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Deformation Theory of CR-Structures and Its Application to Deformations of Isolated Singularities II

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Introduction

Deformations of an analytic variety with only isolated singular points induce deformations of strongly pseudo-convex CR structures on its link. It is M. Kuranishi who initiated to consider deformations of compact strongly pseudo-convex CR structures expecting to describe deformations of isolated singular points of analytic varieties. Since non-equivalent CR manifolds can bound the same isolated singular point, we consider deformations of CR structures up to equivalence weaker than the CR-equivalence. This equivalence is induced from wiggling in a complex manifold and we will call the deformation theory of CR structures under that equivalence the *Kuranishi deformation theory* of CR structures. In [Ku3], [Ku4], M. Kuranishi obtained a C^∞-family of deformations of the CR structure on a compact strongly pseudo-convex CR manifold of real dimension five or higher, continuing his early works on deformations of compact complex structures ([Ku1], [Ku2]). We consider holomorphic families of CR structures. In the first half of this survey, we will review the holomorphically parametrized deformation theory of strongly pseudo-convex CR structures developped by T. Akahori et al. ([Ak1], [Ak2], [Ak3], [Ak4], [Ak-My1], [Ak-My2], [Ak-My3], [Ak-My4], [Bu-Ml], [My1], [My2], [My3]) and its relationship with algebraic deformation theory of isolated singularities ([Do], [Gr], [Tj]).

The relationship between compact strongly pseudo-convex CR manifolds and isolated singularities is based on the fact that an embeddable compact strongly pseudo-convex CR manifold bounds a unique normal Stein complex space ([Ha-La]) and all compact strongly pseudo-convex

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CR manifolds of real dimension five or higher are embeddable ([BM]). In contrast with the higher dimensional case, embeddable three dimensional CR manifolds are rare. (Embeddability is a major problem in the study of three dimensional compact strongly pseudo-convex CR manifolds. Refer to the references of [Bl-Du], [Ep] and [Lm] for papers about that problem.) Recently, J. Bland and C. Epstein generalized the Kuranishi deformation theory to embeddable compact strongly pseudo-convex three dimensional CR structure case and show that the stably embeddable formal deformation theory of a strongly pseudo-convex three dimensional CR structure is isomorphic to the formal deformation theory of the normal isolated surface singularity it bounds ([Bl-Ep]). In the latter half of this survey, we will develop their deformation theory to the actual deformation level in the higher dimensional case and compare it with the deformation theory of normal isolated singularities. By this argument, we will see that the stably embeddable deformation theory of strongly pseudo-convex CR structures fits to the flat deformation theory of normal isolated singularities and then we will complete Kuranishi's program describing the semi-uinversal family of normal isolated singularities of complex dimension three or higher in terms of the CR language.

Other developments in the study of isolated singularities by CR geometry are done in [Lu-Ya] and [Ya]. Refer to [Oh3] for other results in the study of isolated singularities by transcendental methods. The moduli of compact strongly pseudo-convex CR manifolds under CR-equivalence is of natural interest from CR-geometry. It is treated in [Ch-Le] and [Lm] in three dimensional cases. The moduli space of strongly pseudo-convex CR structures on a compact real three-fold divides into two parts; the part of embeddable CR structures and the part of non-embeddable ones. In [Ep], moduli of embeddable CR structures is considered in connection with deformations of isolated singularities, and in [Bl-Du], moduli of non-embeddable ones on S^3 is considered.

In this survey, we consider only the case of real dimension five or higher. In Sections 1–4, we will review the construction of the Kuranishi semi-universal family of compact strongly pseudo-convex CR structures and its relationship with the semi-universal family of normal isolated singularities. The main part of the construction of the Kuranishi semi-universal family was presented in Part I under the assumption $H^2_{\bar\partial_{T'}}(T')=0$. Hence we will only give a modification needed for treating the general case. In Sections 5 and 6, we will work on the stably embeddable deformation theory of strongly pseudo-convex CR structures on a link of a normal isolated singularity.

Notation

(N.1) Let V be a closed normal subvariety of a ball

$$B(c^*) := \{w \in \mathbf{C}^N \mid \sum_{\beta=1}^{N} |w^\beta|^2 < c^*\}$$

such that the origin $0 \in \mathbf{C}^N$ is the only singular point of V and V and $\partial B(c)$ intersect transversally for any $0 < c < c^*$. We denote by U the regular part of V and by ι the natural inclusion map $U \hookrightarrow \mathbf{C}^N$. We will use the following notations throughout this survey:

$$G(a,b) := \{w \in \mathbf{C}^N \mid a < \sum_{\beta=1}^{N} |w^\beta|^2 < b\} \ (0 < a < b < c^*),$$

$$S_c := \{w \in \mathbf{C}^N \mid \sum_{\beta=1}^{N} |w^\beta|^2 = c\} \ (0 < a < c \leq b),$$

$$\Omega(a,b) := V \cap G(a,b), \qquad M_c := V \cap S_c.$$

(N.2) For a holomorphic vector bundle over a CR manifold M, we denote by $A_b^{0,q}(E)$ (resp. $\Gamma(U,E)$) the space of E-valued tangential (0,q)-forms (resp. the space of C^∞-sections of E over a domain $U \subset M$). $A_{b,k}^{0,q}(E)$ (resp. $\Gamma_k(U,E)$ and $\Gamma'_k(U,E)$) the completion of $A_b^{0,q}(E)$ with respect to the Sobolev k-norm (resp. of $\Gamma(U,E)$ with respect to the Sobolev k-norm and the Folland-Stein k-norm).

(N.3) Let S be a germ of a (not necessarily reduced) complex space at the distinguished point $s_o \in S$ and X be a (not necessarily compact) complex manifold (in our argument below, X is a neighbourhood of M_c in U). By a family of deformations of X over (S, s_o), we mean a smooth holomorphic map (in Grothendieck's sense) of complex spaces $\pi : \mathcal{X} \to S$ with $\pi^{-1}(s_o) \simeq X$, that is, for any $x \in \mathcal{X}$ there exist neighbourhoods $\mathcal{W}$ of x in $\mathcal{X}$, W of the origin in $\mathbf{C}^n$ ($n = \dim_{\mathbf{C}} X$) and an isomorphism q so that the diagram

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{\ q\ } & W \times S \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle p_2} \\ S & = & S \end{array}$$

commutes where p_2 denotes the projection onto the second factor. As a local trivialization of a family of deformations of X, we always take a local trivialization of this type. Then we always have a local coordinate $(\zeta^1, \ldots, \zeta^n, s_1, \ldots, s_d)$ of $\mathcal{X}$ such that $(\zeta^1, \ldots, \zeta^n)$ (resp. $(s_1, \ldots, s_d)$) is the coordinate of W (resp. of an ambient space $\mathbf{C}^d$ of S) in the above trivialization.

(N.4) Suppose that X is a locally closed sub-manifold of $\mathbf{C}^N$. By a family of displacements of X in $\mathbf{C}^N$ over a germ (S, s_o), we mean a family of deformations of X over (S, s_o), $\pi : \mathcal{X} \to S$, together with an embedding $\Phi : \mathcal{X} \to \mathbf{C}^N \times S$ such that $\pi = p_2 \circ \Phi$ holds where $p_2 : \mathbf{C}^N \times S \to S$ denotes the projection onto the second factor.

(N.5) There are two approaches to the CR manifolds; the extrinsic approach and the intrinsic one (i.e. treatments as a real submanifold of a complex manifold and as a real manifold equipped with an abstract CR structure, respectively). These approaches are equivalent in the case of compact strongly pseudo-convex CR manifolds of real dimension greater than or equals to five, while there are differences between them in real three dimensional case. Our treatment of deformations of CR structures is based on the intrinsic approach. Refer to [Ta] for the systematic study of this approach. In order to compare deformations of CR structures and that of singularities, we need to take account of the extrinsic approach as well. Refer to [Fo-Ko] or [Ko-Ro] for the extrinsic approach.

Our approach to deformations of normal isolated singularities from deformation theory of CR structures will be done through the following three steps. In each step, we use several fundamental theories. First step: We construct a family of CR structures by a generalized Kodaira-Spencer construction. Refer to [Ko] for the Kodaira-Spencer construction in the case of deformations of compact complex manifolds. The Kodaira-Spencer construction heavily depends on the harmonic theory. Refer to [Fo-Ko] for the standard harmonic theory on a CR manifold. Second step: We prove that the family constructed in the first step is Kuranishi versal. In order to carry out the ideal theoretic argument in the proof, we use the Grauert division theorem. Refer to [Gr] or [Fo-Kn] for the Grauert division theorem. Third step: We compare the family constructed in the first step with the semi-universal family of normal isolated singularities. By the comparison using the Kuranishi semi-universality of the family of CR structures and the semi-versality of the family of isolated singularities, we have a formal isomorphism of their parameter spaces. In order to reach the actual isomorphism, we use the Artin approximation theorem. Refer to [Ar] for the Artin approximation theorem.

We remark that W. Goldmann and J. Millson established a general comparison method. We can compare the above two families directly (without Step 2) using this general method. Refer to [Go-Ml1] and [Go-Ml2] for this general comparison method and to [Bu-Ml] for the approach using this method.

§1. Kuranishi deformation theory of CR structures

Let $M := M_c$ for some $0 < c < c^*$ (cf. (N.1)) and ${}^\circ T''$ be the strongly pseudo-convex CR structure on M induced from the complex structure of V (cf. Part I, sections 2.1 for the notion of a strongly pseudo-convex CR structures). In this section, we formulate some fundamental notions of Kuranishi deformation theory of CR structures on M.

A holomorphic family of CR structures is a notion analogous to a family of complex structures. We fix a splitting Part I, (2.1.6);

$$\mathbf{C}TM = {}^\circ T' + {}^\circ T'' + \mathbf{C}F$$

and denote by $T' = {}^\circ T' + \mathbf{C}F$ the holomorphic tangent bundle of M.

Definition 1.1. *Let T be a germ of a complex subspace of $\mathbf{C}^d$ at the origin defined by an ideal $\mathfrak{I}_T \subset \mathbf{C}\{t_1, \ldots, t_d\}$. A holomorphic family of deformations of the CR structure ${}^\circ T''$ over $(T, 0)$ is $\phi(t) \in A_b^{0,1}(T')[[t_1, \ldots, t_d]] \cap \bigcap_{k>0} A_{b,k}^{0,1}(T')\{t_1, \ldots, t_d\}$ satisfying*

(1) $\phi(0) = 0$,

(2) $P(\phi(t)) \in \mathfrak{I}_T A_{b,k-1}^{0,2}(T')\{t_1, \ldots, t_d\}$ *for all* $k >> 0$

where $P(\phi) = 0$ is the integrability condition (cf. Part I, Section 3.2).

