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On Deformations of Self-Dual Vector Bundles over Quaternionic Manifolds

James F. Glazebrook and Duraiswamy Sundararaman [1]

Abstract.

In this paper we survey a number of results concerned with the deformations of quaternionic structures on classes of quaternionic manifolds and the deformation theory of Hermitian bundles with self-dual connections. The deformations in question are shown to correspond to the deformation theory of complex structures and holomorphic vector bundles over an associated complex manifold referred to as a twistor space. Results related to hypercomplex and hyperkähler manifolds are also discussed.

§0. Introduction.

In this paper we consider classes of quaternionic manifolds M and deformations of Hermitian vector bundles on M that possess self-dual connections in the sense of **[18]**, and survey a number of results in this area. We also describe how the Kuranishi deformation theory for general G-structures is applied in this context. The deformation theory is applied firstly in the quaternionic category and then secondly in the holomorphic category of associated (almost) complex manifolds which are often referred to as *twistor spaces.*

The original problems can be traced back to the classification of $SU(2)$–bundles with self–dual connection on 4–manifolds (see e.g., **[1]** **[7]**). A broad generalization of the latter case to that of quaternionic Kähler manifolds was the subject of **[18]** (see also **[10]**, **[21]**). The classification problem becomes much more difficult and results are only known at present for special quaternionic bundles which carry a self–dual connection (see §2).

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Our references to progress on this problem, are **[5]**, **[6]**, **[15]**, **[10]**, **[18]**, **[21]**, **[22]**, **[23]**. In a different direction, the case of foliated quaternionic structures has been considered in **[11]**, **[12]**.

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§1. G–structures and their deformations.

Following **[13]**, let M be a smooth manifold, $\dim_{\mathbb{R}} M = n$, and let $\mathbf{G} \subset GL(n,\mathbb{R})$ be a connected Lie group. A **G**–*structure* on M is given by a reduction of the principal tangent frame bundle of TM from $GL(n,\mathbb{R})$ to $\mathbf{G}$. If $GL(n,\mathbb{R}) \to B \to M$ denotes the principal tangent frame bundle, then let $\mathbf{G} \to B_{\mathbf{G}} \to M$ be the principal bundle resulting from this reduction. Consider a local diffeomorphism $f : M \to M$ lifting to a bundle automorphism $f_* : B \to B$. We say that f is a local **G**–automorphism if $f_*(B_{\mathbf{G}}) \subset B_{\mathbf{G}}$. For an open set U in M, let $X \in C^\infty_U(TM)$ be a vector field which generates a local 1-parameter group $f(t) = \exp(tX)$ of local diffeomorphisms. Let $U \subset \mathbb{R}^\nu$ be an open neighbourhood of O in $\mathbb{R}^\nu$ with parameter $t = (t_1, \ldots, t_\nu)$ and let $\mathcal{W} \xrightarrow{\omega} U$ be a smooth fibre bundle with fibre M. The structure group of $T\mathcal{W}$ may be defined as follows: consider the group of all matrices $\begin{bmatrix} a & c \\ 0 & b \end{bmatrix}$ where $a \in GL(n,\mathbb{R})$, $b \in GL(\nu,\mathbb{R})$ and $c \in \mathrm{Hom}(\mathbb{R}^\nu,\mathbb{R}^n)$. Let $\mathbf{G}^*$ be the group of all matrices of this type where $a \in \mathbf{G}$ (one can take $\mathbf{G}' \subset \mathbf{G}^*$ to consist of the subgroups where $c = 0$). For a given $\mathbf{G}^*$–structure on $\mathcal{W}$ there exists on the fibre $M_t = \omega^{-1}(t)$ an induced **G**–structure

$$G \longrightarrow B_{\mathbf{G}}(t) \longrightarrow M_t$$

For an open set $\mathcal{U} \subset M$, there is a natural **G**–structure on $\mathcal{W} \times \mathcal{U}$ induced from that on $\mathcal{W}$. If $\mathcal{W}$ possesses a $\mathbf{G}^*$–structure, then $\omega : \mathcal{W} \to \mathcal{U}$ is a deformation of the **G**–structure on M if:

i) there exists a **G**–diffeomorphism between M and $M_0 = \omega^{-1}(0)$, and
ii) the bundle $\mathcal{W} \xrightarrow{\omega} \mathcal{U}$ is locally trivial.

§2. Quaternionic manifolds and their twistor spaces.

Let us now take $n = 4m \quad (m > 1)$. We shall say that M is *almost quaternionic* if the principal tangent frame bundle $B_{\mathbf{G}}$ of M is equipped with a

$$GL(m,\mathbb{H})\,GL(1,\mathbb{H}) := GL(m,\mathbb{H}) \times_{\mathbb{R}} GL(1,\mathbb{H})$$

connection (see **[25]**, **[26]**). We shall denote this $\mathbf{G}$-structure on M by $\mathbf{G}_M$. Observe that $\mathbf{G}_M$ is the same group as $GL(m,\mathbb{H})\,Sp(1) := GL(m,\mathbb{H}) \times_{\mathbb{Z}_2} Sp(1)$. Equivalently, there is a distinguished rank 3 subbundle $\mathbb{G} \subset \mathrm{End}(TM)$ having a local basis $\{I, J, K\}$ satisfying the usual quaternion identities

$$I^2 = J^2 = K^2 = -1, \quad IJ = K = -JI, \ etc.$$

Let E and H respectively denote the vector bundles associated to the fundamental representations of $GL(m,\mathbb{H})$ and $Sp(1)$ respectively on $\mathbb{C}^{2m}$ and $\mathbb{C}^2$. Then taking the following tensor product over $\mathbb{C}$, we have

