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A Method of Prolongation of Tangential Cauchy-Riemann Equations

Chong-Kyu Han and Jae-Nyun Yoo*

§0. Introduction

In this paper we present a method of prolongation of tangential Cauchy-Riemann equations. The technique is, roughly speaking, separating the holomorphic derivatives of CR functions from their complex conjugates and applying the tangential Cauchy-Riemann operators to the holomorphic part. Using this method we show that under generic assumptions mappings of a CR manifold into a CR manifold of higher dimension satisfy a certain Pfaffian system in the jet space, which implies the rigidity and the regularity of CR mappings.

Let M be a differentiable manifold of dimension $2m+1$. A CR structure on M is a subbundle $\mathcal{V}$ of the complexified tangent bundle $T_{\mathbb{C}}M$ having the following properties:

i) each fiber is of complex dimension m,

ii) $\mathcal{V} \cap \bar{\mathcal{V}} = \{ 0 \}$,

iii) $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$ (integrability).

It is well known that if $(M, \mathcal{V})$ is real analytic (C^ω) M is locally embeddable into $\mathbb{C}^{m+1}$ as a real hypersurface. In this paper we are concerned with CR mappings of M into a C^ω real hypersurface N of $\mathbb{C}^{n+1}$, $n \geq m$. Let N be a C^ω real hypersurface of nondegenerate Levi form in $\mathbb{C}^{n+1}$ defined by $r(z, \bar{z}) = 0$, where $z = (z_1, \cdots, z_{n+1})$. Let A and B be $(n+1)$-tuple of nonnegative integers and let $z^A = z_1^{a_1} \cdots z_{n+1}^{a_{n+1}}$ if $A = (a_1, \cdots, a_{n+1})$. After a holomorphic change of coordinates $r(z, \bar{z})$ can be written as

$$r(z, \bar{z}) = z_{n+1} + \bar{z}_{n+1} + \sum_{j=1}^{n} \lambda_j z_j \bar{z}_j + \sum_{A,B} c_{AB} z^A \bar{z}^B, \tag{0.1}$$

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where each λ_j is either 1 or -1 and each term in the last summand is of weight ≥ 3. Weight of a term $c_{AB}z^A\bar{z}^B$ is $\sum_{j=1}^n(a_j+b_j)+2(a_{n+1}+b_{n+1})$ as in [Ch-M]. Now let $\{L_1,\cdots,L_m\}$ be a local basis for $\mathcal{V}$. A mapping $f=(f_1,\cdots,f_{n+1}):M\to N$ is a CR mapping if and only if

$$\bar{L}_i f_j = 0, \quad i=1,\cdots,m, \quad j=1,\cdots,n+1 \tag{2.1}$$

(tangential Cauchy-Riemann equations).

For an m-tuple of non-negative integers $\alpha=(\alpha_1,\cdots,\alpha_m)$ let

$$L^\alpha = L_1^{\alpha_1}\cdots L_m^{\alpha_m}.$$

Our main result is

Theorem 2.1. *Let M^{2m+1} be a C^ω CR manifold of nondegenerate Levi form. Let $\{L_1,\cdots,L_m\}$ be C^ω independent sections of the CR structure bundle $\mathcal{V}$. Let N be a C^ω real hypersurface in $\mathbb{C}^{n+1}$, $n\geq m$, defined by $r(z,\bar{z})=0$, where $r(z,\bar{z})$ is normalized as in (0.1). Let $f:M\to N$ be a CR mapping. Suppose that for some positive integer k the vectors $\{L^\alpha f:|\alpha|\leq k\}$ together with $(0,\cdots,0,1)$ span $\mathbb{C}^{n+1}$. Then f satisfies a complete system of order $2k+1$. Thus, f is determined by $2k$-jet at a point and f is C^ω provided that $f\in C^{2k+1}$.*

If $m=n$, $f:M\to N$ is a CR equivalence, then $\{L_1f,\cdots,L_nf\}$ together with $(0,\cdots,0,1)$ span $\mathbb{C}^{n+1}$, thus $k=1$ and f satisfies a complete system of order 3. In the case $n>m$, the fact that f is dertermined by finite jet at a point is the local rigidity of CR mappings. If M is a real hypersurface in $\mathbb{C}^{m+1}$, a CR mapping f extends holomorphically to a neighborhood of M if and only if f is analytic, thus , if f satisfies the hypothesis of Theorem 2.1 then f extends holomorphically . One can also apply the argument of the Lewy-Pinchuk reflection principle [Forst] to (2.4) of §2, to get the holomorphic extension of f.