We will simply denote it by $\phi(t)$ $(t \in (T, 0))$.

An embedding of M into a family of complex manifolds induces a family of CR structures.

Definition 1.2. *Let S be a germ of a complex subspace of $\mathbf{C}^{d'}$ at the origin defined by an ideal $\mathfrak{I}_S \subset \mathbf{C}\{s_1, \ldots, s_{d'}\}$. Let $\pi : \mathcal{U} \to S$ be a family of complex manifolds. A holomorphic family of embeddings of M into that family is a mapping $F : M \times S \to \mathcal{U}$ with $\pi \circ F = p_2$ where p_2 denotes the projection onto the second factor, which is described locally as follows: Let $\{(\mathcal{W}_i, (\zeta_i^1, \ldots, \zeta_i^n, s_1, \ldots, s_{d'}))\}_{i \in \Lambda}$ be a system of local coordinates of $\mathcal{U}$ as in (N.3). Let $\{U_i\}_{i \in \Lambda}$ be an open covering of M such that $F(U_i \times S) \subset \mathcal{W}_i$. If F is described by $\zeta_i^\alpha = F_i^\alpha(x_i, s)$ $(\alpha = 1, \ldots, n)$ on $U_i \times S$ with respect to the above local coordinate of $\mathcal{W}_i$ and the local coordinate $(x_1^1, \ldots, x_i^{2n-1})$ of U_i, then*

(1) $F_i^\alpha(s) \in \Gamma(U_i, 1)[[s_1, \ldots, s_{d'}]] \cap \Gamma_k(U_i, 1)\{s_1, \ldots, s_{d'}\}$ $(\alpha = 1, \ldots, n)$ *for all* $k >> 0$,

(2) $F_i^\alpha(s) - f_{ij}^\alpha(F_j(s), s) \in \mathfrak{I}_S\Gamma_k(U_i \cap U_j, 1)\{s_1, \ldots, s_{d'}\}$ ($\alpha = 1, \ldots, n$) *for all* $k >> 0$,

where $\zeta_i^\alpha = f_{ij}^\alpha(\zeta_j, s)$ ($\alpha = 1, \ldots, n$) *is the coordinate transformation on* $\mathcal{W}_i \cap \mathcal{W}_j$.

The holomorphic family $\phi(s)$ ($s \in (S, 0)$) of CR structures induced by F is characterized by

$$(\bar{\partial}_b - \phi(s))F_i^\alpha(s) \in \mathfrak{I}_S A_b^{0,1}(U_i, 1)[[s_1, \ldots, s_{d'}]].$$

We call this $\phi(s)$ ($s \in (S, s_o)$) the family of CR structures induced by F.

Since the Kuranishi CR deformation theory is arranged suitable for deformation theory of normal isolated singularities, we do not consider CR structures up to CR isomorphism but consider them up to wiggling in an ambient complex manifold. The following notion of the versality is reasonable for our deformation theory.

Definition 1.3. *A holomorphic family* $\phi(t)$ ($t \in (T, 0)$) *of deformations of* $^\circ T''$ *is Kuranishi versal if it has the following property: For any family of deformations of complex manifolds* $\pi : \mathcal{U} \to S$ *over* (S, s_o) *such that* $\pi^{-1}(s_o)$ *is a neighbourhood of* M *in* U, *there exist a holomorphic map of germs* $\tau : (S, s_o) \to (T, 0)$ *and a holomorphic family of embeddings* $F : M \times S \to \mathcal{U}$ *such that* $F_{|M \times s_o} = \iota$ *holds and the holomorphic family of CR structures induced by* F *coincides with* $\phi(\tau(s))$ ($s \in (S, s_o)$).

We will examine the fist derivative of a family of CR structures. Let $\phi(t)$ ($t \in (T, 0)$) be a holomorphic family of deformations of $^\circ T''$. Since the linear term of $P(\phi)$ is $\bar{\partial}_{T'}\phi$, we have $\bar{\partial}_{T'}v(\phi(t)) = 0$ for $v \in T_0T$ where we denote by T_0T the Zariski tangent space of T at 0. Next, if $\phi(t)$ ($t \in (T, 0)$) and $\psi(t)$ ($t \in (T, 0)$) are holomorphic families of CR structures induced by holomorphic families of embeddings into a family of complex manifolds $F, G : M \times T \to \mathcal{U}$ respectively, then we have

$$\{\sum_{\alpha=1}^n v(F_i^\alpha(t) - G_i^\alpha(t))\frac{\partial}{\partial \zeta_i^\alpha}\} \in A_b^0(T'U_{|M})$$

and

$$\rho^{1,0}(v(\phi(t) - \psi(t))) = \bar{\partial}_b(\sum_{\alpha=1}^n v(F_i^\alpha(t) - G_i^\alpha(t))\frac{\partial}{\partial \zeta_i^\alpha})$$

where $\rho^{1,0} : \mathbf{C}TU_{|M} \to T^{'}U_{|M}$ denotes the projection onto the (1,0)-part. Hence, it is natural to call the $\bar{\partial}_{T'}$ cohomology class of $v(\phi(t))$ the infinitesimal deformation class of the family $\phi(t)$ $(t \in (T,0))$ along the direction $v \in T_0T$.

Definition 1.4. *For a holomorphic family $\phi(t)$ $(t \in (T,0))$ of deformations of CR structures, the infinitesimal deformation map is a linear map $\rho : T_0T \to H^1_{\bar{\partial}_{T'}}(T^{'})$ given by $\rho(v) :=$ the cohomology class of $v(\phi(t))$. A holomorphic family is called effective if its infinitesimal deformation map is injective.*

An effective and Kuranishi versal family is called a Kuranishi semi-universal family.

§2. Construction of the Kuranishi semi-universal family of CR structures

In this section, we consider how to construct the Kuranishi semi-universal family of CR structures on M.

(I) First, we consider the case of $\dim_{\mathbf{R}} M \geq 7$ and will review the construction of the Kuranishi semi-universal family in [Ak3] and [Ak-My1]. However we modify the ideal theoretic argument in [Ak-My1] by using the Grauert division theorem instead of a small tric there, because the adaptation of the division theorem is the most relevant way to treat the case of non-reduced parameter spaces. We first try to construct it using J. J. Kohn's solution of the $\bar{\partial}_b$-Neumann problem (cf. [Fo-Ko]). Though it works only on the formal family level, we will try it, because this is a straightforward analogue of the standard Kodaira-Spencer construction in the case of deformations of complex structures on a compact complex manifold and, by this consideration, we will well understand the naturality of the adaptation of the sub-complex $(\Gamma(M, E_q), \bar{\partial}_q)$ in [Ak3] (cf. Part I, Section 3). The $\bar{\partial}_b$-Neumann Hodge decomposition which we will use is

$$\eta = \rho_{T'}\eta + \bar{\partial}_{T'}\bar{\partial}^*_{T'}N_{T'}\eta + \bar{\partial}^*_{T'}\bar{\partial}_{T'}N_{T'}\eta \quad \text{for } \eta \in A^{0,2}_b(T^{'}), \tag{2.1}$$

where $\rho_{T'}$ denotes the orthogonal projection onto the harmonic space $\mathbf{H}^2_b(T^{'})$ (cf. [Fo-Ko, Theorem 5.4.12]). Let $d = \dim H^1_{\bar{\partial}_{T'}}(T^{'})$ and $\phi_1, \ldots, \phi_d$ be $\bar{\partial}_{T'}$-closed forms which give cohomology basis of $H^1_{\bar{\partial}_{T'}}(T^{'})$.

Set

$$\phi_1(t_1,\ldots,t_d) := \sum_{\sigma=1}^{d} \phi_\sigma t_\sigma, \tag{2.2}$$

$$\phi_\mu(t_1,\ldots,t_d) := \mu\text{-th homogeneous term of}$$

$$-\bar{\partial}^*_{T'} N_{T'} P(\phi^{(\mu-1)}(t_1,\ldots,t_d))$$

(cf. Part I, section 3.2 for the definition of $P(\phi)$),

$$\phi^{(\mu)}(t_1,\ldots,t_d) := \phi^{(\mu-1)}(t_1,\ldots,t_d) + \phi_\mu(t_1,\ldots,t_d)$$

and

$$\hat{\phi}(t_1,\ldots,t_d) := \lim_{\mu\to\infty}\phi^{(\mu)}(t_1,\ldots,t_d).$$

Thus we have $\hat{\phi}(t_1,\ldots,t_d) \in A_b^{0,1}(T')[[t_1,\ldots,t_d]]$ (we will denote it simply by $\hat{\phi}(t)$) satisfying

$$\hat{\phi}(t) + \bar{\partial}^*_{T'} N_{T'} R(\hat{\phi}(t)) = \phi_1(t) \tag{2.3}$$

where $R(\phi) := R_2(\phi) + R_3(\phi)$. Take a basis $e_1,\ldots,e_\ell$ of $\mathbf{H}^2_b(T')$, then we have $\hat{b}_1(t),\ldots,\hat{b}_\ell(t) \in \mathbf{C}[[t_1,\ldots,t_d]]$ such that

$$\rho_{T'} P(\hat{\phi}(t)) = \sum_{\gamma=1}^{\ell} \hat{b}_\gamma(t) e_\gamma.$$

Denote by $\hat{\mathfrak{I}}$ an ideal of $\mathbf{C}[[t_1,\ldots,t_d]]$ generated by $\hat{b}_1(t),\ldots,\hat{b}_r(t)$.

Proposition 2.1.

$$P(\hat{\phi}(t)) \in \hat{\mathfrak{I}} A_b^{0,2}(T')[[t_1,\ldots,t_d]].$$

Proof. By (2.3) and using the Hodge decomposition (2.1), we have

$$P(\hat{\phi}(t)) = \rho_{T'} R(\hat{\phi}(t)) + \bar{\partial}^*_{T'} \bar{\partial}_{T'} N_{T'} R(\hat{\phi}(t)).$$

Using the fact $\bar{\partial}^\phi_b P(\phi) = 0$ (cf. Part I, Lemma 3.7.2), we can prove by induction on μ that

$$\bar{\partial}_{T'} R(\hat{\phi}(t)) \in (\hat{\mathfrak{I}} + \mathfrak{m}^{\mu+1}) A_b^{0,3}(T')[[t_1,\ldots,t_d]]$$

holds for all $\mu \geq 1$ where $\mathfrak{m}$ denotes the maximal ideal of $\mathbf{C}\{t_1, \dots, t_d\}$.
Q.E.D.