$$T^*_{\mathbb{C}}M \cong E \otimes H$$

and

$$\Lambda^2 T^*_{\mathbb{C}}M \cong S^2E + \wedge^2 E \otimes S^2 H$$

If $g_M(IX, IY) = g_M(X,Y)$ for any local section I of $\mathbb{G}$ satisfying $I^2 = -1$, then M is said to be *quaternionic Hermitian* whereby $\mathbf{G}_M$ reduces to the group $Sp(m)\,Sp(1)$ and the above decomposition of 2-forms is refined to

$$\Lambda^2 T^*_{\mathbb{C}}M \cong S^2H + S^2E + \Lambda^2_0 E \otimes S^2 H$$

where $\Lambda^2 E \cong \mathbb{R} + \Lambda^2_0 E$ is the decomposition into irreducible $Sp(m)$-modules. The *fundamental 4-form* of M is a global 4-form Ω defined locally by

$$\Omega = \omega_I \wedge \omega_I + \omega_J \wedge \omega_J + \omega_K \wedge \omega_K$$

for I, J, K a local basis as before and $\omega_I(X,Y) = g_M(X, IY)$ is the local 2-form associated to I, etc. A 2-form F on M is said to be *self–dual* if

$$*F = c_i F \wedge \Omega^{m-1}$$

where $*$ denotes the Hodge star operator on M and for $1 \leq i \leq 3$, the c_i are constants corresponding to the eigenspaces of $\Lambda^2 T^*_{\mathbb{C}}M$ (see **[9]**). We shall be interested in complex vector bundles $V \to M$ for which the connection ∇ on V has a curvature 2-form R_∇ which is $c_2 = c_{S^2E}$-self-dual; specifically, $R_\nabla \in \Omega^2(M, \mathrm{End}\, V \otimes S^2E)$. It will be convenient for us to call such a connection simply *self-dual*, generalizing the situation in dimension 4 (see e.g., **[1]**, **[7]**).

The twistor space Z associated to M is the 2-sphere bundle

$$Z = S^2(\mathbb{G}) = \{I \in \mathbb{G} : \|I\| = 1\}$$

giving rise to the twistor fibration $\pi : Z \to M$ with fibre S^2. Since M is almost quaternionic, it can be shown that Z is an almost complex manifold with almost complex structure $\mathcal{I}$ for each choice of a $\mathbf{G}_M$-connection $\nabla_{\mathbf{G}}$ on M ([**25**]). This connection is determined by a choice of a horizontal distribution on the principal frame bundle $B_{\mathbf{G}}$. Then Z can be regarded as the bundle associated to the structure $\mathbf{G}_M$ via the adjoint action of the $GL(1,\mathbb{H})$ factor on S^2. Equivalently, $Z = \mathbb{P}_{\mathbb{C}}(H)$. Note that there is a canonical $GL(2m+1,\mathbb{C})$–structure $\mathbf{G}_Z$ on Z whereby the horizontal distribution on $B_{\mathbf{G}}$ determines that on Z. The almost complex structure $\mathcal{I}$ at $I \in Z$ is taken to be I on horizontal tangent vectors and multiplication by ι on vertical tangent vectors with respect to π. Now $\mathcal{I}$ depends on the torsion of $\nabla_{\mathbf{G}_M}$ alone where the integrability of $\mathcal{I}$ is equivalent to $\nabla_{\mathbf{G}}$ being torsion-free. In the terminology of [**25**], M is (integrable) *quaternionic* if M admits a torsion–free $\nabla_{\mathbf{G}}$-connection. Thus the 'integrability' of M as a quaternionic manifold is equivalent to Z being a complex manifold (with $\mathcal{I}$ integrable). For a given parameter value t as in §1, we shall denote by $\mathcal{D}$ the space of torsion–free $\nabla_{\mathbf{G}}$–connections on M. Each $I \in Z_x$ determines a decomposition into complex types

$$\Lambda^r(T_x^*M)_{\mathbb{C}} = \mathcal{A}_I^{r,0} \oplus \mathcal{A}_I^{r-1,1} \oplus \cdots \oplus \mathcal{A}_I^{0,r}$$

with $\mathcal{A}_{-I}^{r,0} = \mathcal{A}_I^{0,r} = \overline{\mathcal{A}_I^{0,r}}$. In particular for $r = 2$,

$$\Lambda^2 T_{\mathbb{C}}^* M = \mathcal{A}_I^{2,0} + \mathcal{A}_I^{1,1} + \mathcal{A}_I^{0,2}$$

we have

$$S^2E = \bigcap_{I \in Z} \mathcal{A}_I^{1,1}$$

Recalling that at $I \in Z$, the complex structure $\mathcal{I}$ on Z is equivalent to I on horizontal vectors, it can be shown that if $V \longrightarrow M$ is a complex vector bundle with connection ∇ whose curvature R_∇ is a self-dual 2–form, then π^*V is a holomorphic vector bundle on Z [**18**]. Here, the complex structure $\tilde{\mathcal{I}}$ on π^*V is obtained by taking $\pi^*\nabla$ to give the local splitting

$$T(\pi^*V) = TZ \oplus \mathbb{C}^r \ \ (r = \operatorname{rank} V)$$

and then take $\tilde{\mathcal{I}} = (\mathcal{I}, \iota)$ where ι denotes the usual almost complex structure on $\mathbb{C}^r$. Let