§1. Complete systems

Let f be a smooth (C^∞) mapping of an open subset X of $\mathbb{R}^n$ into an open subset U of $\mathbb{R}^m$. In this section we use superscripts for each components of vectors, thus $x=(x^1,\cdots,x^n)$ and $u=(u^1,\cdots,u^m)$ are the standard coordinates of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and $f(x)=(f^1(x),\cdots,f^m(x))$.

Let U_k be the space of all the different k-th order partial derivatives of the component of f at a point x. Set $U^{(q)} = U \times U_1 \times \cdots \times U_q$ be the Cartesian product space whose coordinates represent all the derivatives of a mapping $u = f(x)$ of all orders from 0 to q. A point in $U^{(q)}$ will be denoted by $u^{(q)}$.

The space $J^q(X, U) = X \times U^{(q)}$ is called the q-th order jet space of the space $X \times U$. If $f : X \to U$ is smooth, let $(j^q f)(x) = (x, f(x), \partial^\alpha(x) : |\alpha| \le q)$, then $j^q f$ is a smooth section of $J^q(X, U)$ called the q-graph of f.

Consider a system of partial differential equations of order q ($q \ge 1$) for unknown functions $u = (u^1, \cdots, u^m)$ of independent variables $x = (x^1, \cdots, x^n)$,

$$\Delta_\lambda(x, u^{(q)}) = 0, \qquad \lambda = 1, \cdots, l, \tag{1.1}$$

where $\Delta_\lambda(x, u^{(q)})$ are smooth functions in their arguments. Then $\Delta = (\Delta_1, \cdots, \Delta_l)$ is a smooth map from $X \times U^{(q)}$ into $\mathbb{R}^l$, so that the given system of partial differential equations describes the subset $\mathcal{S}_\Delta$ of zeros of Δ_λ in $X \times U^{(q)}$, called the solution subvariety of (1.1). Thus, a smooth solution of (1.1) is a smooth map $f : X \to U$ whose q-graph is contained in $\mathcal{S}_\Delta$.

A differential function $P(x, u^{(q)})$ of order q defined on $X \times U^{(q)}$ is a smooth function of x, u, and derivatives of u up to order q. The total derivatives of $P(x, u^{(q)})$ with respect to x^i is the unique smooth function defined by

$$D_i P(x, u^{(q+1)}) := \frac{\partial P}{\partial x^i} + \sum_{a=1}^{m} \sum_J u^a_{J,i} \frac{\partial P}{\partial u^a_J},$$

where $J = (j_1, \cdots, j_n)$ is a multi-index such that $|J| \le q$ and $J, i = (j_1, \cdots, j_i + 1, \cdots, j_n)$. For each nonnegative integer r, the rth-prolongation $\Delta^{(r)}$ of the system (1.1) is the system consisting of all the total derivatives of(1.1) of order up to r. Let $(\Delta^{(r)})$ be the ideal generated by $\Delta^{(r)}$ of the ring of differential functions on $X \times U^{(q+r)}$. If $\widetilde{\Delta} \in (\Delta^{(r)})$ for some r, the equation

$$\widetilde{\Delta}(x, u^{(q+r)}) = 0 \tag{1.2}$$

is called a prolongation of(1.1). Note that any smooth solution of (1.1) must satisfy (1.2). If k is the order of the highest derivative involved in $\widetilde{\Delta}$, we call (1.2) a prolongation of order k.

We now define the complete system.

