6], [12] is that the map

$$t_0(g) = s(g)^N$$

is independent of the choice of s and is well-defined on conjugacy classes. Together with Proposition 2.7 this suggests the ingredients of the lifting theory should be:

(L.1) $H = G = GL(n, \mathbb{F})$,

(L.2) Let $t_0(g) = s(g)^N$, where $s : G \to \tilde{G}$ is any section. Then $t(g) = t_0(g)u(g)$ where $u(g)$ is trivial if N is odd, and $u : G \to \pm 1 \in \mu_N(\mathbb{F})$ is a certain map ([11], §2) if N is even.

(L.3) $$\Phi(h, g) = \begin{cases} \frac{|\Delta(h)|}{|\Delta(g)|} = \frac{|\Delta(h)|}{|\Delta(h^N)|} & \text{if } p(g) = h^N \\ 0 & \textit{otherwise} \end{cases}$$

The need to modify t_0 by u is a subtle and crucial point of the theory ([12], §2).

Assume $(n - 1 + 2nc, N) = 1$. Then t_* is defined by (3.1)(a), with a constant c_T on the right hand side (which could be absorbed in the transfer factors) depending on the torus containing g.

In general some modification is necessary if $Z(\tilde{G})$ is not generated by $t(Z(G))$ and $\mu_N(\mathbb{F})$, so an auxiliary choice of character of $Z(\tilde{G})$ is necessary. See ([12], Proposition 5.6) for details.

Conjecture 5.3 [10]. *If π is an irreducible representation of $G = GL(n, \mathbb{F})$ then $t_*(\pi) = 0$, or $t_*(\pi) = \pm\tilde{\pi}$ for some irreducible genuine representation $\tilde{\pi}$ of $\tilde{G}$.*

Theorem 5.4 [10]. *The conjecture holds for π tempered.*

In the case $\mathbb{F} = \mathbb{C}$, $G = GL(n, \mathbb{C})$ there are essentially no covers to consider (the derived group $SL(n, \mathbb{C})$ is simply connected). Nevertheless for any N t_* may be defined as in (3.1)(a) taking $GL(n, \mathbb{C})$ to itself [29].

Theorem 5.6 [29]. *If π is unitary then $t_*(\pi) = 0$ or $t_*(\pi) = \pm\tilde{\pi}$ for $\tilde{\pi}$ an irreducible unitary representation of $GL(n, \mathbb{C})$.*

Theorem 5.7 [3]. *Let $\mathbb{F} = \mathbb{R}$, $N = 2$.*

(1) *If π is irreducible and unitary then $t_*(\pi) = 0$ or $t_*(\pi) = \pm\tilde{\pi}$ for $\tilde{\pi}$ an irreducible unitary genuine representation of $\tilde{G}$.*

(2) *If π is one-dimensional then $t_*(\pi)$ is a minimal unitary genuine representation $\tilde{\pi}$ with infinitesimal character $\rho/2$ or zero.*

(3) *Let L be a theta-stable Levi factor in G, π_L a virtual character for L. Then (notation below) t_* "commutes with the Euler characteristic of cohomological induction":*

$$t_*(\hat{R}_L^G(\pi_L)) = \hat{R}_{\tilde{L}}^{\tilde{G}}(t_*^L(\pi_L))$$

In (3), $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ is a theta-stable parabolic subalgebra of $\mathfrak{g} = Lie(G)_{\mathbb{C}}$. Let

$$\hat{R}_L^G(\pi_L) = \sum(-1)^k R_{\mathfrak{q}}^k(\pi_L)$$

where $R_{\mathfrak{q}}^k$ is the derived functor in degree k, and π_L is a virtual character of L. Taking $\tilde{L} = p^{-1}(L)$, $\hat{R}_{\tilde{L}}^{\tilde{G}}$ is defined similarly. Finally t_*^L is Kazhdan-Patterson lifting defined for L, which is a product of copies $GL(m,\mathbb{R})$ and $GL(m,\mathbb{C})$.

Remark. In Theorems 5.6 and 5.7 $t_*(\pi)$ is computed explicitly. In particular there is an explicit condition for when it is 0.

Remark. Statement (3) is a cohomological version of the fact that, very generally in the setting of (3.1), t_* commutes with ordinary parabolic induction: with the obvious notation

$$t_*(Ind_M^G(\pi_M)) = Ind_M^G(t_*^M(\pi_M))$$

This follows from the induced character formula.