Proposition 2.2. *$\hat{\phi}(t)$ is formally Kuranishi versal, that is, for any family of deformations of a neighborhood of M in U there exist τ and F in Definition 1.3 as formal power series in s.*

Proof. Let S be a germ of an analytic sub-space of $\mathbf{C}^{d'}$ defined by an ideal $\mathfrak{I}_S \subset \mathbf{C}\{s_1, \dots, s_{d'}\}$ and let $\pi : \mathcal{U} \to S$ be a family of complex manifolds such that $\pi^{-1}(0)$ is a neighbourhood of M in U and $\{(\zeta_i^1, \dots, \zeta_i^n, s_1, \dots, s_{d'})\}_{i \in \Lambda}$ be a system of local coordinates of $\mathcal{U}$, as in (N.3), with the coordinate transformation $\zeta_i^\alpha = f_{ij}^\alpha(\zeta_j, s)$ $(\alpha = 1, \dots, n)$. Then we will construct $\{\hat{F}_i^\alpha(s)\}$ $(\alpha = 1, \dots, n)$ and $\hat{\tau}(s)$ by solving the following equations inductively, where we denote by $F_{i|\mu}^\alpha(s)$ and $\tau_\mu(s)$ the homogeneous terms of $\hat{F}_i^\alpha(s)$ and $\hat{\tau}(s)$ of degree μ respectively and denote $F_i^{(\mu)\,\alpha}(s) = F_{i|0}^\alpha(s) + \cdots + F_{i|\mu}^\alpha(s)$ and $\tau^{(\mu)}(s) = \tau_0(s) + \cdots + \tau_\mu(s)$.

$$(2.4)_\mu \quad F_i^{(\mu-1)\,\alpha}(s) - f_{ij}^\alpha(F_j^{(\mu-1)}(s), s) \in (\mathfrak{I}_S + \mathfrak{m}^\mu)\Gamma(U_i \cap U_j, 1)[[s_1, \dots, s_{d'}]],$$

$$(2.5)_\mu \quad (\bar{\partial}_b - \hat{\phi}(\tau^{(\mu-1)}(s)))F_i^{(\mu-1)\,\alpha}(s) \in (\mathfrak{I}_S + \mathfrak{m}^\mu)\Gamma(U_i, 1)[[s_1, \dots, s_{d'}]],$$

$$(2.6)_\mu \quad \hat{b}_\gamma(\tau^{(\mu-1)}(s)) \in (\mathfrak{I}_S + \mathfrak{m}^\mu)\mathbf{C}[[s_1, \dots, s_{d'}]] \ (\gamma = 1, \dots, \ell).$$

Let $F_i^{(0)\,\alpha}(s) = z_i^\alpha$ $(\alpha = 1, \dots, n)$ and $\tau_0(s) = 0$. Suppose that $F_i^{(\mu-1)\,\alpha}(s)$ and $\tau^{(\mu-1)}(s)$ are obtained such that $(2.4)_{\mu-1}$–$(2.6)_{\mu-1}$ hold.

Let $F_{i|\mu}^{'\,\alpha}(s)$ be the solution of

$$(2.7) \quad -\sum_{\alpha=1}^n \{F_{i|\mu}^{'\,\alpha}(s) - \sum_{\gamma=1}^n \frac{\partial f_{ij}^\alpha}{\partial z_j^\gamma}(\zeta_j, 0) F_{j|\mu}^{'\,\gamma}(s)\} \frac{\partial}{\partial z_i^\alpha} \equiv \sum_{\alpha=1}^n \{F_i^{(\mu-1)\,\alpha}(s) - f_{ij}^\alpha(F_j^{(\mu-1)\,\alpha}(s), s)\} \frac{\partial}{\partial z_i^\alpha}$$
$$\mod (\mathfrak{I}_S + \mathfrak{m}^{\mu+1})\Gamma(U_i, T'X_{|M})[[s_1, \dots, s_{d'}]].$$

Then, there exists $\theta(s) \in A_b^{0,1}(T'X_{|M})[[s_1, \dots, s_{d'}]]$ such that

$$\theta(s) - (\bar{\partial}_b - \hat{\phi}(\tau^{(\mu-1)}(s))) \left(\sum_{\alpha=1}^n F_i^{(\mu-1)\,\alpha}(s) \frac{\partial}{\partial z_i^\alpha} \right) \in (\mathfrak{I}_S + \mathfrak{m}^{\mu+1}) A_b^{0,1}(U_i, T'X_{|M})[[s_1, \dots, s_{d'}]].$$

By a direct calculation, we have

Lemma 2.3.

$$(\bar{\partial}_b - \phi)^2 = -P(\phi) - Q(\phi)$$

where $Q(\phi) \in A_b^{0,2}({}^{\phi}T'')$ *is given by*

$$Q(\phi)(\bar{X}, \bar{Y})f := (\bar{\partial}_b - \phi)f([\bar{X}, \phi(\bar{Y})]_{\circ T''} + [\phi(\bar{X}), \bar{Y}]_{\circ T''} - [\phi(\bar{X}), \phi(\bar{Y})]_{\circ T''})$$

for a function f *on* M.

Lemma 2.4.

(1) $\hat{b}_\gamma(\tau^{(\mu-1)}(s)) \in (\hat{\mathfrak{J}}_S + \mathfrak{m}^{\mu+1})\mathbf{C}[[s_1, \ldots, s_{d'}]]$ $(\gamma = 1, \ldots, r)$,

(2) $\bar{\partial}_b\theta(s) \in (\hat{\mathfrak{J}}_S + \mathfrak{m}^{\mu+1})A_b^{0,2}(U_i, T'X_{|M})[[s_1, \ldots, s_{d'}]]$.

Proof. (1) By Lemma 2.3, we have

$$\bar{\partial}_b\theta(s) + \rho^{1,0}P(\hat{\phi}(\tau^{(\mu-1)}(s))) \in (\hat{\mathfrak{J}}_S + \mathfrak{m}^{\mu+1})A_b^{0,2}(T'X_{|M})[[s_1, \ldots, s_{d'}]]$$

where $\rho^{1,0} : \mathbf{C}TU_{|M} \to T'U_{|M}$ denotes the projection onto the (1,0)-part. Hence

$$\rho_{T'}P(\hat{\phi}(\tau^{(\mu-1)}(s))) \in (\hat{\mathfrak{J}}_S + \mathfrak{m}^{\mu+1})A_b^{0,2}(T')[[s_1, \ldots, s_{d'}]].$$

(2) follows from (1) Q.E.D.

Let $F_\mu''{}^{\alpha}(s)$ and $\tau_\mu(s)$ be the solutions of

$$\bar{\partial}_b\{\sum_{\alpha=1}^{n} F_\mu''{}^{\alpha}(s)\frac{\partial}{\partial z_i^\alpha}\} - (\sum_{\sigma=1}^{d} \phi_\sigma\tau_\mu^\sigma(s)) \tag{2.8}$$

$$\equiv -(\bar{\partial}_b - \hat{\phi}(\tau^{(\mu-1)}(s))\left(\sum_{\alpha=1}^{n}(F_i^{(\mu-1)\,\alpha}(s) + F_{i|\mu}'{}^{\alpha}(s))\frac{\partial}{\partial z_i^\alpha}\right)$$

$$\bmod (\mathfrak{J}_S + \mathfrak{m}^{\mu+1})A_b^{0,1}(T'X_{|M})[[s_1, \ldots, s_{d'}]].$$

Then it is clear that

$$\begin{cases} F_i^{(\mu)\,\alpha}(s) := F_i^{(\mu-1)\,\alpha}(s) + F_{i|\mu}'{}^{\alpha}(s) + F_\mu''{}^{\alpha}(s) \\ \tau^{(\mu)}(s) := \tau^{(\mu-1)}(s) + \tau_\mu(s) \end{cases}$$

satisfy $(2.4)_\mu$–$(2.6)_\mu$. Since the solvability of the equation (2.8) is assured by Lemma 2.4, we have $\{F_i^{(\mu)\ \alpha}(s)\}$ and $\tau^{(\mu)}(s)$ for all $\mu \geq 0$. Q.E.D.

In the proof of Proposition 2.2, we used the Grauert division theorem (cf. [Gr]) in order to solve the linear equations (2.7) and (2.8).

In order to show the convergence of $\hat{\phi}(t)$, $\{\hat{b}_\gamma(t)\}_{1\leq\gamma\leq\ell}$, $\hat{\tau}(s)$ and $\{\hat{F}_i^\alpha(s)\}$ with respect to the Sobolev norm, we need the coercive estimate for the Neumann problem. Though $\bar{\partial}_b$-Neumann problem is not the case, we remark that the following weak-coercive estimate

$$(2.9) \qquad ||\vartheta_{T'} N_{T'} \xi||_k' \leq c||\xi||_k \text{ for } \xi \in \Gamma(M, T')$$

is enough for the convergence of them with respect the Folland-Stein norm $||\ ||_k'$ (cf. Part I, section 3.5 for the definition of the Folland-Stein norm), as long as $\hat{\phi}(t)$ is $^\circ T'$-valued.

In fact, if $\hat{\phi}(t) \in A_b^{0,1}(^\circ T')[[t_1, \ldots, t_d]]$ is assured in the above construction, we have

$$||\phi_\mu(t)||_k' << ||\vartheta_{T'} N_{T'} R(\phi^{(\mu-1)}(t))||_k' << C||\phi^{(\mu-1)}(t)||_k'^2$$

by (2.9) and Part I, Lemma 3.6.3 (we should remark that the estimate in Part I, Lemma 3.6.3 holds for all $\phi \in A_b^{0,1}(^\circ T')$). Where we use the same notation $A(t) << B(t)$ as in Part I, Section 3.7.

Taking account of the following Lemma together with Part I, Theorems 3.3.2 and 3.5.2, we can trace the above construciton relying on the complex $(\Gamma(M, E_q), \bar{\partial}_q)$ instead of $(A_b^{0,q}(T'), \bar{\partial}_{T'})$ and obtain $\hat{\phi}(t)$ which is $A_b^{0,1}(^\circ T')$-valued and satisfies (2.2) and (2.3). (This is the reson why the sub-complex $(\Gamma(M, E_q), \bar{\partial}_q)$ of $(A_b^{0,q}(T'), \bar{\partial}_{T'})$ was introduced in [Ak3].)