Definition 1.1. *We say that a $C^k\,(k \geq q)$ solution f of* (1.1) *satisfies a complete system of order k if there exist prologations of* (1.1) *of order k*

$$\widetilde{\Delta}_\nu(x, u^{(k)}) = 0, \quad \nu = 1, \cdots, N \tag{1.3}$$

which can be solved for all the k-th order partial derivatives as smooth functions of lower order derivatives of f, namely, for each $a = 1, \cdots, m$ and for each multi-indix J with $|J| = k$,

$$f^a_J = H^a_J(x, f^{(p)} : p < k) \tag{1.4}$$

for some function H^a_J which is smooth in its arguments.

The idea of complete system is found in the so-called equivalence problem of E. Cartan: Let G be a Lie-subgroup of $GL(n;\mathbb{R})$ and $\pi : Y \to E$ be a principal fibre bundle with the structure group G over a manifold E of dimension n. The equivalence problem is finding canonically a system of differential 1-forms

$$\omega^1, \cdots, \omega^N, \quad \text{where} \quad N = n + \text{dim}G, \tag{1.5}$$

so that a mapping $f : E \to \tilde{E}$ preserves the G-structure if and only if there exists a mapping $F; Y \to \tilde{Y}$, which is a lift of f,that is, $\tilde{\pi}\circ F = f\circ\pi$, and such that

$$F^*\tilde{\omega}^i = \omega^i, \quad i = 1, \cdots N, \tag{1.6}$$

where $\tilde{\pi} : \tilde{Y} \to \tilde{E}$ is principal fibre bundle of the same structure group G and $\tilde{\omega}^i$ are the corresponding 1-forms on $\tilde{Y}$. (see [Burns,],[BS]). (1.5) is called a complete system of invariants of the G-structure and (1.6) is a complete system of order 1 for F in the sense of Definition 1.1. It turns out that (1.6) is equivalent to a complete system of order 2 for f (see [H3]).

Now we recall that solving the given system of partial differential equations (1.1) is equivalent to finding an integral manifold of the corresponding exterior differential system

$$du^a_I - \sum_{i=1}^{n} u^a_{I,i} dx^i = 0$$

for all multi-index I with $|I| < q$ and $a = 1, \cdots, m$, with an independence condition $dx_1 \wedge \cdots \wedge dx_n \neq 0$ on $\mathcal{S}_\Delta$(see [BCGGG]). If a solution of (1.1) satisfies a complete system of order k then we have the following Pfaffian system on $J^{k-1}(X,U)$:

$$
(1.7) \qquad \begin{cases} du^a - \sum_{j=1}^{n} u^a_j dx^j = 0, & \\ \vdots & \\ du^a_I - \sum_{j=1}^{n} u^a_{I,j} dx^j = 0, & |I| = k-2, \\ du^a_I - \sum_{i=1}^{n} H^a_{I,i} dx^i = 0, & |I| = k-1. \end{cases}
$$

with an independence condition $dx^1 \wedge \cdots \wedge dx^n \neq 0$, where $H^a_{I,i}$ are as in (1.4). Thus , a solution $u = f(x)$ of (1.1) of class C^k satisfies a complete system of order k if and only if

$$(x) \mapsto (x, f(x), \partial_J f(x) : |J| \leq k-1)$$

is an integral manifold of the Pfaffian system (1.7). In particular, we have

Proposition 1.2. *Let f be a solution of* (1.1) *of class C^k. Suppose that f satisfies a complete system* (1.4)*, then f is determined by* $(k-1)$ *jet at a point and f is C^∞. Furthermore, if* (1.1) *is real analytic and each H^a_J is real analytic then f is real analytic.*

§2. Prolongation of CR mappings

In this section we shall prove the following

Theorem 2.1. *Let M^{2m+1} be a C^ω CR manifold of nondegenerate Levi form. Let $\{L_1, \cdots, L_m\}$ be C^ω independent sections of the CR structure bundle $\mathcal{V}$. Let N be a C^ω real hypersurface in $\mathbb{C}^{n+1}$, $n \geq m$, defined by $r(z, \bar{z}) = 0$, where $r(z, \bar{z})$ is normalized as in* (0.1)*. Let $f : M \to N$ be a CR mapping. Suppose that for some positive integer k the vectors $\{L^\alpha f : |\alpha| \leq k\}$ together with $(0, \cdots, 0, 1)$ span $\mathbb{C}^{n+1}$. Then f satisfies a complete system of order $2k+1$. Thus, f is determined by $2k$-jet at a point and f is C^ω provided that $f \in C^{2k+1}$.*