(2) Unramified Representations

Let $\mathbb{F}$ be a p-adic field and fix a split, simply connected algebraic group $\mathbb{G}$ defined over the ring of integers R of $\mathbb{F}$. Let $G = \mathbb{G}(\mathbb{F})$ and $\tilde{G} = G[N]$ as in Section 2. Assume $(p,N) = 1$, so that the cover splits over $G(R)$ and hence over an Iwahori subgroup I. Choose a splitting $s : I \to \tilde{G}$. Fix a character χ of $\tilde{I} = p^{-1}(I)$ such that $\chi|_{s(I)} = 1$ and χ restricted to $\mu_N(\mathbb{F})$ is injective. Consider the category $R(\tilde{G},\chi)$ of smooth representations (π, V) of $\tilde{G}$ such that V is generated by

(5.8)(a) $$V^{\tilde{I},\chi} = \{v \in V \mid \pi(g)v = \chi(g)v \text{ for all } g \in \tilde{I}\}.$$

Now let $\mathcal{H}(\tilde{G},\chi)$ be the space of compactly supported functions on $\tilde{G}$ satisfying

(5.8)(b) $$f(xgy) = \chi(xy)f(g) \quad (x,y \in \tilde{I}, g \in \tilde{G}).$$

We call this the *Iwahori Hecke algebra* for $\tilde{G}$ (more precisely for the genuine representations of $\tilde{G}$ in which $\mu_N(\mathbb{F})$ acts by χ.)

As in the linear case the map $V \to V^{\tilde{I},\chi}$ induces an equivalence of categories between $R(\tilde{G},\chi)$ and the category of finite dimensional representations of $\mathcal{H}(\tilde{G},\chi)$.

A global correspondence of automorphic representations between a linear group H and a non-linear group $\tilde{G}$ should be given at the unramified places by a correspondence of unramified representations of H and $\tilde{G}$. It is natural to realize this correspondence via an isomorphism between $\mathcal{H}(\tilde{G},\chi)$ and the standard Iwahori Hecke algebra $\mathcal{H}(H)$. This is the case for $GL(n)$ ([12], §3), and was considered in general by Savin in [25]. Huang [9] computed the induced correspondence of unramified representations and showed that unitarity is preserved.

We will only state a qualitative version of [25] here.

Theorem 5.9 [25]. *Let $\tilde{G} = G[N]$. For any χ there is a split linear group H defined over $\mathbb{F}$ and an isomorphism*

$$\mathcal{H}(\tilde{G},\chi) \simeq \mathcal{H}(H).$$

The group H is independent of χ and depends on N. With the exception of some cases with $G = SL(n)$ or $SO(4n+2)$, H may be taken to be either G or the split form of the dual group ${}^LG$, depending on N.

One consequence of this result is that it gives a candidate for the group H of Section 4 in these cases.

(3) <u>The metaplectic group</u>

Let $\mathbb{F}$ be a local field of characteristic 0. Let $G = Sp(2n,\mathbb{F})$ and $N = 2$, so $\tilde{G} = \widetilde{Sp}(2n,\mathbb{F})$ is the metaplectic two-fold cover of $Sp(2n,\mathbb{F})$.

The Cartan subgroups $\tilde{T}$ of $\widetilde{Sp}(2n,\mathbb{F})$ are abelian. The map on conjugacy classes is *not* the squaring map as in example (2).

The ingredients are as follows.

(L.1) $H = SO(2n+1,\mathbb{F})$ (the split orthogonal group)

(L.2) Suppose $g \in Sp(2n,\mathbb{F})_0$. Then we define t by the condition: $t(g) = h \in SO(2n+1,\mathbb{F})$ if g,h have the same non-trivial (i.e. $\neq 1$) eigenvalues. This defines a bijection

$$Sp(2n,\mathbb{F})_0/\sim_{st} \overset{1-1}{\longleftrightarrow} SO(2n+1,\mathbb{F})_0/\sim_{st}$$

If $g \in \widetilde{Sp}(2n,\mathbb{F})$ define $t(g) = t(p(g))$.

(L.3) Fix a non-trivial character $\psi : \mathbb{F} \to S^1$. Let $\omega(\psi) = \omega = \omega_e \oplus \omega_o$ be the corresponding oscillator representation. Let

$$\Phi(g) = \Theta_{\omega_e}(g) - \Theta_{\omega_o}(g) \quad (g \in \widetilde{Sp}(2n, \mathbb{F})_0).$$

Evidence that $\Phi(g)$ is a reasonable definition of the transfer factor is provided by:

Lemma 5.10.

$$|\Phi(g)| = \frac{|\Delta_{SO}(h)|}{|\Delta_{Sp}(g)|}$$

where $t(g) = h$.