$$A^r = \Lambda^r E \oplus S^r H \subset (\Lambda^r T^* M)_{\mathbb{C}}$$

be associated to the irreducible $\mathbf{G}_M$ ($= GL(m,\mathbb{H})\, Sp(1)$)–submodule of $\Lambda^r T^* M$ of highest weight in the $Sp(1)$ factor. It is sometimes convenient to write the decomposition of A^r in the following way: let B^r denote the subbundle of $\Lambda^r T^*M$ formed by the sum of $\mathbf{G}_M$–components distinct from A^r. For $2 \le r \le 2m$, we shall set

$$\Lambda^r(T^*M)_{\mathbb{C}} = A^r \oplus B^r$$

With regards to the above decomposition of 2–forms, we see that $B^2 = S^2E$ and thus if V has a self–dual connection ∇, its curvature R_∇ is B^2-valued. Likewise, if V has a c_1–self–dual connection implying that R_∇ is valued in S^2H, then we say that ∇ is *anti-self-dual.* We shall restrict our attention mainly to those connections which are *self–dual.* For $z \in \pi^{-1}(x)$, we have

$$(A^r)_x = \sum_x \mathcal{A}_x^{0,q}$$

where $A_x^0 \cong \mathbb{C}$, $A_x^1 = (T_x^*M)_{\mathbb{C}}$, $A_x^2 = (S^2H + \Lambda_0^2 E \otimes S^2H)_x$, etc.

Let $\eta^r : (\Lambda^r T^*M)_{\mathbb{C}} \to A^r$ be the projection and set $D = \eta \circ d$. Then if M is quaternionic, the complex

$$0 \longrightarrow A^0 \xrightarrow{D=d} A^1 \xrightarrow{D} A^2 \longrightarrow \cdots \longrightarrow A^{2m}$$

is an elliptic complex on M (i.e. $D^2 = 0$). There is a direct relationship with the Dolbeault complex

$$0 \longrightarrow \mathcal{A}^{0,0} \xrightarrow{\bar\partial} \mathcal{A}^{0,1} \xrightarrow{\bar\partial} \mathcal{A}^{0,2} \xrightarrow{\bar\partial} \cdots \longrightarrow \mathcal{A}^{0,2m+1} \longrightarrow 0$$

on Z. Specifically, we have a short exact sequence of vector bundles

$$0 \longrightarrow \mathcal{A}_{\text{hor}}^{0,1} \longrightarrow \mathcal{A}^{0,1} \longrightarrow \mathcal{A}_{\text{ver}}^{0,1} \longrightarrow 0$$

involving $(0,1)$–forms horizontal and vertical with respect to π. Taking exterior powers leads to

$$0 \longrightarrow \mathcal{A}_{\text{hor}}^{0,q} \xrightarrow{\alpha} \mathcal{A}^{0,q} \xrightarrow{\beta} \mathcal{A}_{\text{ver}}^{0,1} \otimes \mathcal{A}_{\text{hor}}^{0,q-1} \longrightarrow 0$$

and the restriction of $\mathcal{A}^{0,q}$ to each fibre $\pi^{-1}(x)$ has a holomorphic structure such that:

a) $\bar\partial_{\text{ver}} = \beta \circ \bar\partial \circ \alpha$ restricted to $\pi^{-1}(x)$ is the usual $\bar\partial$–operator with coefficients in $\mathcal{A}_{\text{hor}}^{0,q}$

b) α induces an isomorphism

$$\alpha^\# : (A^r)_x \xrightarrow{\cong} H^0(\pi^{-1}(x), \mathcal{O}(\mathcal{A}_{\text{hor}}^{0,r})) = \ker(\bar\partial_{\text{ver}} \mid \pi^{-1}(x))$$

c) if $\omega \in C^\infty(A^r)$, then $\alpha^\#(\omega) = (\pi^*\omega)^{0,r}$ and $\alpha^\# \circ D = \bar\partial \circ \alpha^\#$.

Theorem 2.1. [18] *Let $V \to M$ be a complex vector bundle with self–dual connection. On extending D to $A^r(V) = A^r \otimes V$, the complex*

$$0 \longrightarrow A^0(V) \xrightarrow{D} A^1(V) \xrightarrow{D} A^2(V) \cdots \longrightarrow A^{2m}(V) \longrightarrow 0$$

is elliptic and

$$H^r(Z, \mathcal{O}(\pi^* V)) = \begin{cases} \dfrac{\ker(D \mid A^r(V))}{D(A^{r-1}(V))} & \textit{for} \quad 0 \le r \le 2m \\ 0 \quad \textit{for} \quad r = 2m+1. \end{cases}$$

§3. Quaternionic deformations of vector bundles with self–dual connections.

Let M be a compact quaternionic manifold ($\dim_{\mathbb{R}} M = 4m$). Recall from §2 that we have the rank 3 subbundle $\mathbb{G} \subset \operatorname{End} TM$. Let G be a connected Lie subgroup of $GL(N, \mathbb{H})\, Sp(1)$, for some N. Of interest to us are the cases for which G is one of $GL(N, \mathbb{H})\, Sp(1)$, $GL(N, \mathbb{H})$, $Sp(N)$ or $Sp(N)\, Sp(1)$. Henceforth, we assume that G is one of these listed subgroups. Let us now consider a principal G–bundle $G \to P \to M$. In this case, the adjoint bundle of P, denoted $\operatorname{Ad} P = P \times_{\operatorname{Ad}} \mathfrak{g}$, contains a rank 3 subbundle $\mathbb{G}_P$ corresponding to the adjoint representation of $Sp(1)$ which descends to $\mathbb{G}$.