Lemma 2.5. *([My1, Proposition 1.1]) For $\phi \in \Gamma(M, E_1)$, $P(\phi)$ is in $\Gamma(M, E_2)$.*

Hence, we obtain convergent $\phi(t)$ and $\{b_\gamma(t)\}_{1\leq\gamma\leq\ell}$, by modifying the construction as above using $\big(\Gamma(M, E_q), \bar{\partial}_q\big)$ and the Hodge decomposition in Part I, Theorem 3.5.2 instead of $\left(A_b^{0,q}(T'), \bar{\partial}_{T'}\right)$ and the standard $\bar{\partial}_b$-Neumann Hodge decomposition (2.1) respectively. If we set

$$\mathfrak{I} := (b_1(t), \ldots, b_\ell(t)) \subset \mathbf{C}\{t_1, \ldots, t_d\}.$$

then the Grauert division theorem ([Gr]) says that Proposition 2.1 implies

$$P(\phi(t)) \in \mathfrak{I} A_{b,k}^{0,2}(T')\{t_1, \ldots, t_d\} \text{ for all } k >> 0.$$

The proof of convergence of $\hat{\tau}(s)$ and $\{\hat{F}_i^\alpha(s)\}$ is done by the same calculation as in [My3, Note], since $\hat{\phi}(t) \in A_b^{0,1}({}^\circ T^{'})[[t_1, \dots, t_d]]$ holds. And we have, by $(2.4)_\mu$–$(2.6)_\mu$ $(\mu \geq 0)$,

$$F_i^\alpha(s) - f_{ij}^\alpha(F_j(s), s) \in \mathfrak{I}_S\Gamma_k(U_i \cap U_j, 1)\{s_1, \dots, s_{d'}\} \ (\alpha = 1, \dots, n)$$

for all $k >> 0$,

$$(\bar{\partial}_b - \phi(\tau(s)))F_i^\alpha(s) \in \mathfrak{I}_S\Gamma_k(U_i, 1)\{s_1, \dots, s_{d'}\} \ (\alpha = 1, \dots, n)$$

for all $k >> 0$,

$$b_\gamma(\tau(s)) \in \mathfrak{I}_S \ (\gamma = 1, \dots, \ell).$$

Hence we have a Kuranishi semi-universal family of deformations of ${}^\circ T^{''}$.

The parameter space of that semi-universal family is described as $b^{-1}(0)$ by means of the holomorphic map $b : H^1_{\bar{\partial}_b}(T^{'}) \supset D \to \mathbf{H}^2 \simeq H^2_{\bar{\partial}_b}(T^{'})$ given by $h(t) = \rho P(\phi(t))$ where $\rho : \Gamma(M, E_2) \to \mathbf{H}^2$ is the orthogonal projection onto the harmonic space $\mathbf{H}^2 \subset \Gamma(M, E_2)$ (cf. Part I, section 2.5).

(II) Next, we consider the case of $\dim_{\mathbf{R}} M = 5$. In this case, $H^2_{\bar{\partial}_{T'}}(T^{'})$ may be infinite dimensional. However, the $\bar{\partial}_b$-Neumann harmonic space $\mathbf{H}_b^2(T^{'})$ is a closed subspace of the L^2-completion $A_{b,0}^{0,2}(T^{'})$ of $A_b^{0,2}(T^{'})$ and the projection operator onto it makes sense. The Hodge decompositions at degree 2 are obtained as follows using the $\bar{\partial}_b$-Neumann operators at degree 1

$$\eta = \rho_{T'}\eta + \bar{\partial}_{T'} N_{T'} \bar{\partial}^*_{T'} \eta \quad \text{for } \eta \in A_b^{0,2}(T^{'}), \tag{2.10}$$

where $\rho_{T'}$ denotes the orthogonal projection onto $\mathbf{H}_b^2(T^{'})$. The construction of $\hat{\phi}(t) \in A_b^{0,1}(T^{'})[[t_1, \dots, t_d]]$ in part (I) can be carried out using the decomposition (2.10) as follows: Let

$$\phi_1(t_1, \dots, t_d) := \sum_{\sigma=1}^d \phi_\sigma t_\sigma,$$

$$\phi_\mu(t) := \mu\text{-th homogeneous term of } - N_{T'}\bar{\partial}^*_{T'} P(\phi^{(\mu-1)}(t)),$$

$$\phi^{(\mu)}(t_1, \dots, t_d) := \phi^{(\mu-1)}(t_1, \dots, t_d) + \phi_\mu(t_1, \dots, t_d)$$

and

$$\hat{\phi}(t_1, \dots, t_d) := \lim_{\mu\to\infty} \phi^{(\mu)}(t_1, \dots, t_d).$$

And let $\{\hat{b}_\lambda(t)\}_{\lambda\in\Lambda}$ be given by

$$\hat{b}_\lambda(t) = \left(\rho_{T'} P(\hat{\phi}(t)), e_\lambda\right)$$

for an orthonormal basis $\{e_\lambda\}_{\lambda\in\Lambda}$ of $A^{0,2}_{b,0}(T')$. Let $\hat{\mathfrak{J}}$ be an ideal of $\mathbf{C}[[t_1,\ldots,t_d]]$ generated by $\{\hat{b}_\lambda(t)\}_{\lambda\in\Lambda}$. If we note that $P(\hat{\phi}(t)) \in \hat{\mathfrak{J}}A^{0,2}_b(T')[[t_1,\ldots,t_d]]$ is equivalent to $\left(P(\hat{\phi}(t)), e_\lambda\right) \in \hat{\mathfrak{J}}$ for all $\lambda \in \Lambda$, Proposition 2.1 also holds and Proposition 2.2 can be proved by the same argument. Therefore the construction of a formally Kuranishi semi-universal formal family in part (I) of this section is also valid for the case of $\dim_{\mathbf{R}} M = 5$.

In the case of normal strongly pseudo-convex CR manifolds of real-dimension 5, T. Akahori constructed $\hat{\phi}(t) \in A^{0,1}_b(T')[[t_1,\ldots,t_d]]$ which is convergent with respect to the $\| \|''_k$-norm (cf. [Ak4]).

§3. Smoothness of the Kuranishi semi-universal families

In this section, we consider the problem of when the parameter space of the Kuranishi semi-universal family of CR structures on M is smooth. We denote the parameter space of the Kuranishi semi-universal family by T_{CR} (in five dimensional case, we denote the parameter space of the formally Kuranishi semi-universal formal family by $\hat{T}_{CR}$).

(I) The Kodaira-Spencer-type smoothness. By the construction of T_{CR} or $\hat{T}_{CR}$ in §2, it is clear that if $H^2_{\bar{\partial}_{T'}}(T') = 0$ then T_{CR} ($\hat{T}_{CR}$ in five dimensional case) is smooth.

(II) The Bogomolov-type smoothness. The Bogomolov smoothness theorem is a smoothness theorem based on the other principle: In the case of deformations of a compact Kähler manifolds, if the canonical bundle K_X is trivial, then by the inner product with a non-vanishing holomorphic $(n,0)$-form, the integrability condition $P(\phi)$ is converted to an equation of ordinary differential $(n-1,1)$-forms. Using the pure Hodge structure on a compact Kähler manifold, the converted equation is solved without obstructions. Hence the parameter space of the semi-universal family is smooth.

On a strongly pseudo-convex CR manifold, there does not exists a natural pure Hodge structure much less a $(\partial_b, \bar{\partial}_b)$-double complex. In [Ak-My2], we introduced a sub-space $F^{p,q} \subset A^{p,q}_b(T')$. Let θ be a real contact form (that is, a non-vanishing real 1-form which annihilates

${}^{\circ}T' + {}^{\circ}T'')$ and let

$$F^{p,q} := \{\theta \wedge \alpha \in \theta \wedge \Gamma(M, \wedge^{p-1}({}^{\circ}T')^* \wedge \wedge^q({}^{\circ}T'')^*) \mid d\theta \wedge \alpha = 0\}.$$

Then a double-complex $(F^{p,q}; \partial, \bar\partial)$ is naturally induced. The higher part of the total simple complex of $(F^{p,q}; \partial, \bar\partial)$ coincides with (the higher part of) the Rumin complex (cf. [Ru] for the Rumin complex). If there exists a $\bar\partial_b$-closed non-vanishing $(n,0)$-form (i.e. there exists a non-vanishing $\omega \in \Gamma(M, \wedge^n(T')^*)$ satisfying $\bar\partial_{\wedge^n(T')^*}\omega = 0$ or equivalently there exists a non-vanishing $\omega \in \Gamma(M, \wedge^n(T'U_{|M})^*)$ satisfying $\bar\partial_b\omega = 0$), the inner product with that $(n,0)$-form induces an isomorphism of complexes

$$\iota : (\Gamma(M, E_q), \bar\partial_q) \simeq (F^{n-1,q}, \bar\partial),$$

where $(\Gamma(M, E_q), \bar\partial_q)$ is the sub-complex of $(A_b^{0,q}(T'), \bar\partial_{T'})$ introduced by T. Akahori (cf. Part I, Section 3). Hence, the only difference from the compact Kähler case is the lack of the pure Hodge structure on $(F^{p,q}; \partial, \bar\partial)$. Because of this lack, the analogue of the Bogomolov smoothness does not necessarily hold in deformations of CR structures (cf. [Ak-My3]). Hence, we consider unobstructedness of a subspace of $H^1_{\bar\partial_{T'}}(T')$, where we call a subspace *unobstructed* if there exists a holomorphic family of CR structures whose infinitesimal deformation space coincides with that space.

Let $I^{p,q} := Z^{p,q}_{\partial} \cap Z^{p,q}_{\bar\partial} / Z^{p,q}_{\partial} \cap \bar\partial F^{p,q-1}$ and $J^{p,q} := \partial F^{p-1,q} \cap Z^{p,q}_{\bar\partial} / \partial F^{p-1,q} \cap \bar\partial F^{p,q-1}$ with denoting $Z^{p,q}_{\partial} := \mathrm{Ker}\partial \cap F^{p,q}$ and $Z^{p,q}_{\bar\partial} := \mathrm{Ker}\bar\partial \cap F^{p,q}$.

Theorem 3.1. *([Ak-My2], [Ak-My4]) Suppose that $\dim_{\mathbf{R}} M \geq 7$. If $J^{n-1,2} = 0$ then $\iota^{-1}(I^{n-1,1})$ is unobstructed.*

Further developments in connection with deformations of isolated singularities are done in [My5] and [My6] using the Hodge structure on a strongly pseudo-convex domain ([De], [Oh1], [Oh2], [Oh-Ta]).

§4. Deformation theory of normal isolated singularities

In this section, we review briefly deformation theory of normal isolated singularities. Refer to [Tj] and [Gr] for details.

Let V be a germ of an analytic variety with a unique singular point o. In this article, we assume that V is a normal complex space.