Here Δ_{SO} and Δ_{Sp} are the Weyl denominators for $SO(2n+1, \mathbb{F})$ and $Sp(2n, \mathbb{F})$ respectively.

Proof. Recall $|\Delta(g)| = |\prod_{\alpha \in \Delta}(1 - \alpha(g))|^{\frac{1}{2}}$ where Δ is the set of roots. Suppose g has eigenvalues $\{x_i\}$. The roots are then x_i^2 and x_i/x_j. For $SO(2n+1, \mathbb{F})$ the roots evaluated at $t(g)$ are x_i and x_i/x_j. In the quotient the x_i/x_j terms cancel to give

$$\begin{aligned}\frac{|\Delta_{SO}(h)|}{|\Delta_{Sp}(g)|} &= \frac{|\prod(1 - x_i)|^{\frac{1}{2}}}{|\prod(1 - x_i^2)|^{\frac{1}{2}}} \\ &= |\prod(1 + x_i)|^{-\frac{1}{2}} \\ &= |det(1 + g)|^{-\frac{1}{2}}.\end{aligned}$$

By a result of Howe for $\tilde{g} \in \widetilde{Sp}(2n, \mathbb{F})$, with $g = p(\tilde{g})$, $|\Theta_\omega(\tilde{g})| = |det(1 - g)|^{-\frac{1}{2}}$. Let $z \in \widetilde{Sp}(2n, \mathbb{F})$ be a central element satisfying $p(z) = -I$. The central characters of ω_e and ω_o have opposite sign on z. It follows that

$$|\Phi(\tilde{g})| = |\Theta_\omega(z\tilde{g})| = |det(1 - (-g))|^{-\frac{1}{2}},$$

proving the result.

Definition 5.11. Let π_H be a stable virtual character of $SO(2n + 1, \mathbb{F})$. Define

$$t_*(\Theta_{\pi_H})(g) = \Phi(g)\Theta_{\pi_H}(t(g))$$

for $g \in \widetilde{Sp}(2n, \mathbb{F})_0$. This is a genuine conjugation invariant function on $\widetilde{Sp}(2n, \mathbb{F})_0$.

Conjecture 5.12. *t_* is a bijection from*

$$\{\textit{stable virtual characters of } SO(2n+1, \mathbb{F})\}$$

to

$$\{\text{“stable” genuine virtual characters of } \widetilde{Sp}(2n,\mathbb{F})\}.$$

The notion of stable for $\widetilde{Sp}(2n,\mathbb{F})$ requires definition. It reduces to $\widetilde{SL}(2,\mathbb{F})$, and should be a condition on each elliptic torus. (It is it not the same as restriction from $\widetilde{GL}(2,\mathbb{F})$.) Over $\mathbb{R}$ it comes down to defining an action of the full Weyl group on a compact Cartan subgroup $\tilde{T}_c$.

Remark 5.13. Suppose one assumes the conjecture for some unknown transfer factor Φ. Let $\pi_H = \mathbb{C}$ (the trivial representation); the conjecture gives

$$t_*(\mathbb{C})(g) = \Phi(g) \tag{5.14)(a}$$

which says the transfer factor $\Phi(g)$ is equal to the character of the small virtual representation $t_*(\mathbb{C})$. Note that the subtleties in the definition of the transfer factor are accounted for by the character of $t_*(\mathbb{C})$.

The same idea in the general setting of (3.2)(b) gives

$$t_*(\mathbb{C}) = \sum_{\{h|t(h)=g\}/\sim} \Phi(h,g) \tag{5.14)(b}$$

i.e. the transfer factor averaged over H should be a small virtual character of G. This might be useful in *defining* the transfer factors.

Theorem 5.15 [1]. *The conjecture is true for* $\mathbb{F} = \mathbb{R}$.

Theorem 5.16 [26]. *The conjecture is true for* $\mathbb{F}$ *p-adic,* $n = 1$.

Theorem 5.17. *The conjecture is true for the minimal principal series of* $\widetilde{Sp}(2n,\mathbb{F})$ *and* $SO(2n+1,\mathbb{F})$.