We shall consider those G–bundles on M whose associated vector bundle $V \to M$ has a self–dual connection ∇_V (recall with curvature $R_\nabla \in \Omega^2(M, S^2 E \otimes \operatorname{Ad} P)$). In the terminology of **[25]**, $(V, \nabla_V) \to M$ is said to be a *quaternionic vector bundle.*

Definition 3.1. *Let $V \to M$ be a quaternionic vector bundle associated to the principal G–bundle $G \to P \to M$. A quaternionic deformation of the bundle $G \to P \to G$ is specified by a deformation of the self–dual connection ∇_V within the space of G–connections together with a deformation M_t of the $\mathbf{G}_M$–structure of M through torsion–free $\mathbf{G}_M$ connections in $\mathcal{D}$.*

Note that since a self–dual connection on V induces the same on $\mathbb{G}_P$, we obtain a family of complexes $(A^*(\mathbb{G}_P), D_t)$ for each torsion–free connection $D_t \in \mathcal{D}$.

Recalling that the pull–back of V by π^* to Z gives a holomorphic vector bundle $(\tilde{V}, \tilde{J}) \to (Z, J)$ associated to a holomorphic principal bundle $\tilde{P}$ on Z. We can implement the simultaneous deformations of $(\tilde{V}, \tilde{J})$ and (Z, J) in the holomorphic category. This was studied in

[27] **[28]** for complex principal bundles and we shall outline one of the main results. We commence by considering the (complexified) Atiyah sequence

$$0 \to \mathrm{Ad}_{\mathbb{C}}\, \tilde{P} \to T_{\mathbb{C}}\tilde{P}/G_{\mathbb{C}} \to T_{\mathbb{C}}Z \to 0$$

At the level of sheaves, let us agree to write the corresponding exact sequence as

$$0 \to \mathrm{Ad}\, \tilde{P} \to \tilde{Q} \to TZ \to 0$$

Tensoring this sequence with $\mathcal{A}^{0,r}$ on Z, we set

$$T_1^r = \mathrm{Ad}\, \tilde{P} \otimes \mathcal{A}^{0,r},\ T_2^r = \tilde{Q} \otimes \mathcal{A}^{0,r},\ T_3^r = TZ \otimes \mathcal{A}^{0,r}$$

and summing each term over $r \geq 0$, we obtain the exact sequence

$$0 \longrightarrow T_1 \longrightarrow T_2 \xrightarrow{h} T_3 \longrightarrow 0$$

where the operators $\bar{\partial}$ and $[\ ,\ ]$ are extended to each T_i in the usual way.

Definition 3.2. *An almost complex principal bundle structure on $G_{\mathbb{C}} \to \tilde{P} \to Z$ is a pair $(\tilde{J}, J)$ where $\tilde{J}$ (respectively J) is an almost complex structure on $\tilde{P}$ (respectively Z), such that*

i) *$\tilde{J}$ is $G_{\mathbb{C}}$–invariant*
ii) *the almost complex structure on $\tilde{P}/G_{\mathbb{C}}$ induced by $\tilde{J}$ is J, and*
iii) *$\tilde{J}$ restricted to each fibre gives the integrable almost complex structure on $G_{\mathbb{C}}$.*

Proposition 3.3. **[27]**, **[28]** *There is a bijective correspondence between the almost complex structures $(\tilde{J}, J)$ on $G_{\mathbb{C}} \to \tilde{P} \to Z$ which are sufficiently close to given fixed almost complex structures $(\tilde{J}_0, J_0)$ and elements $\psi \in T_1^1$ close to 0 and satisfying $h(\psi) = \varphi$ where φ is taken relative to J. The integrability condition is*

$$\bar{\partial}\psi - \frac{1}{2}[\psi, \psi] = 0.$$

To see how Proposition 3.3 can be applied to this situation, we recall from **[11]**, **[12]** that a vector field X on M is said to be a *quaternionic vector field* if via its infinitesimal automorphisms it preserves the quaternionic structure $\mathbf{G}_M$ of M. For $m = 1$, $GL(1, \mathbb{H})\, GL(1, \mathbb{H})$ is the conformal group $CO(4)$ on a 4–manifold. The twistor correspondence yields the following (see e.g., **[11]**, **[24]**):

Proposition 3.4. *If X is a quaternionic vector field on M then X induces a holomorphic vector field Y on Z such that $\pi_*(Y) = X$. Conversely, a projectable holomorphic vector field on Z induces a quaternionic vector field on M.*

Lemma 3.5. *An infinitesimal $\mathbf{G}_M$–automorphism of M lifts via π to an infinitesimal $\mathbf{G}_Z$–holomorphic automorphism of Z and conversely on π–related vector fields.*

Following the discussion in §1, let $\mathcal{W} \xrightarrow{\omega} \mathcal{U}$ be a $\mathbf{G}_M$–deformation and $\tilde{\mathcal{W}} \xrightarrow{\tilde{\omega}} \tilde{\mathcal{U}}$ be a $\mathbf{G}_Z$–deformation.