Definition 4.1. *A family of deformations of V over a germ (S, s_o) is a flat holomorphic mapping of germs $f : \mathcal{V} \to S$ with $f^{-1}(s_o) \simeq V$.*

The equivalence of two families are defined by the equivalence of the two flat holomorphic mappings and the notion of versality is defined in a usual manner. The infinitesimal deformation map is a map $\rho : T_{s_o}S \to \mathrm{Ext}^1(\Omega^1_V, \mathcal{O}_V)$ and a holomorphic family is called effective if the infinitesimal deformation map is injective. An effective and versal family is called a semi-universal family. It is shown in [Tj] that the obstruction space is $\mathrm{Ext}^2(\Omega^1_V, \mathcal{O}_V)$ and H. Grauert ([Gr]) proved the existence of the semi-universal family.

We may assume that V is a closed subvariety of a ball $B(c^*)$ in $\mathbf{C}^N$ defined by $\tilde{h}_1 = \cdots = \tilde{h}_{m_1} = 0$ and o is the origin of $\mathbf{C}^N$. Denote $B := B(c^*)$, $\Omega := \Omega(a, b)$, and $M := M_c$ for some fixed $0 < a < c \le b < c^*$.

We recall Tjurina's description of $\mathrm{Ext}^q(\Omega^1_V, \mathcal{O}_V)$ (cf. [Tj]): The sheaf of germs of Kähler differentials Ω^1_V is given by $\Omega^1_V := \Omega^1_B/\Omega'_{|V}$ where Ω' is the sub-sheaf of Ω^1_B consisting of germs of forms ω such that $\omega = \sum_\lambda f_\lambda d\tilde{h}_\lambda + \sum_\lambda \tilde{h}_\lambda \phi_\lambda$ with $f_\lambda \in \mathcal{O}_B$ and $\phi_\lambda \in \Omega^1_B$. Hence, we have a free resolution of Ω^1_V,

$$0 \leftarrow \Omega^1_V \leftarrow \Omega^1_B \otimes \mathcal{O}_V \overset{d_0}{\leftarrow} \mathcal{O}_V^{m_1} \overset{d_1}{\leftarrow} \mathcal{O}_V^{m_2} \overset{d_2}{\leftarrow} \mathcal{O}_V^{m_3} \overset{d_3}{\leftarrow} \cdots$$

where $d_0(u_1, \ldots, u_{m_1}) := \sum_\lambda u_\lambda d\tilde{h}_\lambda$.

$\mathrm{Ext}^*(\Omega^1_V, \mathcal{O}_V)$ is the cohomology groups of the following complex:

$$0 \to H^0(V, \Theta_B \otimes \mathcal{O}_V) \overset{d_0^*}{\to} H^0(V, \mathcal{O}_V^{m_1}) \overset{d_1^*}{\to} H^0(V, \mathcal{O}_V^{m_2}) \overset{d_2^*}{\to} H^0(V, \mathcal{O}_V^{m_3}) \to .$$

Since V is normal, this complex is quasi-isomorphic to the following complex:

$$0 \to H^0(\Omega, \Theta_B \otimes \mathcal{O}_\Omega) \overset{d_0^*}{\to} H^0(\Omega, \mathcal{O}_\Omega^{m_1}) \overset{d_1^*}{\to} H^0(\Omega, \mathcal{O}_\Omega^{m_2}) \overset{d_2^*}{\to} H^0(\Omega, \mathcal{O}_\Omega^{m_3}) \to$$

where we note that $d_0^*(v) = (v(\tilde{h}_1), \ldots, v(\tilde{h}_{m_1}))$ $v \in H^0(\Omega, \Theta_B \otimes \mathcal{O}_\Omega)$. Using the commutative diagram,

$$\begin{array}{ccccccccc} 0 & \to & \Theta_\Omega & \overset{F}{\to} & \Theta_B \otimes \mathcal{O}_\Omega & \to & N_{\Omega/B} & \to & 0 \\ & & \| & & \| & & \downarrow & & \\ 0 & \to & \Theta_\Omega & \overset{F}{\to} & \Theta_B \otimes \mathcal{O}_\Omega & \overset{d_0^*}{\to} & \mathcal{O}_\Omega^{m_1} & \to & \cdots \end{array}$$

where $F : \Theta_\Omega \to \Theta_B \otimes \mathcal{O}_\Omega$ denotes the differential of the natural embedding $\iota : \Omega \to G$ and $N_{\Omega/G}$ is the normal bundle of Ω in G, we have

Proposition 4.1.

(1) $\mathrm{Ext}^0(\Omega^1_V, \mathcal{O}_V) \simeq H^0(\Omega, \Theta_\Omega)$,

(2) $\mathrm{Ext}^1(\Omega^1_V, \mathcal{O}_V) \simeq \mathrm{Ker}\{H^1(\Omega, \Theta_\Omega) \to H^1(\Omega, \Theta_B \otimes \mathcal{O}_\Omega)\}$,

(3) $\mathrm{Ext}^2(\Omega^1_V, \mathcal{O}_V) \simeq \mathrm{Ker}\{H^1(\Omega, N_{\Omega/B}) \to H^1(\Omega, \mathcal{O}_\Omega^{m_1})\}$.

Using [Ya, pp.81-82] and [Hö, Theorem 3.4.9] and noting that depth $\mathcal{O}_{V,o} \geq r$ is equivalent to $H^q(V \setminus o, \mathcal{O}_{V \setminus o}) = 0$ $(1 \leq q \leq r-2)$ (cf. [Ba]), we have

Theorem 4.2. *If* depth $\mathcal{O}_{V,o} \geq 3$ *and* $\dim_{\mathbf{C}} V \geq 4$,

(1) $\mathrm{Ext}^1(\Omega^1_V, \mathcal{O}_V) \simeq H^1_{\bar\partial_b}(T' U_{|M})$,

(2) $\mathrm{Ext}^2(\Omega^1_V, \mathcal{O}_V) \simeq H^1_{\bar\partial_b}(N_{\Omega/G|M}) \subset H^2_{\bar\partial_b}(T' U_{|M})$.

Remark. (Cf. [Bl-Ep], Propositions 6.1 and 6.2 below.) In the case of $\dim_{\mathbf{C}} V = 2$:

(1) $\mathrm{Ext}^1(\Omega^1_V, \mathcal{O}_V)$ is a finite dimensional subspace of $H^1_{\bar\partial_b}(T' U_{|M})$, though the latter space is infinite dimensional.

(2) $\mathrm{Ext}^2(\Omega^1_V, \mathcal{O}_V)$ is a finite dimensional subspace of $H^1_{\bar\partial_b}(N_{\Omega/G|M})$.

Suppose that $\dim_{\mathbf{C}} V \geq 4$. Take a model of the semi-universal family of deformations of V, say $f : \mathcal{V} \to S$, such that $f^{-1}(s_o) \simeq V$ $(s_o \in S)$ as germs at the singular point o. We may assume that $\Omega \subset f^{-1}(s_o)$ and let $\phi(t)$ $(t \in (T_{CR}, 0))$ be the Kuranishi semi-universal family of deformations of CR structures on M constructed in §2. Based on Theorem 4.2, the following comparison theorem is proved.

Theorem 4.3. *([Bu-Ml], [My2]) If $\dim_{\mathbf{C}} V \geq 4$ and depth $\mathcal{O}_{V,o} \geq 3$, then $(T_{CR}, 0) \simeq (S, s_o)$ and the holomorphic family $\phi(t)$ $(t \in (T_{CR}, 0))$ is induced by a holomorphic family of embeddings F:*

$$\begin{array}{ccc} M \times T_{CR} & \overset{F}{\hookrightarrow} & \mathcal{V} \\ \downarrow p_2 & & \downarrow f \\ T_{CR} & \simeq & S. \end{array}$$

§5. Stably embeddable deformations of CR structures

A compact strongly pseudo-convex CR manifold arises as a boundary of a Stein space if and only if it is embedded in a complex Euclidean space ([Ha-La]). Such CR manifolds are called embeddable CR manifolds. While all compact strongly pseudo-convex CR manifold of real dimension five or higher are embeddable ([BM]), embeddable CR manifolds are rare in the three dimensional case. In [Bl-Ep], J. Bland and C. Epstein formulated deformation theory of embeddable three dimensional CR structures and showed that it is equivalent on the formal deformation level to the deformation theory of normal isolated surface singularities.

In the higher dimensional case, though all compact strongly pseudo-convex CR manifolds bound Stein spaces, there are differences between the Kuranishi deformation theory of CR structures and the deformation theory of normal isolated singularities, unless depth$\mathcal{O}_{V,o} \geq 3$ (cf. [Bu-Ml, §10]). The Kuranishi deformation theory of CR structures would correspond to the non-flat deformation theory (cf. [Es]), while the flat deformation theory corresponds to a special deformation theory of CR structures. Recently, [My6] generalizes the stably embeddable deformation theory of three dimensional CR structures in [Bl-Ep] to the higher dimensional complex structure case, and shows that it fits to the deformation theory of normal isolated singularities. In this section and the next one, we consider the CR-version of [My6] and complete the Kuranishi program describing the semi-universal family of normal isolated singularities in terms of the CR-language, in the case of complex dimension three or higher.

Let V be as at the beginning of §4 and use the same notation about M, ${}^{\circ}T^{''}$, B and Ω as in §4.

Definition 5.1. *Let T be a germ of a complex subspace of $\mathbf{C}^d$ at the origin defined by an ideal $\mathfrak{I}_T \subset \mathbf{C}\{t_1, \ldots, t_d\}$. A stably embeddable family of deformations of the CR structure ${}^{\circ}T^{''}$ in $\mathbf{C}^N$ over $(T, 0)$ is a holomorphic family $\phi(t)$ of deformations of ${}^{\circ}T^{''}$ over $(T, 0)$ such that there exists*

$$g(t) \in A_b^0(T^{'}\mathbf{C}^N{}_{|M}))[[t_1, \ldots, t_d]] \cap \cap_{k>0} A_{b,k}^0(T^{'}\mathbf{C}^N{}_{|M}))\{t_1, \ldots, t_d\}$$

satisfying

$$(\bar{\partial}_b - \phi(t))(\iota + g(t)) \in \mathfrak{I}_T A_{b,k}^{0,1}(T^{'}\mathbf{C}^N{}_{|M})\{t_1, \ldots, t_d\}$$

for all $k >> 0$.

We consider a stably embeddable family of CR structures up to wiggling in an ambient complex manifold. Hence, we take the following notion of versality.

Definition 5.2. *A stably embeddable family $\phi(t)$ $(t \in (T,0))$ of deformations of ${}^{\circ}T''$ in $\mathbf{C}^N$ is Kuranishi versal if it has the following property: For any family of displacements (in $\mathbf{C}^N$) of a neighbourhood of M in U, over a germ (S, s_o), say*

$$\begin{array}{ccc} \mathcal{U} & \hookrightarrow & \mathbf{C}^N \times S \\ \downarrow \pi & & \downarrow p_2 \\ S & = & S, \end{array}$$

there exist a holomorphic map $\tau : (S, s_o) \to (T,0)$ and a holomorphic family of embeddings $F : M \times S \to \mathcal{U}$ such that $F_{|M\times 0} = \iota$ and the holomorphic family of CR structures induced by F coincides with $\phi(\tau(s))$ $(s \in (S, s_o))$.