Proof. $f = (f_1, \cdots, f_{n+1})$ is a CR mapping of M into N if and only if

$$\begin{aligned} &\bar{L}_i f_j = 0, \quad \text{for each} \quad i = 1, \cdots, m \quad \text{and} \quad j = 1, \cdots, n+1 \\ &\text{(tangential Cauchy-Riemann equations)} \end{aligned} \tag{2.1}$$

and $r \circ f = 0$, that is,

$$f_{n+1} + \bar{f}_{n+1} + \sum_{j=1}^{n} \lambda_j f_j \bar{f}_j + \sum_{A,B} c_{AB} f^A \bar{f}^B = 0, \tag{2.2}$$

where $f^A := f_1^{a_1} \cdots f_{n+1}^{a_{n+1}}$ and the terms in the last summand are of weight ≥ 3. Let $\alpha = (\alpha^1, \cdots, \alpha^m)$ be a m-tuple of non-negative integers. Apply $\bar{L}^\alpha$ to (2.2), then by (2.1) we have

$$\bar{L}^\alpha \bar{f}_{n+1} + \sum_{j=1}^{n} \lambda_j f_j \bar{L}^\alpha \bar{f}_j + \sum c_{AB} f^A (\bar{L}^\alpha \bar{f}^B) = 0. \tag{2.3}$$

Since the set of vectors $\{\bar{L}^\alpha \bar{f} : |\alpha| \leq k\}$ and $(0, \cdots, 0, 1)$ contains $(n+1)$ linearly independent vectors, we can solve (2.2) and (2.3) for $(f_1, \cdots, f_{n+1})$ in terms of $\bar{L}^\alpha \bar{f}$, $|\alpha| \leq k$, to get

$$f_j = H_j(\bar{L}^\alpha \bar{f} : |\alpha| \leq k), \quad \text{for each} \quad j = 1, \cdots, n+1, \tag{2.4}$$

where each H_j is an analytic function of the arguments in the parenthesis.

Let $\beta = (\beta_1, \cdots, \beta_m)$ be any multi-index. Apply L^β to (2.4). Then we have

$$L^\beta f_j = L^\beta H_j(\bar{L}^\alpha \bar{f} : |\alpha| \leq k). \tag{2.5}$$

Now let T be a C^ω real vector field on M which is transversal to the $\mathcal{V} \oplus \bar{\mathcal{V}}$, so that the set $\{T, L_j, \bar{L}_j, j = 1, \cdots, m\}$ forms a basis of the complexified tangent space of M. Let $[L_j, \bar{L}_k] = \sqrt{-1}\rho_{j\bar{k}} T \mod (\mathcal{V}, \bar{\mathcal{V}})$. Then $(\rho_{j\bar{k}})$, $j, k = 1, \cdots, m$ is a non-degenerate hermitian matrix. We may assume that $[\rho_{j\bar{k}}(0)]$ is diagonal at the refence point $0 \in M$. In the right hand side of (2.5), each time we apply L_i to $H_j(\bar{L}^\alpha \bar{f} : |\alpha| \leq k)$, computations by chain rule show that T-directional derivatives occurs when commuting L and $\bar{L}$, and by (2.1) the total order of the derivatives remains $\leq k$, for example,

$$
\begin{aligned}
\bar{L}_1 L_1 f_j &= (L_1 \bar{L}_1 - [L_1, \bar{L}_1]) f_j \\
&= \{L_1 \bar{L}_1 - (\sqrt{-1}\rho_{1\bar{1}} T + \sum_{i=1}^{m} (a_i L_i + b_i \bar{L}_i))\} f_j \\
&\quad \text{for some functions } a_i \text{ and } b_i \\
&= -\sqrt{-1}\rho_{1\bar{1}} T f_j + \sum_{i=1}^{m} a_i L_i f_j \quad \text{by (2.1).}
\end{aligned} \tag{2.6}
$$