Proposition 3.6. *A $\mathbf{G}_M$–deformation induces a $\mathbf{G}_Z$–deformation and conversely on π–related vector fields. In particular, the diagram below is commutative:*

$$\begin{array}{ccc} \tilde{\mathcal{W}} & \xrightarrow{\tilde{\omega}} & \tilde{\mathcal{U}} \\ {\scriptstyle w}\downarrow & & \downarrow{\scriptstyle u} \\ \mathcal{W} & \xrightarrow{\omega} & \mathcal{U} \end{array}$$

Denoting the $\tilde{V}$–valued forms on Z by $\mathcal{A}^*(\tilde{V})$ we let $\nabla''(\tilde{V})$ denote the set of $\mathbb{C}$–linear maps

$$\nabla'' : \mathcal{A}^0(\tilde{V}) \to \mathcal{A}^{0,1}(\tilde{V})$$

satisfying

$$\nabla''(fs) = (d''f)s + f \cdot \nabla'' s$$

for $s \in \mathcal{A}^0(\tilde{V})$, $f \in \mathcal{A}^0$. Each ∇'' extends uniquely to a $\mathbb{C}$–linear map

$$\nabla'' : \mathcal{A}^{p,q}(\tilde{V}) \to \mathcal{A}^{p,q+1}(\tilde{V}),\ p, q \geq 0$$

satisfying

$$\nabla''(\psi\nu) = d''\psi \wedge \nu + (-1)^{r+s}\psi \wedge \nabla''\nu$$

for $\nu \in \mathcal{A}^{p,q}(\tilde{V})$, $\psi \in \mathcal{A}^{r,s}$. The set $\nabla''(\tilde{V})$ is an affine space which can be identified with the infinite dimensional vector space $\mathcal{A}^{0,1}(\operatorname{End}\tilde{V}) \cong \mathcal{A}^{0,1}(\operatorname{Ad}\tilde{P})$. Let $\mathcal{H}''(\tilde{V}) \subset \nabla''(\tilde{V})$ be the set of those ∇'' satisfying the integrability condition $\nabla'' \circ \nabla'' = 0$. The set $\mathcal{H}''(\tilde{V})$ can be regarded as the set of holomorphic bundle structures on $\tilde{V}$. The group $GL(\tilde{V})$ of C^∞ bundle automorphisms of $\tilde{V}$ (inducing the identity transformation on Z) acts on $\nabla''(\tilde{V})$ and maps $\mathcal{H}''(\tilde{V})$ to itself. Two holomorphic structures ∇_1'' and ∇_2'' of $\tilde{V}$ are said to be equivalent if they lie on the same $GL(\tilde{V})$–orbit. The moduli space of holomorphic structures on $\tilde{V}$ is the quotient space $\mathcal{H}''(\tilde{V})/\,GL(\tilde{V})$.

For a given parameter s say, the family of such connections $\{\nabla_V\}_s$, as a deformation space, is $H^1(\mathbb{G}_P)$ **[18]** . Thus for given parameter values s and t, we may consider the space

$$\{\{(\nabla_V)_s\},\{D_t\}\} := \{\mathcal{Q}_{s,t}\} \cong H^1(\mathbb{G}_P) \times \mathcal{D}$$

Let $\mathcal{J}$ denote the space of almost complex structures on Z and let $\mathcal{R} = \mathcal{H}''(\tilde{V})/\,GL(\tilde{V}) \times \mathcal{J}$.

Proposition 3.7. *For ψ, φ as in Proposition 3.3 and $J \in \mathcal{J}$ integrable, the space $\{\mathcal{Q}_{s,t}\}$ injects into*

$$T_{(\nabla'',J)}\mathcal{R} = \{\psi \in T_1^1 : h(\psi) = \varphi,\ \bar{\partial}\psi - \frac{1}{2}[\psi,\psi] = 0\}.$$

§4. Complex vector bundles over quaternionic Kähler manifolds.

Henceforth, we assume that M is compact and further, assume that M is *quaternionic Kähler* which by definition means that the linear holonomy of M is contained in the subgroup $Sp(m)\,Sp(1) \subset SO(4m)$. Note then that E becomes the bundle associated to the fundamental representation of $Sp(m)$ on $\mathbb{C}^{2m}$. For a complex vector bundle $V \longrightarrow M$ with connection ∇, one often considers the *Yang-Mills* functional

$$\mathrm{YM}(\nabla) = \frac{1}{2}\int_M \|R_\nabla\|^2 d\nu_M$$

along with the following topological invariant (or 'instanton number')

$$k = \frac{1}{8\pi^2}\int_M \mathrm{Tr}(R_\nabla \wedge R_\nabla) \wedge \Omega^{m-1}$$

which is equal to $\langle p_1(V), [\Omega^{m-1}]\rangle$. With regards to the decomposition in §2, we can write the above as

$$k = -\frac{1}{8\pi^2}\int_M \sum_{i=1}^{3} c_i \|R_i\|^2 \Omega^m.$$

When $R_\nabla = R_i$, we have $\mathrm{YM}(\nabla) = 4\pi^2|k/c_i|$ and the functional YM is minimized when ∇ is c_1 or c_2–self–dual (anti–self–dual or self–dual respectively) (see **[9]**, **[18]**). In proceeding, we consider two classes of vector bundles relevant to the previous discussion:

$\mathbb{V}_M$:= {Pairs (V, ∇_V) where $V \to M$ is a complex vector bundle and ∇_V is a self–dual Hermitian connection on V.}

$\mathbb{V}_Z$:= {Pairs $(\tilde{V}, \nabla_{\tilde{V}})$ where $\tilde{V} \to Z$ is a holomorphic vector bundle with Hermitian $(1,0)$–connection $\nabla_{\tilde{V}}$ and Hermitian metric $h(\ ,\ )$. The bundle $\tilde{V}$ is flat restricted to the fibres of π and is assumed to be endowed with a 'real'-structure $\tau : Z \to Z$ (see e.g. [**1**]) which lifts to a bundle automorphism $\tilde{\tau} : \tilde{V} \to \tilde{V}$. A bundle map $\sigma : \tilde{V} \to \tilde{V}$ is then defined fibrewise by

$$f \in \tilde{V}_z \to \sigma(f) \in \tilde{V}^*_{\tau(z)}$$

where $\sigma(f)(s) := h(f, \tilde{\tau}(f))$ for each $s \in \tilde{V}_{\tau(z)}$. The map σ is an antiholomorphic bundle automorphism.}

A fundamental result is the following:

Theorem 4.1. *The assignment*

$$\mathbb{V}_M \ni (V, \nabla_V) \to (\pi^* V, \pi^* \nabla_V) \in \mathbb{V}_Z$$

defines a bijective correspondence between $\mathbb{V}_M$ *and* $\mathbb{V}_Z$.