Let $(\phi(t)$ $(t \in (T,0))$ be a stably embeddable family of deformations of ${}^{\circ}T''$ with which $g(t)$ is associated. Since

$$P(\phi(t)) = \bar{\partial}_{T'}\phi(t) + R_2(\phi(t)) + R_3(\phi(t)) = 0 \text{ and } (\bar{\partial}_b - \phi(t))(\iota + g(t)) = 0,$$

we have

$$\bar{\partial}_{T'} v(\phi(t)) = 0 \qquad \text{and} \qquad \bar{\partial}_b v(g(t)) - d\iota v(\phi(t)) = 0$$

for $v \in T_0T$. Hence $v(\phi(t))$ is $\bar{\partial}_{T'}$-closed and $d\iota v(\phi(t))$ is $\bar{\partial}_b$-exact.

Definition 5.3. *For a stably embeddable family $\phi(t)$ $(t \in (T,0))$ of deformations of a CR structure, the infinitesimal deformation map is the linear map*

$$\rho : T_0T \to \operatorname{Ker}\{H_b^1(T') \to H_b^1(T'\mathbf{C}^N{}_{|M})\}$$

given by $\rho(v) :=$ the cohomology class of $v(\phi(t))$. A holomorphic family is called effective if its infinitesimal deformation map is injective.

An effective and Kuranishi versal family is called a Kuranishi semi-universal family.

§6. Construction of the Kuranishi semi-universal family of stably embeddable deformations of CR structures

In this section, we will use the same notation as in §5. In [M6], we introduced a double-complex $K_{\bar\Omega}^{\bullet,\bullet}$:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & H^0(\bar\Omega, T'B_{|U}) & \xrightarrow{H} & H^0(\bar\Omega, \oplus^{m_1} 1_U) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow i & & \downarrow i & & \\
0 & \longrightarrow & A^0_{\bar\Omega}(T'U) & \xrightarrow{F} & A^0_{\bar\Omega}(T'B_{|U}) & \xrightarrow{H} & A^0_{\bar\Omega}(\oplus^{m_1} 1) & \longrightarrow & 0 \\
 & & \downarrow \bar\partial & & \downarrow \bar\partial & & \downarrow \bar\partial & & \\
0 & \longrightarrow & A^{0,1}_{\bar\Omega}(T'U) & \xrightarrow{F} & A^{0,1}_{\bar\Omega}(T'B_{|U}) & \xrightarrow{H} & A^{0,1}_{\bar\Omega}(\oplus^{m_1} 1) & \longrightarrow & 0 \\
 & & \downarrow \bar\partial & & \downarrow \bar\partial & & \downarrow \bar\partial & & \\
0 & \longrightarrow & A^{0,2}_{\bar\Omega}(T'U) & \xrightarrow{F} & A^{0,2}_{\bar\Omega}(T'B_{|U}) & \xrightarrow{H} & A^{0,2}_{\bar\Omega}(\oplus^{m_1} 1) & \longrightarrow & 0 \\
 & & \downarrow \bar\partial & & \downarrow \bar\partial & & \downarrow \bar\partial & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

where $K_{\bar\Omega}^{0,0} = A^0_{\bar\Omega}(T'U)$ and we denote $H^0(\bar\Omega, E) := \{u \in A^0_{\bar\Omega}(E) \mid \bar\partial u = 0\}$ for a holomorphic vector bundle E over U, i denotes the inclusion map and F (resp. H) is the differential of the natural embedding $\iota : U \to B$ (resp. the homomorphism given by $H(v) = (v(\tilde h_1), \ldots, v(\tilde h_{m_1}))$ for $v \in T'B_{|U}$).

Proposition 6.1 ([My7]).

$$\mathrm{Ext}^q(\Omega^1_V, \mathcal{O}_V) \simeq H^q(K_{\bar\Omega}^{\bullet,\bullet}) \ (q = 1, 2).$$

As the CR-version of $K_{\bar\Omega}^{\bullet,\bullet}$, we consider the following double complex $K_M^{\bullet,\bullet}$;

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & H^0_b(T'B_{|M}) & \xrightarrow{H} & H^0_b(\oplus^{m_1} 1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow i & & \downarrow i & & \\
0 & \longrightarrow & A^0_b(T'U_{|M}) & \xrightarrow{F} & A^0_b(T'B_{|M}) & \xrightarrow{H} & A^0_b(\oplus^{m_1} 1) & \longrightarrow & 0 \\
 & & \downarrow \bar\partial_b & & \downarrow \bar\partial_b & & \downarrow \bar\partial_b & & \\
0 & \longrightarrow & A^{0,1}_b(T'U_{|M}) & \xrightarrow{F} & A^{0,1}_b(T'B_{|M}) & \xrightarrow{H} & A^{0,1}_b(\oplus^{m_1} 1) & \longrightarrow & 0 \\
 & & \downarrow \bar\partial_b & & \downarrow \bar\partial_b & & \downarrow \bar\partial_b & & \\
0 & \longrightarrow & A^{0,2}_b(T'U_{|M}) & \xrightarrow{F} & A^{0,2}_b(T'B_{|M}) & \xrightarrow{H} & A^{0,2}_b(\oplus^{m_1} 1) & \longrightarrow & 0 \\
 & & \downarrow \bar\partial_b & & \downarrow \bar\partial_b & & \downarrow \bar\partial_b & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

where $K_M^{0,0} = A_b^0(T'U_{|M})$, $H_b^0(E)$ denotes the space of all CR-sections of a holomorphic vector bundle E on M and we denote by the same symbol F the composite of the projection $\rho^{1,0} : T' \to T'U_{|M}$ and $F : T'U_{|M} \to T'B_{|M}$, and i and H are the same as above.

The analytic restrictions $\tau : A_{\bar{\Omega}}^{0,q}(T'U) \to A_b^{0,q}(T'U_{|M})$ and $\tau : A_{\bar{\Omega}}^{0,q} \to A_b^{0,q}$ induce a homomorphism of double complexes

$$\tau : K_{\bar{\Omega}}^{\bullet,\bullet} \to K_M^{\bullet,\bullet}.$$

Proposition 6.2. *τ induces an isomorphism*

$$H^q(K_{\bar{\Omega}}^{\bullet,\bullet}) \simeq H^q(K_M^{\bullet,\bullet})$$

for $q = 1, 2$.

For the proof, we use the following lemma.

Lemma 6.3. *Let M be a real hypersurface of a complex manifold U and $\gamma : E_2 \to E_3$ a surjective homomorphism of C^∞-vector bundles over U. We suppose that there exists a splitting $j : E_3 \to E_2$. Then, for any $u_3 \in \Gamma(U, E_3)$ and $\mathbf{u}_2 \in \Gamma(M, E_{2|M})$ with $\gamma(\mathbf{u}_2) = u_{3|M}$, there exists $u_2 \in \Gamma(U, E_2)$ such that $\gamma(u_2) = u_3$ and $u_{2|M} = \mathbf{u}_2$.*

Proof. We may assume that $\Omega = M \times (-\epsilon, \epsilon)$. Let $u_2(x,t) := j(u_3(x,t)) - j(u_3(x,0)) + \mathbf{u}_2(x)$. Then $u_2(x,0) := \mathbf{u}_2(x)$ and $\gamma(u_2(x,t)) = u_3(x,t) - u_3(x,0) + \gamma(\mathbf{u}_2(x)) = u_3(x,t)$. Q.E.D.

Proof of Proposition 6.2. The case of $q = 2$: First, we prove the surjectivity. Let

$$(\varphi_2, \mathbf{g}_1, \mathbf{h}_0) \in A_b^{0,2}(T'U_{|M}) \oplus A_b^{0,1}(T'B_{|M}) \oplus A_b^0(\oplus^{m_1} 1_M)$$

satisfies $d(\varphi_2, \mathbf{g}_1, \mathbf{h}_0) = (0,0,0)$. We will find

$$(\bar{\phi}_2, \bar{g}_1, \bar{h}_0) \in A_{\bar{\Omega}}^{0,2}(T'U) \oplus A_{\bar{\Omega}}^{0,1}(T'B_{|U}) \oplus A_{\bar{\Omega}}^0(\oplus^{m_1} 1_U)$$

satisfying $d(\bar{\phi}_2, \bar{g}_1, \bar{h}_0) = (0,0,0)$ and $\tau(\bar{\phi}_2, \bar{g}_1, \bar{h}_0) = (\varphi_2, \mathbf{g}_1, \mathbf{h}_0)$. Applying [Ko-Ro, Theorem 7.5] to $\mathbf{k}_0 := \gamma \mathbf{h}_0 \in H_b^0(E_{|M})$, there exists $\bar{k}_0 \in H^0(\bar{\Omega}, E)$ such that $\bar{k}_0 = \mathbf{k}_0$ where we denote by E the quotient bundle $\oplus^{m_1} 1_U / N_{U/B}$. By Lemma 6.3, there exists $h_0 \in A_{\bar{\Omega}}^0(\oplus^{m_1} 1_U)$ such that $\gamma(h_0) = k_0$ and $h_{0|M} = \mathbf{h}_0$. Since $\gamma(\bar{\partial} h_0) = \bar{\partial} k_0 = 0$, by Lemma 6.3, there exists $g_1 \in A_{\bar{\Omega}}^{0,1}(T'B_{|U})$ such that $\alpha(g_1) = \beta^{-1}(\bar{\partial} h_0)$ and $\tau g_1 = \mathbf{g}_1$. Since $\beta \circ \alpha(\bar{\partial} g_1) = 0$, by Lemma 6.3, there exists

$\phi_2 \in A^{0,2}_{\bar\Omega}(T'U)$ such that $F\phi_2 = \bar\partial g_1$ and $\tau\phi_2 = \varphi_2$. Next, we prove the injectivity. Let

$$(\bar\phi_2, \bar g_1, \bar h_0) \in A^{0,2}_{\bar\Omega}(T'U) \oplus A^{0,1}_{\bar\Omega}(T'B_{|U}) \oplus A^{0}_{\bar\Omega}(\oplus^{m_1} 1_U)$$

and suppose $\tau(\bar\phi_2, \bar g_1, \bar h_0) = d(\varphi_1, \mathbf{g}_0, \mathbf{h}_{-1})$ where

$$(\varphi_1, \mathbf{g}_0, \mathbf{h}_{-1}) \in A^{0,1}_b(T'U_{|M}) \oplus A^0_b(T'B_{|M}) \oplus H^0_b(\oplus^{m_1} 1_M).$$

By the Lewy extension theorem, we have $\bar h_{-1} \in H^0(\bar\Omega, \oplus^{m_1}\mathcal{O}_U)$ such that $\bar h_{-1|M} = \mathbf{h}_{-1}$. Since $\bar\partial\gamma(h_0 - h_{-1}) = \gamma\bar\partial h_0 = \gamma\beta\alpha(g_1) = 0$ and $\gamma(h_0 - h_{-1})_{|M} = \gamma\beta\alpha(\mathbf{g}_0) = 0$, $\gamma(h_0 - h_{-1}) = 0$. Hence, by Lemma 6.3, there exists $g_0 \in A^0_{\bar\Omega}(T'B_{|U})$ such that $\alpha(g_0) = h_0 - h_{-1}$ and $g_{0|M} = \mathbf{g}_0$. Since $\beta\alpha(g_1 - \bar\partial g_0) = \bar\partial h_0 - \bar\partial(h_0 - h_{-1}) = 0$, $g_1 - \bar\partial g_0 \in A^{0,1}_{\bar\Omega}(T'U)$. Hence, if we set $F\phi_1 := -g_1 + \bar\partial g_0$, then $\bar\partial\phi_1 = \phi_2$ because $F\bar\partial\phi_1 = -\bar\partial g_1 = F\phi_2$.

