Advanced Studies in Pure Mathematics 25, 1997
CR-Geometry and Overdetermined Systems
pp. 406–418

A Complex Frobenius Problem

Sidney M. Webster

Dedicated to Professor Kuranishi

Introduction

A complex Frobenius structure on a smooth (real) manifold M is a smooth complex vector sub-bundle E of the complexified tangent bundle $T(M)\otimes\mathbf{C}$ which satisfies the integrability condition $[E,E]\subseteq E$ (i.e. the set of local sections of E is closed under Lie bracket). Such a structure is also termed formally integrable, or involutive. The bracket of a section of E with a section of the complex conjugate bundle $\overline{E}$, taken mod $E+\overline{E}$, gives the Levi-form of the structure. If $E\cap\overline{E}=0$, then E is a CR structure; it is an almost complex structure if also $E+\overline{E}=T(M)\otimes\mathbf{C}$. The integrability problem is to find independent functions, the differentials of which span the sub-bundle $E^{\perp}$ of complex covectors annihilating E. The problem of local solvability is to establish a Poincaré lemma in the natural de Rham-Dolbeault complex associated to the differential ideal generated by sections of $E^{\perp}$.

The case of identically vanishing Levi-form was already treated in works by Nirenberg [13] and Hörmander [7]. The Mizohata operator [5], [12] on $\mathbf{R}^2$ gives perhaps the simplest complex Frobenius structure for which local solvability fails (it is also important for the canonical transformation theory of partial differential equations [8]). Nirenberg [14] has shown that local integrability fails for small perturbations of this structure. Certain interesting higher dimensional analogues have been studied by Trèves [18]. These are structures on $\mathbf{R}^{n+1}$ induced by (local) maps $f:\mathbf{R}^{n+1}\to\mathbf{C}$, $df\neq 0$. The topology of the fibers $f^{-1}(a)$ plays a key role in the questions of local solvability and of local integrability for small perturbations of these structures. In his recent book [19] Trèves also treats the integrability and solvability problem for a variety of important structures.

Received May 9, 1995.
Partially supported by NSF Grant DMS-9201966.

Here we consider local structures which are modeled on those induced by *generic folds* $F : M \to \mathbf{C}^n$, where M is an open subset of $\mathbf{R}^{2n}$. The fiber is always a pair of points which coalesce along a smooth hypersurface $N \subset M$. For $n = 1$, F essentially induces the Mizohata structure [17]. For $n \geq 2$ the case where $F(N)$ is part of the boundary of a strongly pseudoconvex domain $D \subset \mathbf{C}^n$ is of special interest. F is an *interior fold* if $F(M) \subseteq \overline{D}$, and an *exterior fold* if $F(M) \cap D = \emptyset$.

Our **main result** implies that *complex Frobenius structures which are small perturbations of strongly pseudoconvex interior folds are locally integrable, if* $n \geq 2$. It is false for $n = 1$ by Nirenberg's example. We show by counterexample ((1.7) below) that local integrability also fails for small perturbations of strongly pseudoconvex *exterior* folds ($n = 2$). Thus, fiber topology does not suffice to determine the main properties of fold-like structures. The integrability results given here for them are more similar to those known for CR structures of hypersurface type. We refer specifically to the positive embedding results of Kuranishi [11] and Akahori [1], and the counterexamples of Nirenberg [14] and Jacobowitz and Trèves [10]. Though we point out that there is no unresolved dimension as there is (presently) for CR structures.

Originally we had hoped that the current problem would be more similar to the Kuranishi embedding problem and be amenable to the methods of [20]; but we were unable to construct an exact homotopy formula, i. e. one valid without shrinking the domain. However, it turned out that one could establish the above integrability result, and much more easily, by reducing it to a theorem of Hanges and Jacobowitz [6]. One drawback to this method is that it only yields C^∞ regularity. Thus we were unable to address the question of C^k regularity, one of out original aims. However, we hope that the work will shed some further light of the integrability problem.

The main result is proved also without the aid of the Poincaré lemma. In fact we use our integrability theorem to reduce it to the case of an interior fold, which we carry out in section 1, for the admissible degrees. We are indebted to Cordaro and Trèves [3] for a helpful remark in this respect. In section 2 we characterize intrinsically the various formal generic fold-like structures and give a useful normalization for them. This is used in section 3 together with [6] to derive the main result.