Following [**26**], if M has positive scalar curvature, then Z admits a Kähler-Einstein metric of positive scalar curvature (the model example is to take $M = \mathbb{H}P^m$, m-dimensional quaternionic projective space with corresponding $Z = \mathbb{C}P^{2m+1}$). When M has positive scalar curvature, a pair $(\pi^* V, \pi^* \nabla_V)$ arising from Theorem 4.1 on Z, is a holomorphic vector bundle with Ricci-flat Hermitian-Einstein connection [**21**]. Let $\mathbb{V}^h_Z$ denote elements of $\mathbb{V}_Z$ endowed with these extra properties. Then we have:

Corollary 4.2. *Let M be a compact quaternionic Kähler manifold with positive scalar curvature. Then the assignment*

$$\mathbb{V}_M \ni (V, \nabla_V) \to (\pi^* V, \pi^* \nabla_V) \in \mathbb{V}^h_Z$$

defines a bijective correspondence between $\mathbb{V}_M$ *and* $\mathbb{V}^h_Z$.

§5. Hermitian–Einstein vector bundles and the Kuranishi space

Continuing from the end of the last section, let M be a compact quaternionic –Kähler manifold of positive scalar curvature and $\tilde{V} \to Z$ a holomorphic vector bundle. For a Hermitian (metric) structure h on

$\tilde{V}$, let $\mathcal{D}(\tilde{V},h)$ denote the set of connections on $\tilde{V}$ preserving h and let $U(\tilde{V},h)$ be the subgroup of $GL(\tilde{V})$ consisting of unitary automorphisms of $(\tilde{V},h)$. We now consider the set

$$\mathcal{H}(\tilde{V},h)=\{\nabla\in\mathcal{D}(\tilde{V},h):\nabla=\nabla'+\nabla'',\ \nabla''\in\mathcal{H}''(\tilde{V})\}$$

If $\nabla\in\mathcal{H}(\tilde{V},h)$, then ∇'' defines a unique holomorphic structure in $\tilde{V}$ such that ∇'' is the $\bar{\partial}$–operator on $\tilde{V}$–valued (p,q)–forms. Accordingly, ∇ is the Hermitian connection of $(\tilde{V},h)$ with respect to this holomorphic structure.

Consider now the subset $\mathcal{E}(\tilde{V},h)$ of Hermitian–Einstein connections on $\tilde{V}$:

$$\mathcal{E}(\tilde{V},h)=\{\nabla\in\mathcal{H}(\tilde{V},h):\iota\Lambda R(\nabla)=cI_{\tilde{V}}\}$$

where $R(\nabla)$ denotes the curvature of ∇ and $\Lambda:\mathcal{A}^{p,q}\longrightarrow\mathcal{A}^{p-1,q-1}$ is $(1,1)$contraction relative to the Kähler form. The space $\mathcal{E}(\tilde{V},h)/U(\tilde{V},h)$ is the moduli space of Hermitian-Einstein structures on $\tilde{V}$ and we have an injective map

$$\mathcal{E}(\tilde{V},h)\longrightarrow\mathcal{H}''(\tilde{V})/\,GL(\tilde{V})$$

Let us now consider the tangent spaces to these moduli spaces at given connections. Firstly, the tangent space to $\mathcal{H}''(\tilde{V})/\,GL(\tilde{V})$ at ∇'', is given by

$$H^{0,1}(Z,\operatorname{End}\tilde{V}^{\nabla''})=\frac{\{\alpha\in\mathcal{A}^{0,1}(\operatorname{End}\tilde{V}):\nabla''\alpha=0\}}{\{\nabla''f:f\in\mathcal{A}^{0}(\operatorname{End}\tilde{V})\}}.$$

The tangent space of $\mathcal{E}(\tilde{V},h)/U(\tilde{V},h)$ at ∇ is given by

$$\mathbf{H}^1=\frac{\{\alpha\in\mathcal{A}^1(\operatorname{End}(\tilde{V},h):\nabla\alpha\in\mathcal{A}^{1,1}(\operatorname{End}(\tilde{V},h))\ \mathit{and}\ \Lambda\nabla\alpha=0\}}{\{\nabla f:f\in\mathcal{A}^0(\operatorname{End}(\tilde{V},h))\}}$$

$$\cong\{\alpha\in\mathcal{A}^1(\operatorname{End}(\tilde{V},h):\nabla\alpha\in\mathcal{A}^{1,1}(\operatorname{End}(\tilde{V},h),\ \Lambda\nabla\alpha=0\ \mathit{and}\ \nabla^*\alpha=0\}$$

Let $\alpha=\alpha'+\alpha''\in\mathcal{A}^1(\operatorname{End}(\tilde{V},h))$. The assignment $\alpha\rightarrow\alpha''$ gives an isomorphism of $\mathbf{H}^1$ onto the space of harmonic $(0,1)$–forms with values in $\operatorname{End}(\tilde{V},h)$, leading to

$$\begin{aligned}\mathbf{H}^1&\cong H^{0,1}(Z,\operatorname{End}(\tilde{V}^{\nabla''}))\\&=\{\alpha\in\mathcal{A}^{0,1}(\operatorname{End}\tilde{V}):\nabla''\alpha''=0\ \mathit{and}\ \nabla''^*\alpha''=0\}.\end{aligned}$$