Proof. Let T be a split Cartan subgroup of $Sp(2n,\mathbb{F})$ and let $B \supset T$ be a Borel subgroup. Let $\tilde{B}$, $\tilde{T}$ be the inverse images of B and T in $\widetilde{Sp}(2n,\mathbb{F})$; $\tilde{T}$ is abelian. For $\tilde{\chi}$ a genuine character of $\tilde{T}$ the genuine induced representation $\pi = \pi(\tilde{\chi})$ is defined as usual.

Similarly let $B' = T'N'$ be a Borel subgroup of the split group $SO(2n+1,\mathbb{F})$. For χ' a character of T' let $\pi' = \pi'(\chi')$ be the corresponding induced representation.

The Weyl group W of T in $Sp(2n,\mathbb{F})$ acts naturally on $\tilde{T}$ and by the induced character formula [5] for $g \in \tilde{T} \cap \tilde{G}_0$

$$\Theta_\pi(g) = \frac{1}{|\Delta_{Sp}(g)|}\sum_W \tilde{\chi}(w\cdot g) \tag{5.18}$$

Write $g=(p(g),\epsilon)$ in cocycle notation ($\epsilon=\pm 1$). Let

$$x_1,\dots,x_n,x_1^{-1},,\dots,x_n^{-1}$$

be the eigenvalues of $p(g)$. Fix a unitary (additive) character ψ of $\mathbb{F}$, and let $\gamma(x,\psi)$ be the corresponding gamma factor of Weil ([22], Appendix). Define a genuine character of $\tilde{T}$ by

$$\tag{5.19} \begin{aligned}\tilde{\chi}_0(g) &= \gamma(\textstyle\prod x_i,\psi)\epsilon \\ &= \textstyle\prod_i \gamma(x_i,\psi)\prod_{i<j}(x_i,x_j)_{\mathbb{F}}\epsilon\end{aligned}$$

where $(,)_{\mathbb{F}}$ is the Hilbert symbol. A cocycle calculation shows that $\tilde{\chi}_0(w\cdot g)=\tilde{\chi}_0(g)$ for all $w\in W$. The character $\chi=\tilde{\chi}_0^{-1}\tilde{\chi}$ factors to T and (5.18) becomes

$$\tag{5.20} \Theta_\pi(g)=\frac{\tilde{\chi}_0(g)}{|\Delta_{Sp}(g)|}\sum_W \chi(w\cdot p(g))$$

Let $h=t(g)$, so h and g have the same eigenvalues (h has eigenvalue 1 in addition). With the natural isomorphism $\phi: T'\simeq T$ (take any such isomorphism which preserves eigenvalues) let $\chi'=\chi\circ\phi$. Let $W'\simeq W$ be the Weyl group of T' in $SO(2n+1,\mathbb{F})$. Then

$$\tag{5.21} \begin{aligned}\Theta_\pi(g) &= \frac{\tilde{\chi}_0(g)}{|\Delta_{Sp}(g)|}\sum_W \chi(w\cdot p(g)) \\ &= \frac{\tilde{\chi}_0(g)}{|\Delta_{Sp}(g)|}\sum_{W'} \chi'(w'h) \\ &= \tilde{\chi}_0(g)\frac{|\Delta_{SO}(h)|}{|\Delta_{Sp}(g)|}\frac{1}{|\Delta_{SO}(h)|}\sum_{W'} \chi'(w'h) \\ &= \tilde{\chi}_0(g)\frac{|\Delta_{SO}(h)|}{|\Delta_{Sp}(g)|}\Theta_{\pi'}(h).\end{aligned}$$

It only remains to prove the factor

$$\tag{5.22} \tilde{\chi}_0(g)\frac{|\Delta_{SO}(h)|}{|\Delta_{Sp}(g)|}$$

is equal to $\Phi(g)$. In absolute value this is Lemma 5.10. Suppose $n=1, p(g)=diag(x,\frac{1}{x}), g=(p(g),\epsilon)$. The character of the oscillator representation has been computed in [26]. The result is:

$$\tag{5.23} \Phi(g)=\Theta_{\omega_e}(g)-\Theta_{\omega_o}(g)=\frac{\gamma(x,\psi)\epsilon}{|(1+x)(1+\frac{1}{x})|^{\frac{1}{2}}}$$

which equals (5.22) since $\tilde{\chi}_0 = \gamma(\chi, \psi)\epsilon$ by (5.19).

The general case follows since the character of the oscillator representation representation is essentially diagonal, as is (5.22). More precisely by ([22], Proposition 3.8 and Corollary 5.6) Φ is diagonal with the exception of a cross term $\prod_{i<j}(x_i, x_j)$ which is accounted for by (5.19). This completes the proof.