§1. Generic folds

Let $z = (z', z^n = x^n + iy^n)$ be complex coordinates on $\mathbf{C}^n$ and consider a (local) domain with boundary

$$(1.1) \qquad D : y^n > h(z', x^n) \ , \ \partial D : y^n = h(z', x^n) \ ,$$

where h is a smooth real function with $h(0) = 0$, $dh(0) = 0$. We also use (z', t, s) as coordinates on $\mathbf{R}^{2n} \cong \mathbf{C}^{n-1} \times \mathbf{R}^2$ and define a map $F : M \to \mathbf{C}^n$ by

$$(1.2) \qquad F(z', t, s) = (z', z^n = t + i(\frac{1}{2}s^2 + h(z', t))),$$

where M is a suitable neighborhood of 0 in $\mathbf{R}^{2n}$ which is symmetric about N,

$$(1.3) \qquad N = M \cap \{s = 0\}.$$

F is a *generic fold* of M onto $\overline{D}$ with $F(N) = \partial D$.

The complex one-forms

$$(1.4) \qquad \begin{aligned} \theta^\alpha = dz^\alpha \ , \theta^n = dz^n, \\ dz^n = (1 + ih_t)dt + i(sds + h_\alpha dz^\alpha + h_{\overline{\alpha}} d\overline{z}^\alpha), \end{aligned}$$

($h_\alpha = \partial h/\partial z^\alpha$, etc.) span an n-dimensional sub-bundle $E^\perp$ of the complex cotangent bundle. (We use the index ranges

$$(1.5) \qquad 1 \le \alpha, \beta, \gamma \le n - 1 \, ; \, 1 \le i, j, k \le n \, ,$$

and the summation convention for repeated indices). We let E be the sub-bundle of complex vectors annihilated by $E^\perp$. With $\{\theta^\alpha, \theta^n, d\overline{z}^\alpha, ds\}$ as a basis of complex covectors, we get dual complex complex vector fields $X_{\overline{\alpha}}$, $X_{\overline{n}}$ spanning E, for which

$$df \equiv X_{\overline{\alpha}} f d\overline{z}^\alpha + X_{\overline{n}} f ds, \text{ mod } (\theta^j),$$

for any smooth function f. We readily compute

$$(1.6) \qquad X_{\overline{\alpha}} = \partial_{\overline{\alpha}} - i\frac{h_{\overline{\alpha}}}{1 + ih_t}\partial_t \, , \, X_{\overline{n}} = \partial_s - i\frac{s}{1 + ih_t}\partial_t \, .$$

These vectors together with their complex conjugates,

$$X_j \equiv \overline{X_{\overline{j}}},$$

are linearly independent except along N, where $X_{\overline{n}}$ becomes real and transverse and spans $E \cap \overline{E}$. Along N we have the bracket relations

$$[X_\alpha, X_{\overline{\beta}}]|_N = -ig_{\alpha\overline{\beta}}\partial_t\,, [X_\alpha, X_{\overline{n}}]|_N = 0\,, [X_n, X_{\overline{n}}]|_N = -ig_{n\overline{n}}\partial_t\,.$$

Here $g_{n\overline{n}} = [2(1+h_t^2)]^{-1}$, and $g_{\alpha\overline{\beta}}$ corresponds to the Levi-form of ∂D under the equivalence $F|_N$. The full matrix $g_{i\overline{j}}$, defined along N, represents the Levi-form of the complex Frobenius structure E. It is positive definite if D is strongly pseudoconvex, in which case F is an interior fold. F is an exterior fold relative to $\mathbf{C}^n - D$ if the latter domain is strongly pseudoconvex.

Next we consider some abstractly defined structures E on $\mathbf{R}^4$ with coordinates $(z = x+iy, t, s)$. E is the span of the complex vector fields

$$X_{\overline{1}} = \partial_{\overline{z}}\,, X_{\overline{2}} = \partial_s - is(1+s\xi+z)\partial_t\,, \tag{1.7}$$

where $\xi(t,s)$ is a smooth real valued function defined near 0. Clearly, $[X_{\overline{1}}, X_{\overline{2}}] = 0$, $[X_1, X_{\overline{1}}] = 0$, and $[X_1, X_{\overline{2}}] = -i\partial_t$. Thus E is formally integrable, and $E \cap \overline{E}$ is non-zero only along the hypersurface $N : x = -s - s^2\xi(t,s)$, where it is spanned by the transverse vector $X_{\overline{2}} = \mathrm{Re}(X_{\overline{2}})$. The Levi-form is non-degenerate indefinite along N. In [14] Nirenberg has constructed a function ξ for which $(X_{\overline{2}}|_{z=0})u = 0$ has no non-constant solution $u(t,s)$. Thus, if $X_{\overline{j}}F = 0$, $j = 1, 2$, then $(dF \wedge dz)(0) = 0$, so that $(dF_1 \wedge dF_2)(0) = 0$. Hence, the structure E, which is a formally integrable strictly pseudo convex *exterior* fold is *not* actually integrable near 0 for this choice of ξ.