The case of $q = 1$: First, we prove the surjectivity. Let

$$(\varphi_1, \mathbf{g}_0, \mathbf{h}_{-1}) \in A^{0,1}_b(T'U_{|M}) \oplus A^0_b(T'B_{|M}) \oplus H^0_b(\oplus^{m_1} 1_M)$$

satisfies $d(\varphi_1, \mathbf{g}_0, \mathbf{h}_{-1}) = (0,0,0)$. We will find

$$(\bar\phi_1, \bar g_0, \bar h_{-1}) \in A^{0,1}_{\bar\Omega}(T'U) \oplus A^0_{\bar\Omega}(T'B) \oplus H^0(\bar\Omega, \oplus^{m_1}\mathcal{O}_U)$$

satisfying $d(\bar\phi_1, \bar g_0, \bar h_{-1}) = (0,0,0)$ and $\tau(\bar\phi_1, \bar g_0, \bar h_{-1}) = (\varphi_1, \mathbf{g}_0, \mathbf{h}_{-1})$. By the Lewy extension theorem, there exists $\bar h_{-1} \in H^0(\bar\Omega, \oplus^{m_1}\mathcal{O}_U)$ such that $\bar h_{-1|M} = \mathbf{h}_{-1}$. If we set $\bar k_{-1} := \gamma(\bar h_{-1}) \in H^0(\bar\Omega, E)$, $\bar k_{-1} = 0$ because $k_{-1|M} = 0$. Hence $\bar h_{-1} \in H^0(\bar\Omega, N_{U/B})$ and by Lemma 6.3, there exists $\bar g_0 \in A^0_{\bar\Omega}(T'B)$ such that $\beta\alpha(\bar g_0) = \bar h_{-1}$ and $\bar g_{0|M} = \mathbf{g}_0$. Since $\beta\alpha(\bar\partial\bar g_0) = \bar\partial\bar h_{-1} = 0$, $\bar\partial\bar g_0 \in A^{0,1}_{\bar\Omega}(T'U)$. Hence, there exists $\bar\phi_1 \in A^{0,1}_{\bar\Omega}(T'U)$ such that $F\bar\phi_1 = \bar\partial\bar g_0$ and $\tau\bar\phi_1 = \varphi_1$. Next, we prove the injectivity. Let

$$(\bar\phi_1, \bar g_0, \bar h_{-1}) \in A^{0,1}_{\bar\Omega}(T'U) \oplus A^0_{\bar\Omega}(T'B) \oplus H^0(\bar\Omega, \oplus^{m_1}\mathcal{O}_U)$$

and suppose $\tau(\bar\phi_1, \bar g_0, \bar h_{-1}) = d(\varphi_0, \mathbf{g}_{-1})$ where $(\varphi_0, \mathbf{g}_{-1}) \in A^0_b(T'U_{|M}) \oplus H^0_b(T'B_{|M})$. We have $\bar g_{-1} \in H^0(\bar\Omega, T'B)$ such that $\bar g_{-1|M} = \mathbf{g}_{-1}$ by the Lewy extension theorem. Since $(\bar h_{-1} - H(\bar g_{-1}))_{|M} = 0$, $\bar h_{-1} - H(\bar g_{-1}) = 0$. Since $\beta\alpha(\bar g_0 - \bar g_{-1}) = \bar h_{-1} - H(\bar g_{-1}) = 0$, there exists $\bar\phi_0 \in A^0_{\bar\Omega}(T'U)$ such that $F\bar\phi_0 = \bar g_0 - \bar g_{-1}$ and $\bar\phi_{0|M} = \varphi_0$. Since $F\bar\phi_1 = \bar\partial\bar g_0 = F\bar\partial\bar\phi_0$, we have $\bar\phi_1 = \bar\partial\bar\phi_0$. Q.E.D

Let $(K^\bullet_M, d)$ be the total simple complex of the double complex $K^{\bullet,\bullet}_M$.

Proposition 6.4. *If* $\dim_{\mathbf{R}} M \geq 5$, *there exist operators* $Z : K_M^q \to \operatorname{Ker} d$ *and* $Q : \operatorname{Ker} d \to K_M^{q-1}$ $(q = 1, 2)$, *satisfying*

(1) $Z_{|\mathrm{Ker} d} = id_{\mathrm{Ker} d}$,

(2) $d \circ Q \circ d = d$.

Hence, if we set $\mathcal{H}_M^q := (1 - d \circ Q) \circ Z(K_M^q)$ and $\rho_{\mathcal{H}} := (1 - d \circ Q) \circ Z : K_M^q \to \mathcal{H}_M^q$, then we have

Corollary 6.5. *For* $q = 1, 2$,

(1) *The natural homomorphism* $\mathcal{H}_M^q \to H^q(K_M^{\bullet,\bullet})$ *is an isomorphism,*

(2) *a homotopy formula* $u = \rho_{\mathcal{H}} u + d \circ Q \circ Z u + (1 - Z) u$ *holds for* $u \in K_M^q$.

The existence of Z and Q is proved by a parallel argument of [My6, §4] with the $\bar{\partial}$-analysis on $\bar{\Omega}$ replaced by the $\bar{\partial}_b$-analysis on M, where we use the standard $\bar{\partial}_b$-Neumann Hodge decompositions at $A_b^{0,1}(T')$, $A_b^{0,1}(N_{U/B|M})$ and $A_b^{0,1}$; say $\eta = \rho\eta + \bar{\partial}_b \bar{\partial}_b^* N_b \eta + \bar{\partial}_b^* \bar{\partial}_b N_b \eta$, and the ones at $A_b^0(T' B_{|M})$ and A_b^0; say $\eta = \rho\eta + \bar{\partial}_b^* N_b \bar{\partial}_b \eta$. These Hodge decompositions are all possible if $\dim_{\mathbf{R}} M \geq 5$ (cf. [Fo-Ko]). At the same time, we have the following estimates. We denote by $|| \ ||_k$ and $|| \ ||_k'$ the Sobolev norm and the Folland-Stein norm respectively of order k.

Proposition 6.6.

(1) *For* $(a_1, b_0, c_{-1}) \in K_M^{0,1} \oplus K_M^{1,0} \oplus K_M^{2,-1}$, *let* $Z(a_1, b_0, c_{-1}) = (a_1', b_0', c_{-1}')$ *and* $Q(a_1', b_0', c_{-1}') = (a_0'', b_{-1}'')$. *Then*

$$||a_0''||_k' \leq C ||a_1'||_k \leq C' ||a_1||_k$$

holds.

(2) *For* $(a_2, b_1, c_0) \in K_M^{0,2} \oplus K_M^{1,1} \oplus K_M^{2,0}$, *let* $Z(a_2, b_1, c_0) = (a_2', b_1', c_0')$ *and* $Q(a_2', b_1', c_0') = (a_1'', b_0'', c_{-1}'')$. *Then*

$$||a_1''||_k + ||b_0''||_k' \leq C ||b_1'||_k \leq C' ||b_1||_k$$

holds.

Here C *and* C' *denote constants independent of* (a_1, b_0, c_{-1}) *nor* (a_2, b_1, c_0).

Furthermore, the same adjustments as [My6] are possible.

Let

$$
\begin{aligned}
{}^\circ K_M^{0,q} &:= \{a_q \in A_b^{0,q}({}^\circ T') \mid \bar\partial_b a_q \in A_b^{0,q+1}({}^\circ T')\},\\
{}^\circ K_M^{1,q} &:= \{b_q \in A_b^{0,q}({}^\circ \tilde T' B_{|\partial\Omega}) \mid \bar\partial_b b_q \in A_b^{0,q+1}({}^\circ \tilde T' B_{|\partial\Omega})\}
\end{aligned}
$$

where ${}^\circ\tilde T' B$ is the subbundle of $T' B$ given by

$$
{}^\circ\tilde T' B := \{v \in T' B \mid v(\sum_{\beta=1}^{N} |w^\beta|^2) = 0\}.
$$

Then by the parallel argument as in the latter part of [My6, §3], we can construct Z and Q so that the following proposition holds.

Proposition 6.7. *For any cohomology class in $H^1(K_M^\bullet)$ has a representative in ${}^\circ K_M^{0,1} \oplus {}^\circ K_M^{1,0} \oplus K_M^{2,-1}$.*

Proposition 6.8.

$$
\begin{aligned}
&(1)\quad Z(a_2, b_1, c_0) \in {}^\circ K_M^{0,2} \oplus {}^\circ K_M^{1,1} \oplus K_M^{2,0},\\
&\qquad\qquad\qquad\qquad \textit{if } (a_2, b_1, c_0) \in {}^\circ K_M^{0,2} \oplus {}^\circ K_M^{1,1} \oplus K_M^{2,0},\\
&(2)\quad Q(a_2, b_1, c_0) \in {}^\circ K_M^{0,1} \oplus {}^\circ K_M^{1,0} \oplus K_M^{2,-1},\\
&\qquad\qquad\qquad\qquad \textit{if } (a_2, b_1, c_0) \in {}^\circ K_M^{0,2} \oplus {}^\circ K_M^{1,1} \oplus K_M^{2,0}.
\end{aligned}
$$

Using Z and Q, we construct the Kuranishi semi-universal family of stably embeddable deformations of ${}^\circ T''$ by the argument in §2. Though a stably embeddable deformation of ${}^\circ T''$ is represented by $\phi \in K_M^{0,1}$ with which a $g \in K_M^{1,0}$ satisfying $(\bar\partial - \phi)(\iota + g) = 0$ is associated, we consider a triple $(\phi, g, k) \in K_M^{0,1} \oplus K_M^{1,0} \oplus K_M^{2,-1}$ satisfying

$$
\begin{aligned}
&P(\phi, g, k) :=\\
&\Big(\bar\partial_b\phi + R_2(\phi) + R_3(\phi), (\bar\partial - \phi)(\iota + g), (\tilde h + \tilde k) \circ (\iota + g)\Big) = (0,0,0)
\end{aligned}
$$

where $\tilde k$ denotes a holomorphic extension of k over $\bar B(c)$. Note that the holomorphic extension of k is possible in a unique way (cf. [Bl-Ep, Theorem A.1]), and that $\tilde k \circ (\iota + g)$ is considered as a Taylor series. We remark that the last term concerns the equation of the image $(\iota + g)(M)$; that is, $\tilde h + \tilde k$ is the defining equation of the subvariety which $(\iota + g)(M)$ bounds.