More generally it should follow easily that t_* commutes with parabolic induction (this is true over $\mathbb{R}$ and $\mathbb{C}$), which would reduce the conjecture to the discrete series.

(4) <u>$SL(n)$</u>

Let $\mathbb{F}$ be a p-adic field satisfying $|\mu_n(\mathbb{F})| = n$ and let $\widetilde{SL}(n) = SL(n, \mathbb{F})[n]$.

The standard approach to representation theory of $SL(n)$ is via restriction from $GL(n)$. In general this is a difficult problem. For $n = 2$ this was the first case of endoscopy [14] and is highly non-trivial. For some results on the general case see [7], [28].

It is rather surprising that in contrast to the linear case the irreducible representations of $\widetilde{SL}(n)$ may be understood very simply in terms of those of the corresponding cover $\widetilde{GL}(n)$ of $GL(n)$. In fact the restriction of an irreducible representation of $\widetilde{GL}(n)$ to $\widetilde{SL}(n)$ always has $|\mathbb{F}^*/\mathbb{F}^{*n}|$ components which may be distinguished via their central characters. We sketch the situation briefly, details will appear elsewhere.

Let $\widetilde{GL}(n) = GL(n)[n, 0]$ (notation as above), which contains $\widetilde{SL}(n)$ as a normal subgroup. A standard construction in the theory of covering groups is the commutator. If $ghg^{-1}h^{-1} = 1$ then, for any inverse images $\tilde{g}, \tilde{h}$ of g, h, $\eta = \tilde{g}\tilde{h}\tilde{g}^{-1}\tilde{h}^{-1}$ is in $\mu_N(\mathbb{F})$, and we write $\{g, h\} = \eta$. The key point is the commutator formula ([11], proof of Proposition 0.1.1):

$$\begin{aligned} \{xI, g\} &= (x, det(g))_n^{n-1} \\ &= (x, det(g))^{-1} \end{aligned} \tag{5.24}$$

where $(,)_n$ is the n^{th} norm residue symbol of $\mathbb{F}$ [27].

Let Z be the center of $GL(n)$,

$$H = SL(n)Z = \{g \mid det(g) \in \mathbb{F}^{*n}\},$$

$\tilde{H} = p^{-1}(H)$. By (5.24) the center $Z(\tilde{H})$ of $\tilde{H}$ is $\tilde{Z} = p^{-1}(Z)$ and this gives

$$\tilde{H} = \widetilde{SL}(n)Z(\tilde{H}). \tag{5.25}$$

Therefore the representation theory of $\widetilde{SL}(n)$ reduces easily to that of $\tilde{H}$.

Note that $\tilde{H}$ is normal, and has index $m = |\mathbb{F}^*/\mathbb{F}^{*n}|$, in $\widetilde{GL}(n)$. Again from (5.24) we see that *the centralizer of $Z(\tilde{H})$ in $\widetilde{GL}(n)$ is $\tilde{H}$*. That is, if $g \in \widetilde{GL}(n), g \notin \tilde{H}$ then $gzg^{-1} = z'$ for some $z, z' \in Z(\tilde{H}), z \neq z'$, i.e.

$$gzg^{-1} = \eta z \tag{5.26}$$

for some $\eta \in \mu_N(\mathbb{F}), \eta \neq 1$.

Now let σ be any genuine irreducible representation of $\tilde{H}$ with central character χ_σ. Let σ^g be the image of σ under the action of g on $\hat{H}$. Then (5.24) implies

$$g \notin \tilde{H} \Rightarrow \chi_{\sigma^g} \neq \chi_\sigma \tag{5.27}$$

and therefore $\sigma^g \not\simeq \sigma$.

Now let π be an irreducible genuine representation of $\widetilde{GL}(n)$, and suppose σ is a summand of π restricted to $\tilde{H}$. It follows from (5.27) and Clifford theory that

$$\pi|_{\tilde{H}} = \sum \sigma^g \tag{5.28}$$

where the sum runs over $g \in S = \widetilde{GL}(n)/\tilde{H}$.

The determinant induces an isomorphism $S \simeq \mathbb{F}^*/\mathbb{F}^{*n}$. For $\overline{x} \in \mathbb{F}^*/\mathbb{F}^{*n}$ let $x \in \mathbb{F}^*$ be a representative, and let $z_{\overline{x}} = (xI, \zeta)$ be any element of $\tilde{Z}$ satisfying $p(z_{\overline{x}}) = xI$.