We return to the fold structure (1.2),(1.6) and consider the problem of local solvability. We assume that the smooth one-form

$$\varphi = \varphi_{\overline{\alpha}} d\overline{z}^\alpha + \varphi_{\overline{n}} ds\,, \tag{1.8}$$

satisfies the compatibility condition $\overline{\partial}_E\varphi = 0$, or

$$X_{\overline{i}}\varphi_{\overline{j}} - X_{\overline{j}}\varphi_{\overline{i}} = 0\,, 1 \le i, j \le n. \tag{1.9}$$

The problem is to find a smooth function g with $\overline{\partial}_E g = \varphi$, i.e.

$$X_{\overline{j}}g = \varphi_{\overline{j}}\,, 1 \le j \le n. \tag{1.10}$$

We assume, in addition, that φ satisfies

$$\varphi_{\overline{n}} = O(s^\infty)\,, \tag{1.11}$$

along N. More intrinsically, $\varphi_{\overline{n}}$ is the interior product of φ with a smooth non-vanishing section of E which becomes real along N. Thus, (1.11) says that φ is *flat on the real characteristics of E*. By a standard argument (see below) one can always achieve (1.11) by adding an exact form to φ. Grushin's example shows that, even with this extra condition, (1.10) may not have a solution for $n = 1$. However, if $n \geq 2$ and F is a strongly pseudoconvex interior fold, we shall show that (1.9) is sufficient for the existence of a solution to (1.10).

We define open sets and maps

$$M^{\pm} = \{(z', t, s) \in M : \pm s > 0\}\,, F_{\pm} = F|_{M^{\pm}}\,,$$

so that $F_{\pm} : M^{\pm} \to D$ are isomorphisms of Frobenius structures. We set $\varphi^{\pm} = (F_{\pm}^{-1})^* \varphi$, so that

$$\begin{aligned} \varphi^{\pm} &= b^{\pm}_{\overline{\alpha}} d\overline{z}^{\alpha} \pm [2(y^n - h)]^{-1/2} b^{\pm}_{\overline{n}} d(y^n - h) \\ &\equiv (b^{\pm}_{\overline{\alpha}} \pm \frac{-h_{\overline{\alpha}} b^{\pm}_{\overline{n}}}{\sqrt{2(y^n - h)}}) d\overline{z}^{\alpha} \pm \frac{(i - h_{x^n}) b^{\pm}_{\overline{n}}}{2\sqrt{2(y^n - h)}} d\overline{z}^n\,, \end{aligned} \tag{1.12}$$

where $b^{\pm}_{\overline{j}} = \varphi_{\overline{j}} \circ F_{\pm}^{-1}$, and $\equiv$ is congruence mod dz^j. These forms will blow up along ∂D, unless $\varphi_{\overline{n}}$ vanishes along N. This is the motivation for the condition (1.11).

Lemma 1.1. *If (1.9) and (1.11) hold, then the forms $\varphi^{\pm}$ are smooth $\overline{\partial}$-closed (0,1)-forms on the closure $\overline{D}$.*

proof. By (1.9) they are closed, and the $b^{\pm}_{\overline{n}}$ are clearly smooth on $\overline{D}$ and vanish to infinite order on ∂D by (1.11). By the chain rule and (1.9) we have

$$-2i\partial_{\overline{z}^n} b^{+}_{\overline{\alpha}} = (1 + ih_{x^n})(X_{\overline{\alpha}}\varphi_{\overline{n}}) \circ F_{+}^{-1}\,. \tag{1.13}$$

But each $X_{\overline{\alpha}}\varphi_{\overline{n}}$ also vanishes to infinite order along N, so these functions are also smooth up to the boundary. For each fixed z' we apply the Cauchy formula to $b^{+}_{\overline{\alpha}}$ on the domain with counter-clockwise boundary

$$\begin{aligned} D(z') &= \{z_n : |x^n| \leq \delta\,,\ h(z', x^n) \leq y^n \leq h(z', x^n) + \rho\}, \\ \partial D(z') &= a_0(z') - a_{\rho}(z') + c_{\delta}(z') - c_{-\delta}(z')\,, \\ a_{\sigma}(z') &= \{x^n + i(h(z', x^n) + \sigma) :\ -\delta \leq x^n \leq +\delta\}\,, \\ c_{\sigma}(z') &= \{\sigma + iy^n :\ h(z', \sigma) \leq y^n \leq h(z', \sigma) + \rho\}\,, \end{aligned} \tag{1.14}$$

where $\delta > 0$, $\rho > 0$ are sufficiently small. We have ($\zeta^n = \xi^n + i\eta^n$)

(1.15)

$$2\pi i b^{+}_{\overline{\alpha}}(z', z^n) = \int_{\partial D(z')} \frac{b^{+}_{\overline{\alpha}}(z', \zeta^n)}{\zeta^n - z^n} d\zeta^n + \int_{D(z')} \frac{\partial_{\overline{z}^n} b^{+}_{\overline{\alpha}}(z', \zeta^n)}{\zeta^n - z^n} d\zeta^n \wedge d\overline{\zeta}^n\,.$$