Now we introduce notations : for each pair of non-negative integers (p, q) with $p \geq q$, let C_p be the set of C^ω functions in the arguments

$$T^t L^\alpha f_j : t + |\alpha| \leq p, \quad j = 1, \cdots, n+1$$

and $C_{p,q}$ be the subset of C_p of C^ω functions in the arguments

$$T^t L^\alpha f_j : t + |\alpha| \leq p, \quad t \leq q, \quad j = 1, \cdots, n+1,$$

and let $\bar{C}_p, \bar{C}_{p,q}$ be the complex conjugate of C_p and $C_{p,q}$, respectively. Then (2.5) implies that $L^\beta f_j \in \bar{C}_k$, for any multi-index $\beta = (\beta_1, \cdots, \beta_m)$.

In particular, for each $i = 1, \cdots, m$

$$L_i f_j \in \bar{C}_k. \tag{2.7}$$

Apply $\bar{L}_i$ to (2.7), then by the same calculation as in (2.6) we have

$$T f_j \in \bar{C}_{k+1,k}. \tag{2.8}$$

Similarly, for each $i, k = 1, \cdots, m$, we have

$$L_k L_i f_j \in \bar{C}_k. \tag{2.9}$$

Apply $\bar{L}_k$ to (2.9), then by (2.7), (2.8) and (2.9) we have

$$T L_i f_j \in \bar{C}_{k+1,k}. \tag{2.10}$$

Then by induction on $|\alpha|$, we have

$$T L^\alpha f_j \in \bar{C}_{k+1,k}. \tag{2.11}$$

Now apply $\bar{L}_i \bar{L}_k$ to (2.9), then by (2.7) – (2.11) we have

$$T^2 f_j \in \bar{C}_{k+2,k}, \tag{2.12}$$

and by induction on $|\alpha|$, we have

$$T^2 L^\alpha f_j \in \bar{C}_{k+2,k}. \tag{2.13}$$

Then by induction on t, we have

$$T^t L^\alpha f_j \in \bar{C}_{k+t,k}, \quad \text{for each} \quad j = 1, \cdots, n+1, \tag{2.14}$$

which shows that

$$C_{p,q} \subset \bar{C}_{k+q,k}, \quad \text{for any pair } (p,q) \text{ with } p \geq q. \tag{2.15}$$

Taking the complex conjugate of (2.15), we have

$$\bar{C}_{p,q} \subset C_{k+q,k}, \quad \text{for any pair } (p,q) \text{ with } p \geq q. \tag{2.16}$$

In paticular, if $q = k$

$$\bar{C}_{p,k} \subset C_{2k,k} \quad \text{for all} \quad p \geq k. \tag{2.17}$$

Substitute (2.17) in (2.15), to get

$$C_{p,q} \subset C_{2k,k}, \quad \text{for any pair } (p,q) \text{ with } p \geq q. \tag{2.18}$$

In particular, we have

$$C_{2k+1} \subset C_{2k}. \tag{2.19}$$

Now consider the derivatives $T^t L^\alpha \bar{L}^\beta f_j$, where $t + |\alpha| + |\beta| = 2k+1$. If $|\beta| \neq 0$, this is zero by (2.1). If $|\beta| = 0$, then (2.19) shows that $T^t L^\alpha f_j, t + |\alpha| = 2k+1$, can be expressed as a C^ω function in the arguments $T^t L^\beta f_j : t + |\beta| \leq 2k$, thus, f satisfies a complete system of order $2k+1$, which completes the proof.

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Chong-Kyu Han
Department of Mathematics,
Seoul National University,
Seoul, 151-742,
Korea.
ckhan@math.snu.ac.kr

Jae-Nyun Yoo
Department of Mathematics
Pohang University of Science and Technology
Pohang, 790-600,
Korea.
jnyoo@posmath.postech.ac.kr