Proposition 5.29. *Let σ be an irreducible genuine representation of $\tilde{H}$ with central character χ_σ, and suppose σ is contained in the restriction of an irreducible representation π of $\widetilde{GL}(n)$.*

$$\Theta_\sigma(g) = \frac{1}{m} \sum_{\overline{x} \in S} \chi_\sigma(z_x)^{-1} \Theta_\pi(z_x g) \quad (g \in \tilde{H}_0).$$

(This is independent of the choices of $z_{\overline{x}}$, $\overline{x} \in S$.)

Proof. Write $\pi|_{\tilde{H}} = \sum_{y \in S} \sigma^y$. Take the character of both sides at $z_x g$:

$$\begin{aligned} \Theta_\pi(z_x g) &= \sum_y \Theta_{\sigma^y}(z_x g) \\ &= \sum_y \chi_{\sigma^y}(z_x) \Theta_{\sigma^y}(g). \end{aligned} \tag{5.30}$$

Multiply both sides by $\chi_\sigma(z_x)^{-1}$, sum over x, and use orthogonality of characters on S. The result follows.

This reduces computation of irreducible characters of $\widetilde{SL}(n)$ to $\widetilde{GL}(n)$, and hence (via [10]) to $GL(n)$ (at least for tempered representations). In fact there is a natural reduction to $PGL(n)$, as we now sketch.

For $\nu_0 \in \widehat{\mathbb{F}^*}$ let $\chi(\nu_0)(xI;\zeta) = \nu_0(x)\zeta\gamma(x,\psi)$. This is a genuine character of $\tilde{Z}$, and every genuine character arises this way. Each χ_σ is equal to $\chi(\nu_0)$ for some ν_0.

Suppose π is an irreducible representation of $GL(n)$ whose central character satisfies $\chi_\pi(\zeta I) = 1$ for all $\zeta \in \mu_n(\mathbb{F})$. Let $\tilde{\pi}$ be the lift of π to $\widetilde{GL}(n)$ via [10]. By the basic relationship between the central characters of π and $\tilde{\pi}$ [10], the central character of any summand of the restriction of $\tilde{\pi}$ to $\tilde{H}$ is of the form $\chi(\nu_0)$, where ν_0 satisfies

$$\nu_0(x)^n = \chi_\pi(xI). \tag{5.31}$$

The choices of ν_0 satisfying (5.31) parametrize the summands of the restriction of $\tilde{\pi}$ to $\tilde{H}$.

Definition 5.32. For ν_0 and π satisfying (5.31), let $L(\pi,\nu_0)$ be the summand of $\tilde{\pi}|_{\tilde{H}}$ with central character $\chi(\nu_0)$.

For β_0 a character of $\mathbb{F}^*$ let $\beta(g) = \beta_0(det(g))$. The relation with $PGL(n)$ arises from the observation that $\pi\nu^{-1}$ has trivial central character, i.e. factors to $PGL(n)$. However the map $L(\pi,\nu_0) \to \pi\nu^{-1}$ is not well defined. The Proposition implies

$$L(\pi\beta,\nu_0\beta_0) = L(\pi,\nu_0) \tag{5.33}$$

but only provided $\beta(\zeta) = 1$ for all $\zeta \in \mu_n(\mathbb{F})$, i.e. $\beta \in \widehat{\mathbb{F}^{*n}}$. Let $X = \widehat{\mathbb{F}^*}/\widehat{\mathbb{F}^{*n}} \simeq \mu_n(\mathbb{F})\hat{}$.

This suggests a definition of stability for $\widetilde{SL}(n)$.

Definition 5.34.

$$L_{st}(\pi,\nu_0) = \sum_{\alpha\in X} L(\pi\alpha,\nu_0\alpha_0)$$

Such a virtual character is said to be *stable*.

Now (5.33) implies $L_{st}(\pi\beta,\nu_0\beta) = L_{st}(\pi,\nu_0)$ for *all* β. Hence

$$L_{st}(\pi,\nu_0) \to \pi\nu^{-1}$$

is a well–defined map from stable genuine virtual representations of $\tilde{H}$ to representations of $GL(n)$ with trivial central character, or alternatively $PGL(n)$. Tracing through the maps gives a character identity relating stable genuine virtual representations of $\widetilde{SL}(n)$ to irreducible representations of $PGL(n)$ (again assuming the conjecture of [10], which holds for tempered representations).

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