In the double integral we extend the numerator of the integrand smoothly by zero across ∂D. Then we may change the domain of integration $D(z')$ to one whose boundary is independent of z', except for the upper curve $a_\rho(z')$, which varies smoothly with z'. It's then clear that the double integral gives a function of (z', z^n) which is smooth up to the boundary. In the line integral over $c_\delta(z')$, we make the substitution $\sigma = \eta^n - h(z', \delta)$ to get

$$\int_{\sigma=0}^{\rho} \frac{\varphi_{\overline{\alpha}}(z', \delta, \sqrt{2\sigma})}{\delta + i(\sigma + h(z', \delta)) - z^n} i d\sigma .$$

This is smooth in (z', z^n), as z^n crosses the boundary curve $a_0(z')$. Similarly for $c_{-\delta}(z')$. For the line integral over $a_\rho(z')$ we have

$$\int_{\xi^n=-\delta}^{+\delta} \frac{\varphi_{\overline{\alpha}}(z', \xi^n, \sqrt{2\rho})}{\xi^n + i(h(z', \xi^n) + \rho) - z^n} (1 + i h_{x^n}(z', \xi^n)) d\xi^n .$$

This is also clearly smooth in (z', z^n) across the boundary. For the integral over $a_0(z')$, we set $\rho = 0$ in the above. Then $\varphi_{\overline{\alpha}}(z', \xi^n, 0)$ is a smooth function. Thus we have a Cauchy integral of a smooth function over a smooth curve, both depending smoothly on the parameter z'. By a well-known argument this gives a function of (z', z^n) which is smooth up to the boundary. The same argument for φ^- completes the proof.

To achieve (1.11) we replace φ by $\varphi - \overline{\partial}_E g$, where g is chosen so that $\varphi_{\overline{n}} - X_{\overline{n}} g = O(s^\infty)$. Formally, we set

$$\varphi_{\overline{n}} = \sum_{k=0}^{\infty} w_k(z', t) s^k, \ g = \sum_{k=1}^{\infty} g_k(z', t) s^k,$$

where $k! w_k = \partial_s^k \varphi_{\overline{n}}(z', t, 0)$. We determine the functions g_k successively by $g_1 = w_0$, $2g_2 = w_1$, and

$$\frac{-i}{1 + i h_t} \partial_t g_k + (k + 2) g_{k+2} = w_{k+1}.$$

By the theorem of E. Borel (see [8]) there is a smooth function g with these prescribed s-derivatives along $s = 0$.

Now we assume that D is strongly pseudoconvex, so that we may invoke Kohn's solution of the $\overline{\partial}$-problem [4]. Alternately, and in a more elementary vein, we may employ the local solution operators of Range and Lieb [15]. Thus, there exist functions $g^\pm$ smooth on $\overline{D}$ with $\overline{\partial} g^\pm = \varphi^\pm$. It follows that on ∂D we have the tangential Cauchy-Riemann equations $\overline{\partial}_b (g^+ - g^-) = 0$. By the H. Lewy extension theorem (smooth version), there is a function g^0, holomorphic on D and smooth on $\overline{D}$, with $g^0 = g^+ - g^-$ on ∂D. We replace g^- by $g^- + g^0$, so that $g^+ = g^-$ on ∂D. Then we define g on M by: $g = g^+ \circ F$ on M^+ and $g = g^- \circ F$ on M^-. Clearly, g is continuous on M and smooth on $M^+ \cup M^-$, where it satisfies (1.10). Also, all tangential derivatives $\partial^K_{(z,t)} g$ are continuous on $M^+ \cup N$ and on $M^- \cup N$. If we repeatedly differentiate the equation $X_{\overline{n}} g = \varphi_{\overline{n}}$ and use (1.6), we see that all derivatives of g are continuous across N. Thus, in contrast to the example of Grushin [5], [14] we have the following.

Proposition 1.2 (Poincaré lemma). *Let the smooth one-form φ (1.8) satisfy the compatibility condition (1.9) relative to the fold structure E induced by the map F (1.2). If $n \geq 2$ and if F is a strongly pseudoconvex interior fold, then there exists a smooth function g with $\overline{\partial}_E g = \varphi$.*

The Poincaré lemma also holds for $(0,q)$-forms (mod $E^\perp$) for $2 \leq q \leq n-1$. (That it does not hold for top degree forms follows from a general result of Cordaro and Hounie [2]). We sketch the argument. Relative to the coframe (1.4) the $(0,q)$-form φ has the representation

$$\varphi = \varphi' + ds \wedge \varphi'',$$

where φ', φ'' are q- and $q-1$-forms in $d\overline{z}'$ with coefficients in (z', t, s). In an obvious notation

$$\overline{\partial}_E \varphi = \overline{\partial}'_E \varphi' + ds \wedge (X_{\overline{n}} \varphi' - \overline{\partial}'_E \varphi'').$$