To proceed, let us define:

$$\mathcal{B}^p = \mathcal{A}^p_{\mathbb{R}}(\mathrm{End}(\tilde{V},h))$$

$$\mathcal{B}^{p,q} = \mathcal{A}^{p,q}(\mathrm{End}(\tilde{V},h)) = \mathcal{A}^{p,q} \otimes_{\mathbb{R}} \mathcal{B}^0 = \mathcal{A}^{p,q} \otimes_{\mathbb{R}} \mathcal{A}^0_{\mathbb{R}}(\mathrm{End}(\tilde{V},h))$$

$$\mathcal{B}^2_+ = \mathcal{B}^2 \cap (\mathcal{B}^{2,0} + \mathcal{B}^{0,2} + \mathcal{B}^0\omega) = \{\alpha + \bar{\alpha} + \beta\omega : \alpha \in \mathcal{B}^{2,0} \text{ and } \beta \in \mathcal{B}^0\}$$

$$\mathcal{C}^{0,q} = \mathcal{A}^{0,q}(\mathrm{End}\,\tilde{V}) = \mathcal{A}^{0,q} \otimes_{\mathbb{C}} \mathcal{A}^{0,0}(\mathrm{End}\,\tilde{V}) \ .$$

Consider now the complex

$$\begin{array}{ccccccccc}
0 \to \mathcal{B}^0 & \xrightarrow{\nabla} & \mathcal{B}^1 & \xrightarrow{\nabla_+} & \mathcal{B}^2_+ & \xrightarrow{\nabla_2} & \mathcal{B}^{0,3} & \xrightarrow{\nabla} & \dots\ \mathcal{B}^{0,n} \to 0 \\
\downarrow j_0 & & \downarrow j_1 & & \downarrow j_2 & & \downarrow j_3 & & \downarrow j_n \\
0 \to \mathcal{C}^{0,0} & \xrightarrow{\nabla''} & \mathcal{C}^{0,1} & \xrightarrow{\nabla''} & \mathcal{C}^{0,2} & \xrightarrow{\nabla''} & \mathcal{C}^{0,3} & \xrightarrow{\nabla''} & \dots\ \mathcal{C}^{0,n} \to 0
\end{array}$$

where $(\mathcal{C}^*)$ is elliptic if $\nabla \in \mathcal{H}(\tilde{V},h)$ and $(\mathcal{B}^*)$ is elliptic if $\nabla \in \mathcal{E}(\tilde{V},h)$. We can decompose $\mathcal{B}^2$ as $\mathcal{B}^2 = \mathcal{B}^2_+ \oplus \mathcal{B}^2_-$ where

$$\mathcal{B}^2_- = \{\alpha \in \mathcal{A}^{1,1}(\mathrm{End}(\tilde{V},h)) : \alpha = \bar{\alpha} \text{ and } \Lambda\alpha = 0\}$$

and we have the projections

$$p_+ : \mathcal{B}^2 \to \mathcal{B}^2_+ \qquad p_- : \mathcal{B}^2 \to \mathcal{B}^2_-$$

$$p^{2,0} : \mathcal{B}^2 \to \mathcal{B}^{2,0} \qquad p^{0,2} : \mathcal{B}^2 \to \mathcal{B}^{0,2}$$

where we set $\nabla_+ = p_+ \circ \nabla$ and $\nabla_2 = \nabla'' \circ p^{0,2}$. Then $\mathcal{E}(\tilde{V},h)$ can be expressed as

$$\mathcal{E}(\tilde{V},h) = \{\nabla + \alpha : \alpha \in \mathcal{B}^1 \text{ and } \nabla_+\alpha + p_+(\alpha \wedge \alpha) = 0\}$$

Consider a slice $\nabla + \mathcal{S}_\nabla$ in $\mathcal{E}(\tilde{V},h)$ in which

$$\mathcal{S}_\nabla = \{\alpha \in \mathcal{B}^1 : \nabla_+\alpha + p_+(\alpha \wedge \alpha) = 0 \text{ and } \nabla^*\alpha = 0\}$$

and the condition $\nabla^*\alpha = 0$ states that the slice is orthogonal to the $U(\tilde{V},h)$–orbit of ∇ . The *Kuranishi map* $k : \mathcal{B}^1 \to \mathcal{B}^1$ is defined by

$$k(\alpha) = \alpha + \nabla^*_+ \circ G \circ p_+(\alpha \wedge \alpha)$$

where G denotes the Green's operator. If $\nabla + \alpha \in \mathcal{E}(\tilde{V}, h)$, then it can be shown that $\nabla_+(k(\alpha)) = 0$. On the other hand, we have $\nabla^*(k(\alpha)) = \nabla^*\alpha$. Taking the harmonic forms in $\mathcal{B}^q$,

$$\mathbf{H}^q = \{\beta \in \mathcal{B}^q : \Delta\beta = 0\}$$

we have

$$k(\mathcal{S}_\nabla) \subset \mathbf{H}^1 = \{\beta \in \mathcal{B}^1 : \nabla_+\beta = 0 \text{ and } \nabla^*\beta = 0\}.$$

Letting End^0 denote trace–free skew Hermitian–endomorphisms, we arrive at

Theorem 5.1. [**17**] *Let $\nabla \in \mathcal{E}(\tilde{V}, h)$. If*

$$H^0(Z, \mathrm{End}^0(\tilde{V}^{\nabla''})) = 0 \qquad \text{and} \qquad H^2(Z, \mathrm{End}^0(\tilde{V}^{\nabla''})) = 0$$

then the Kuranishi map k defines a homeomorphism of a neighborhood of 0 in the slice $\mathcal{S}_D$ onto a neighborhood of 0 in $\mathbf{H}^1 \cong H^1(Z, \mathrm{End}(\tilde{V}^{\nabla''}))$.