The construction of the Kuranishi semi-universal family using the complex $(K_M^\bullet, d)$ is parallel to the argument in §2. In fact, by the argument with $(A_b^{0,\bullet}(T'), \bar\partial_b)$, $\rho + \bar\partial_b\bar\partial_b^* N$ and $\bar\partial_b^* N$ replaced by $(K_M^\bullet, d)$, Z and Q respectively, we can prove the existence of

$$(\hat\phi(t), \hat g(t), \hat k(t)) \in (K_M^{0,1} \oplus K_M^{1,0} \oplus K_M^{2,-1})[[t_1, \dots, t_d]]$$

such that

(6.1) $(\hat\phi(0), \hat g(0), \hat k(0)) = (0, 0, 0)$,

(6.2) $(\hat\phi(t), \hat g(t), \hat k(t)) \equiv \sum_{\sigma=1}^d (\phi_\sigma, g_\sigma, k_\sigma) t_\sigma \bmod \mathfrak{m}^2$ where $\{[(\phi_\sigma, g_\sigma, k_\sigma)]\}_{\sigma=1}^d$ is a cohomology basis of $H^1(K_M^{\bullet,\bullet})$ and $\mathfrak{m}$ denotes the maximal ideal of $\mathbf{C}\{t_1, \dots, t_d\}$,

(6.3) there exists an extension $\tilde k(t) \in H^0(\bar B, \mathcal{O}_B)$ of $\hat k(t)$ such that

$$P(\hat\phi(t), \hat g(t), \tilde k(t)) :=$$
$$\left(\bar\partial\hat\phi(t) - \frac{1}{2}[\hat\phi(t), \hat\phi(t)], (\bar\partial - \hat\phi(t))(\iota + \hat g(t)), (\tilde h + \tilde k(t)) \circ (\iota + \hat g(t))\right)$$
$$\in \hat{\mathfrak{I}}(K_M^{0,2} \oplus K_M^{1,1} \oplus K_M^{2,0})[[t_1, \dots, t_d]],$$

where $\hat{\mathfrak{I}}$ is an ideal of $\mathbf{C}[[t_1, \dots, t_d]]$ generated by $\hat b_1(t), \dots, \hat b_\ell(t)$ and $\rho_{\mathcal{H}} P(\hat\phi(t), \hat g(t), \tilde k(t)) = \sum_{\beta=1}^\ell \hat b_\beta(t) e_\beta$ with respect to a basis $e_1, \dots, e_\ell$ of $\mathcal{H}^2$,

(6.4) it is formally Kuranishi versal, that is, there exists τ and F in Definition 5.2 as formal power series in s.

The proof of (6.4) needs to treat an extra term other than the argument in the proof of Proposition 2.2. Let $\pi : \mathcal{U} \to S$ together with an embedding $\Psi : \mathcal{U} \hookrightarrow \mathbf{C}^N \times S$ be a family of displacements (in $\mathbf{C}^N$) of a neighborhood of M in U. Suppose that Ψ is expressed by $w^\beta = \Psi_i^\beta(\zeta_i, s)$ $(\beta = 1, \dots, N)$ with respect to a local coordinate $(\zeta_i^1, \dots, \zeta_i^n, s_1, \dots, s_{d'})$ of $\mathcal{U}$ as in (N.3) and the coordinate $(w^1, \dots, w^N)$ of $\mathbf{C}^N$. By the argument parallel to the proof of Proposition 2.2, we can prove the existence of

$$\hat\tau(s) \in \mathbf{C}^d[[s_1, \dots, s_{d'}]]$$
$$\hat F_i^\alpha(s) \in \Gamma(U_i, T'U_{|M})[[s_1, \dots, s_{d'}]] \ (\alpha = 1, \dots, n)$$
$$\hat\eta^\beta(s) \in H^0(\bar B(c), \mathcal{O}_B)[[s_1, \dots, s_{d'}]] \ (\beta = 1, \dots, N)$$

satisfying

$$(1)\qquad \hat{\tau}(0)=0,\ \hat{F}_i^{\alpha}(0)=id_M,\ \hat{\eta}^{\beta}(0)=w^{\beta}$$

$$(2)\qquad \hat{F}_i^{\alpha}(s)-f_{ij}^{\alpha}(\hat{F}_j(s),s)\equiv 0 \bmod \mathfrak{I}_S$$

$$(3)\qquad \left(\bar{\partial}-\hat{\phi}(\hat{\tau}(s))\right)\hat{F}_i^{\alpha}(s)\equiv 0 \bmod \mathfrak{I}_S$$

$$(4)\qquad \hat{\eta}^{\beta}(\Psi_i(\hat{F}_i(s),s),s)-\hat{g}_i^{\beta}(\hat{\tau}(s))\equiv 0 \bmod \mathfrak{I}_S$$

$$(5)\qquad b_{\gamma}(\hat{\tau}(s))\equiv 0 \bmod \mathfrak{I}_S,$$

where $\zeta^{\alpha}=f_{ij}^{\alpha}(\zeta_j,s)$ $(\alpha=1,\dots,n)$ is the coordinate transformation and $\mathfrak{I}_S$ denotes the defining ideal of S in $\mathbf{C}\{s_1,\dots,s_{d'}\}$. Then (6.4) follows from the existence of the above $\{\hat{F}_i^{\alpha}(s)\}$ and $\hat{\tau}(s)$.

In order to assure the convergence of $\hat{\phi}(t)$, $\hat{g}(t)$, $\hat{b}_{\gamma}(t)$ $(\gamma=1,\dots,\ell)$, $\hat{\tau}(s)$ and $\{\hat{F}_i^{\alpha}(s)\}$, we need the adjustment of Z and Q as above. Indeed, by these adjustment and by starting the construction with the initial term

$$(\hat{\phi}_1(t),\hat{g}_1(t),\hat{k}_1(t))=\sum_{\sigma=1}^{d}(\phi_{\sigma},g_{\sigma},k_{\sigma})t_{\sigma}$$

such that

$$(\phi_{\sigma},g_{\sigma},k_{\sigma})\in {}^{\circ}K_M^{0,1}\oplus {}^{\circ}K_M^{1,0}\oplus K_M^{2,-1}\ (\sigma=1,\dots,d)$$

holds (it is possible by Proposition 6.7),we have $\hat{\phi}(t)\in {}^{\circ}K_M^{0,1}[[t_1,\dots,t_d]]$ which assures the convergence of $\hat{\phi}(t)$ and $\hat{g}(t)$ for the same reason as in §2, using the estimate of Z and Q (cf. Proposition 6.6). The convergence of $\hat{b}_{\gamma}(t)$ follows from the fact that $P_{|\mathcal{H}}:\mathcal{H}\to K_M^{0,2}\oplus K_M^{1,1}$ is injective where P denotes the projection operator of $K_M^{0,2}\oplus K_M^{1,1}\oplus K_M^{2,0}$ onto the first two factors. The convergence of $\hat{\tau}(s)$ and $\{\hat{F}_i^{\alpha}(s)\}$ is proved by the same calculation as in [My3].

Hence we have

Theorem 6.9. *Let V be a locally closed normal Stein subvariety in $\mathbf{C}^N$ and M a link of one singular point V. If $dim_{\mathbf{C}}V\geq 3$, then there exists a Kuranishi semi-universal family of stably embeddable deformations of CR structures on M.*

Let V and M be as in Theorem 6.9 and o the normal isolated singular point which M bounds. Let $f:\mathcal{V}\to S$ be the semi-universal family of flat deformations of the germ (V,o). We may assume that $\mathcal{V}\subset\mathbf{C}^N\times S$ is a subspace and $M\subset f^{-1}(s_o)$ $(s_o\in S)$.

Theorem 6.10. *Let V and M be as in Theorem 6.9 and $f : \mathcal{V} \to S$ as above. Let $\phi(t)$ $(t \in (T, 0))$ be the Kuranishi semi-universal family of stably embeddable deformations of CR structures on M (obtained in Theorem 6.9). Then $(T, 0) \simeq (S, s_o)$ and there exists a holomorphic family of embeddings of M into the family $\mathcal{V} \to S$ such that $\phi(t)$ $(t \in (T, 0))$ is induced from this family of embeddings.*

Outline of the proof. A formal family $\tilde{h}_1 + \tilde{k}_1(t), \ldots, \tilde{h}_{m_1} + \tilde{k}_{m_1}(t)$ of holomorphic functions on $\bar{B}(c)$ (obtained in Theorem 6.9) defines a formal family of subvarieties of $\bar{B}(c)$, say $\hat{\mathcal{V}} \subset \bar{B}(c) \times \hat{T}$, and we can prove that it is a flat family by the same argument as [Bl-Ep, Theorem 5.1]. Hence, we can compare $\phi(t)$ $(t \in (T, 0))$ with $f : \mathcal{V} \to S$ using their Kuranishi semi-universality and formally semi-universality respectively. Theorem 6.10 follows from this comparison taking account of Propositions 6.1 and 6.2.

In the case of $\dim_{\mathbf{C}} V = 2$, our notion of stably embeddable deformations of CR structures is nothing but the one of three dimensional embeddable CR structures in [Bl-Ep]. In fact, $H^1(K_M^\bullet)$ coincides with $\mathrm{Def}_1(M, \bar{Z}, X_0)$ (the space of first order embeddable deformations) in [Bl-Ep], where $\bar{Z}$ denotes the original CR structure on M and X_0 coincides with the embedding ι. However, the construction of the (convergent) semi-universal family of stably embeddable deformations of three dimensional CR structures on M is still open due to the difficulty of the analysis at K_M^2.

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