By a change $\varphi \mapsto \varphi - \overline{\partial}_E \psi'$, where ψ' is a $(0, q-1)$-form without ds, we can achieve $\varphi'' = O(s^\infty)$ as above. Suppose that $\overline{\partial}_E \varphi = 0$. Then the analogue of lemma (1.1) shows that the transplanted forms $\varphi^\pm$ are smooth on $\overline{D}$ and agree along ∂D. There exist smooth $(0, q-1)$-forms $\psi^\pm$ on $\overline{D}$ with $\overline{\partial}\psi^\pm = \varphi^\pm$. We have $\overline{\partial}(\psi^+ - \psi^-) = 0$ along ∂D. If we are below the Lewy unsolvability degree, i. e. $q - 1 < n - 1$, then there is a smooth $(0, q-2)$-form η with $\overline{\partial}\eta = \psi^+ - \psi^-$ along ∂D. We replace ψ^- by $\psi^- + \overline{\partial}\eta$. We patch together the forms $F^*\psi^\pm$ on $M^\pm$ as above to get a smooth form ψ satisfying $\overline{\partial}_E \psi = \varphi$.

§2. Local normalization

We consider n independent smooth (C^∞) complex vector fields $X_{\overline{j}}$, defined on an open set M of $\mathbf{R}^{2n}$ containing 0, and denote their complex conjugates by X_j as before. At each point $x \in M$, E_x is the complex vector space spanned by the $X_{\overline{j}}(x)$. We assume that $E_0 \cap \overline{E}_0$ is one-dimensional. After changing the frame, we may assume that $X_{\overline{n}}(0) =$

$X_n(0)$ spans $E_0 \cap \overline{E}_0$, and that $X_{\overline{n}}(0)$, $X_{\overline{\alpha}}(0)$, $X_\alpha(0)$ (see (1.5)) span $E_0 + \overline{E}_0$. We then choose coordinates $(z^\alpha = x^\alpha + iy^\alpha, t, s)$ so that $X_{\overline{\alpha}}(0) = \partial_{\overline{\alpha}}$, $X_{\overline{n}} = \partial_s$, and ∂_t is transverse to $E + \overline{E}$ at 0. Then we may write

$$X_{\overline{j}} = A_{\overline{j}}^{\overline{\beta}} \partial_{\overline{\beta}} + A_{\overline{j}}^{\overline{n}} \partial_s + A_{\overline{j}}^{\beta} \partial_\beta + B_{\overline{j}} \partial_t \,,$$

where $A_{\overline{j}}^{\overline{i}}(0) = \delta_{\overline{j}}^{\overline{i}}$. After changing the frame via the inverse matrix of $A_{\overline{j}}^{\overline{i}}$, we get

$$\text{(2.1)} \qquad \begin{aligned} X_{\overline{\alpha}} &= \partial_{\overline{\alpha}} + A_{\overline{\alpha}}^{\beta} \partial_\beta + B_{\overline{\alpha}} \partial_t \,, \\ X_{\overline{n}} &= \partial_s + A_{\overline{n}}^{\beta} \partial_\beta + B_{\overline{n}} \partial_t \,, \end{aligned}$$

where $A_{\overline{i}}^{\beta}(0) = 0$, $B_{\overline{j}}(0) = 0$. In this form (2.1) the frame is uniquely determined by the coordinate system (z, t, s), which we refer to as adapted to our structure at 0.

The *degeneracy locus*,

$$\text{(2.2)} \qquad N = \{x \in M : E_x \cap \overline{E}_x \neq 0\} \,,$$

is the set of points where some non-trivial linear combination of the $X_{\overline{j}}$ is real, or equivalently, where the vectors $X_{\overline{j}}, X_j$ are dependent. If we write out the condition $0 = a^j X_{\overline{j}} + b^j X_j$ and first eliminate a^n, we get

$$\text{(2.3)} \qquad N = \{r = 0\} \,, r = \det \begin{bmatrix} \delta_{\overline{\beta}}^{\alpha} & A_{\overline{\beta}}^{\alpha} & -iA_{\overline{n}}^{\alpha} \\ A_{\beta}^{\overline{\alpha}} & \delta_{\beta}^{\overline{\alpha}} & iA_{n}^{\overline{\alpha}} \\ B_\alpha & B_{\overline{\alpha}} & 2\mathrm{Im}(B_{\overline{n}}) \end{bmatrix} .$$

The shift in the bars over indices reflects complex conjugation, and a factor of i has been inserted in the last column of the determinant to make r real. Our *genericity* assumption on the degeneracy is the transversality condition

$$\text{(2.4)} \qquad X_{\overline{n}} r(0) \neq 0 \,.$$

It implies that N is a smooth hypersurface in M. Alternately, we may state it as follows: $dr(0)$, extended to $T_0(M) \otimes \mathbf{C}$ and then restricted to E_0 , is non-zero. It follows that $E \cap (T(N) \otimes \mathbf{C})$ gives a CR structure of real hypersurface type on N. If we additionally restrict the initially chosen frame above so that the $X_{\overline{\alpha}}$ span the (0,1)-vectors of this CR structure, then $\partial_{\overline{\alpha}} r(0) = 0$. We then make the coordinate change

$$z'^\alpha = z^\alpha \,, t' = t \,, s' = r(z, t, s) \,,$$

and the corresponding frame change

$$X'_{\overline{n}} = (r_s + A_{\overline{n}}^{\beta} r_\beta + B_{\overline{n}} r_t)^{-1} X_{\overline{n}} \, , X'_{\overline{\alpha}} = X_{\overline{\alpha}} - (r_{\overline{\alpha}} + A_{\overline{\alpha}}^{\beta} r_\beta + B_{\overline{\alpha}} r_t) X'_{\overline{n}} \, .$$

After this we may assume that N has the form (1.3), and that the vectors $X_{\overline{j}}$ still have the form (2.1).

To preserve these normalizations, we restrict our coordinate changes to the form

$$z'^{\alpha} = z^{\alpha} + f^{\alpha} \, , t' = t + f^0 \, , s' = s + f^n \, , \tag{2.5}$$

where

$$f^{\alpha} = O(2) \, , f^0 = O(2) \, , f^n = s\hat{f}^n \, , \hat{f}^n = O(1).$$

In the prime system there is a unique adapted E-frame

$$\begin{aligned} X'_{\overline{\alpha}} &= \partial'_{\overline{\alpha}} + A'_{\overline{\alpha}}{}^{\beta} \partial'_{\beta} + B'_{\overline{\alpha}} \partial_{t'} \, , \\ X'_{\overline{n}} &= \partial_{s'} + A'_{\overline{n}}{}^{\beta} \partial'_{\beta} + B'_{\overline{n}} \partial_{t'} \, . \end{aligned} \tag{2.6}$$

As in Section 1 of [20] we have the relations

$$\begin{aligned} A_{\overline{\alpha}}^{\beta} + X_{\overline{\alpha}} f^{\beta} &= (\delta_{\overline{\alpha}}^{\overline{\gamma}} + X_{\overline{\alpha}} f^{\overline{\gamma}}) A'_{\overline{\gamma}}{}^{\beta} + X_{\overline{\alpha}} f^n A'_{\overline{n}}{}^{\beta}, \\ B_{\overline{\alpha}} + X_{\overline{\alpha}} f^0 &= (\delta_{\overline{\alpha}}^{\overline{\gamma}} + X_{\overline{\alpha}} f^{\overline{\gamma}}) B'_{\overline{\gamma}} + X_{\overline{\alpha}} f^n B'_{\overline{n}}, \\ A_{\overline{n}}^{\beta} + X_{\overline{n}} f^{\beta} &= X_{\overline{n}} f^{\overline{\gamma}} A'_{\overline{\gamma}}{}^{\beta} + (1 + X_{\overline{n}} f^n) A'_{\overline{n}}{}^{\beta}, \\ B_{\overline{n}} + X_{\overline{n}} f^0 &= X_{\overline{n}} f^{\overline{\gamma}} B'_{\overline{\gamma}} + (1 + X_{\overline{n}} f^n) B'_{\overline{n}}. \end{aligned} \tag{2.7}$$

Exactly as in [20] we may choose $f^{\alpha}(z,t)$, $f^0(z,t)$ to achieve

$$[A_{\overline{\alpha}}^{\beta}]|_N = O(|(z,t)|^2), \tag{2.8}$$

$$[B_{\overline{\alpha}}]|_N = -ib_{\beta\overline{\alpha}} z^{\beta} + B^*_{\overline{\alpha}}(z,t), \; B^*_{\overline{\alpha}} = O(|(z,t)|^2),$$

where the hermitian matrix $b_{\beta\overline{\alpha}}$ represents the Levi-form at 0 of the CR structure on N.

For all further normalizations, we restrict to changes (2.5) with $f = s\hat{f}$, so that N and the functions (2.8) on it remain unchanged. The last two equations of (2.7) give

$$[A_{\overline{n}}^{\beta} + \hat{f}^{\beta} - \hat{f}^{\overline{\gamma}} A_{\overline{\gamma}}^{\beta}]|_N = [(1 + \hat{f}^n) A'_{\overline{n}}{}^{\beta}]|_N,$$

$$[B_{\overline{n}} + \hat{f}^0 - \hat{f}^{\overline{\gamma}} B_{\overline{\gamma}}]|_N = [(1 + \hat{f}^n) B'_{\overline{n}}]|_N .$$

We can choose $\hat{f}^\beta(z,t)$, $\hat{f}^0(z,t)$ uniquely so that the left side of the first equation, and the real part of the left side of the second vanish. After this change we may assume

$$A^\beta_{\overline{n}} = s\hat{A}^\beta_{\overline{n}} \, , B_{\overline{n}} = s\hat{B}_{\overline{n}} \, , \mathrm{Im}(\hat{B}_{\overline{n}}) \neq 0 \, ,$$

since $r = 0$ and $\partial_s r \neq 0$ on N in (2.3). Now we restrict to changes (2.5) with

$$f^\beta = s^2 \hat{f}^\beta \, , f^0 = s^2 \hat{f}^0 \, , f^n = s\hat{f}^n \, . \tag{2.9}$$

Substituting into (2.7), dividing by s, and letting $s \to 0$ gives

$$[\hat{A}^\beta_{\overline{n}} + 2\hat{f}^\beta - 2\hat{f}^{\overline{\gamma}} A^\beta_{\overline{\gamma}}]|_N = [(1 + \hat{f}^n) \hat{A}'^{\,\beta}_{\overline{n}}]|_N \, ,$$

$$[\hat{B}_{\overline{n}} + 2\hat{f}^0 - 2\hat{f}^{\overline{\gamma}} B_{\overline{\gamma}}]|_N = [(1 + \hat{f}^n) \hat{B}'_{\overline{n}}]|_N \, .$$

Thus we can choose $\hat{f}(z,t)$ to make

$$[\hat{A}'^{\,\beta}_{\overline{n}}]|_N = 0 \, , [\mathrm{Re}(\hat{B}'_{\overline{n}})]|_N = 0 \, , [\mathrm{Im}(\hat{B}'_{\overline{n}})]|_N = -1 \, .$$

After these preliminary normalizations we make the coordinate change (2.5),(2.9) with

$$\hat{f}(z,t,s) = \sum_{j=1}^{\infty} \frac{1}{j!} s^j g(s/\varepsilon_j) \hat{f}_j(z,t) \, , \tag{2.10}$$

where $\hat{f} = (\hat{f}^\beta, \hat{f}^0, \hat{f}^n)$. Following [8], vol. I, p. 16, we choose $g(s)$ to be a fixed smooth real valued function of suitably small compact support with $g(s) - 1$ vanishing to infinite order at $s = 0$. The functions $\hat{f}_j$ are successively chosen, depending on the previous choices, to achieve

$$A^\beta_{\overline{n}} = O(s^{j+1}) \, , B_{\overline{n}} + is = O(s^{j+1}) \, . \tag{2.11}$$

This is independent of $\varepsilon_j > 0$ which is then chosen so small that

$$|\partial^K_{(z,t,s)} [s^j g(s/\varepsilon_j) \hat{f}_j(z,t)]| \leq j! 2^{-j} \, , |K| \leq j - 1 \, .$$

The transformation so constructed is smooth, and in the final coordinate system (2.11) holds for every j. Hence, we have established the following.

Lemma 2.1. *There exists a smooth adapted coordinate system (z, t, s) and corresponding frame (2.1) so that (1.3), (2.2) and (2.8) hold, and*

$$A_{\overline{n}}^{\beta} = O(s^{\infty}) \, , B_{\overline{n}} + is = O(s^{\infty}) \, .$$

For the case $n = 1$, see Sjörstrand [16] and Trèves [17].

§3. Structures with positive definite Levi-form

With the normalizations of lemma (2.1) we have the bracket relations at 0

$$[X_i, X_{\overline{j}}]_0 = -2ib_{i\overline{j}}\partial_t \, , b_{\alpha\overline{n}} = 0 \, , b_{n\overline{n}} = 1/2 \, . \tag{3.1}$$

$b_{i\overline{j}}$ represents the Levi-form of the structure E at 0. It will be positive definite precisely when $b_{\alpha\overline{\beta}}$, which represents the Levi-form of the CR structure on N, is positive definite.

Theorem 3.1. *Let E be a complex Frobenius structure of rank n on a neighborhood of 0 in $\mathbf{R}^{2n}$ with $n \geq 2$. Suppose that E has a generic degeneracy with positive definite Levi-form at 0. Then there exists a neighborhood M of 0 and a strongly pseudoconvex interior fold $F : M \to \mathbf{C}^n$ inducing the structure E on M.*

For the proof we assume the normalizations of lemma (2.1) and consider the transformation

$$T : (z, t, s) \mapsto (z, t, y = \frac{1}{2}s^2) \, ,$$

and its restrictions $T_{\pm}$to $M^{\pm}$. On $U = T_{\pm}(M^{\pm}) \subseteq \{y \geq 0\}$ we have the vector fields

$$X'_{\overline{\alpha}} = (T_+)_*(X_{\overline{\alpha}}) \, , X'_{\overline{n}} = (T_+)_*(\frac{1}{s}X_{\overline{n}}) \, .$$

Clearly $X'_{\overline{n}}$ is smooth up to the boundary $y = 0$.

By a Cauchy integral argument similar to the one given in the proof of lemma (1.1), it follows that the $X'_{\overline{\alpha}}$'s are also smooth up to the boundary of U. In fact, we just set $h \equiv 0$ in (1.13),(1.14), and replace $b_{\overline{\alpha}}^{+}$ by $A_{\overline{\alpha}}^{\beta} \circ (T_+)^{-1}$, or $B_{\overline{\alpha}} \circ (T_+)^{-1}$ in (1.15). Hence,we have a smooth almost complex structure with strongly pseudoconvex boundary on U. By the theorem of Hanges and Jacobowitz [6] there is a holomorphic coordinate system $G_+ : U \to \mathbf{C}^n$ which is smooth up to the boundary for a perhaps smaller U containing 0. Similarly, we have a smooth holomorphic

$G_- : U \to \mathbf{C}^n$ for the almost complex structure induced by T_-. The map

$$G_+^{-1} \circ G_- : G_-(\partial U) \to G_+(\partial U)$$

is a CR equivalence. By the smooth version of the H. Lewy extension theorem, it extends to a biholomorphic equivalence $H : G_+(U) \to G_-(U)$, which is smooth up to the boundary. We replace G_- by $H \circ G_-$, and then define

$$F = \begin{cases} G_+ \circ T & on \quad M^+ \\ G_- \circ T & on \quad M^- \end{cases}$$

An argument strictly analogous to that given in the proof of proposition (1.2) shows that F is smooth on M. Since F is an embedding of N, its coordinate functions are independent on M also. Hence, F satisfies the requirements of the theorem.

The hypotheses on the Levi-form can clearly be weakened, since they may be so in the Hanges-Jacobowitz theorem (Catlin), and in the H. Lewy extension theorem (Trepreau).

References

[1] T. Akahori, A new approach to the local embedding theorem of CR structures for $n \geq 4$, Memoirs AMS 67 (1987).

[2] P. Cordaro and J. Hounie, On the local solvability of underdetermined systems of vector fields, Amer. J. Math. 112(1990)243-270.

[3] P. Cordaro and F. Treves, personal communication.

[4] G. B. Folland and J. J. Kohn, The Neumann Problem for the Cauchy-Riemann Complex, Ann. of Math. Studies 75, Princeton University Press (1972).

[5] V. V. Grushin, A certain example of a differential equation without solutions, Math. Notes 10(1971)499-501 (from Mat. Zametki 10 (1971)125-128).

[6] N. Hanges and H. Jacobowitz, A remark on almost complex structures with boundary, Amer. Jour. Math. 111(1989)53-64.

[7] L. Hörmander, The Frobenius-Nirenberg theorem, Ark. Mat.29 (1964) 425-432.

[8] —— , The Analysis of Linear Partial Differential Operators, Vol.1-4, Springer-Verlag, New York (1983-85).

[9] H. Jacobowitz, Global Mizohata structures, J. Geom. Anal. 3 (1992) 153-193.

[10] —— , and F. Treves, Aberrant CR structures, Hokkaido Math. J. 22 (1983) 276-292.

[11] M. Kuranishi, Strongly pseudoconvex CR structures over small balls III, Ann. of Math. 116 (1982) 249-330.

[12] S. Mizohata, Solutions nulles et solutions non analytic, J. Math. Kyoto Univ. 1(1962)271-302.

[13] L. Nirenberg, A complex Frobenius theorem, Seminars on analytic functions I, Princeton (1957)172-189.

[14] ——— , Lectures on linear partial differential equations, CBMS Studies No. 17, AMS(1973).

[15] M. Range and I. Lieb, Lösungsoperatoren für den Cauchy-Riemann-Kpmplex mit C^k-Abschätzungen, Math. Ann. 253 (1980) 145-164.

[16] J. Sjörstrand, Note on a paper of F. Treves concerning Mizohata type operators, Duke Math. Jour. 47(1980)601-608.

[17] F. Treves, Remarks about certain first-order linear PDE in two variables, Comm. PDE 5(4) (1980)381-425.

[18] ——— , On the local solvability and the local integrability of systems of vector fields, Acta Math. 151(1983)1-38.

[19] F. Treves, Hypo-Analytic Structures: Local Theory, Princeton University Press (1992).

[20] S. M. Webster, On the proof of Kuranishi's embedding theorem, Ann. Inst. H. Poincaré, Anal. non-lin., 6(1989)183-207.

Department of Mathematics
University of Chicago
Chicago, IL 60637
U. S. A.