Remark 5.2. *A holomorphic vector bundle $\tilde{V} \to Z$ is said to be simple if every holomorphic endomorphism is a constant. The second named author and independently Miyajima* [**28**] [**19**]*have shown for an algebraic manifold Z there is an isomorphism of (not necessarily reduced) complex vector spaces*

$$\mathcal{M}_{an} \cong \mathcal{M}_{alg}$$

between the moduli spaces of holomorphic simple vector bundles and algebraic simple bundles on Z.

§6. Remarks on Hyperkähler structures and the moduli of hyperholomorphic vector bundles.

A $4n$–dimensional Riemannain manifold is *hyperkähler* if its holonomy group is contained in $Sp(n) \subset SO(4n)$. Equivalently, M is quaternionic Hermitian with I, J and K globally defined and

$$d\omega_I = d\omega_J = d\omega_K = 0.$$

Since $Sp(n) \subset SU(2n)$, a hyperkähler manifold is Ricci–flat [**3**], [**4**].

Theorem 6.1. [**2**] *Let (M, I) be a compact Kähler manifold with (complex) symplectic form ω. Then for any Kähler class $\alpha \in H^2(M, \mathbb{R})$, there exists on M a unique Riemannian metric g_M such that*

1. *g_M is hyperkähler;*
2. *I is a parallel almost–complex structure;*
3. *the Kähler class of (g_M, I) is α.*

For a hyperkähler manifold M, there is an S^2 of almost complex structures

$$\{aI + bJ + cK \ : a^2 + b^2 + c^2 = 1\}$$

and the twistor space

$$Z = M \times \mathbb{C}P^1 \xrightarrow{\pi} M$$

fibres holomorphically over M. Let $p : Z \to \mathbb{C}P^1$ be the natural projection. The antipodal map of the fibres defines a *real* structure on Z (see e.g. **[1]**). Using the three Kähler forms $\omega_I, \omega_J, \omega_K$ along with a stereographic complex coordinate ξ on $\mathbb{C}P^1$, the form

$$\omega = (\omega_I + \imath\omega_K) + 2\xi\omega_I + (\omega_J - \imath\omega_K)$$

is a complex symplectic form on the fibres of p taking values in the line bundle $\mathcal{O}(2)$. The converse is true for a complex manifold Z of complex dimension $2n+1$ which fibres holomorphically $p : Z \to \mathbb{C}P^1$ with the above properties tenable: the parametrization of the (real) holomorphic structures is a (real) $4n$–dimensional hyperkähler manifold whose twistor space is Z (see **[14]**).

The Uhlenbeck–Yau theorem **[29]** says that an indecomposable holomorphic vector bundle V over a compact Kähler manifold M admits a Yang–Mills metric connection if and only if V is stable, and this metric is unique. Applying this to such a bundle $V \to M$ with Hermitian connection Θ over a hyperkähler manifold M yields:

Proposition 6.2. *The metric connection Θ is hyperholomorphic (that is, holomorphic with respect to each of I, J and K) if and only if its curvature R_Θ is of type $(1,1)$ with respect to any complex structure induced by the hyperkähler structure of M.*

It can be shown that such a hyperholomorphic connection Θ is Yang–Mills. In fact, $\Lambda(R_\Theta) = 0$ where Λ denotes contraction with each Kähler $(1,1)$–form. For instance, on a hyperkähler surface a stable holomorphic bundle $\tilde{V}$ with $\deg \tilde{V} = 0$ always admits a unique hyperholomorphic connection **[20]** **[8]**.

Let $\mathcal{S} \to M$ a be locally–free sheaf on and consider the composition of maps:

$$[\ ,\ ] \circ \varpi \ : H^1(\operatorname{End}\mathcal{S}) \times H^1(\operatorname{End}\mathcal{S}) \longrightarrow H^2(\operatorname{End}\mathcal{S})$$

where

$$\varpi \ : H^1(\operatorname{End}\mathcal{S}) \times H^1(\operatorname{End}\mathcal{S}) \longrightarrow H^2(\operatorname{End}\mathcal{S} \otimes \operatorname{End}\mathcal{S})$$

and

$$[\ ,\ ]\ : \mathrm{End}\,\mathcal{S} \times \mathrm{End}\,\mathcal{S} \longrightarrow \mathrm{End}\,\mathcal{S}$$

denotes the commutator map. The above composition gives defines the Yoneda pairing

$$\lambda\ : \mathrm{Ext}^1(\mathcal{S},\mathcal{S}) \times \mathrm{Ext}^1(\mathcal{S},\mathcal{S}) \longrightarrow \mathrm{Ext}^2(\mathcal{S},\mathcal{S})$$

which is an obstruction to the existence of a deformation (see e.g. **[16]**) . Specifically, suppose that $\varrho \in \mathrm{Ext}^1(\mathcal{S},\mathcal{S})$ satisfies $\lambda(\varrho,\varrho) \neq 0$ then there exists no deformation parametrized by a complex space $\mathcal{M}$ such that the image of the Kodaira–Spencer map

$$T_x\mathcal{M} \to \mathrm{Ext}^1(\mathrm{End}\,\mathcal{S})$$

is proportional to λ. In this way the Yoneda pairing turns out to be the only obstruction to the existence of a deformation of a hyperholomorphic vector bundle over a hyperkähler manifold.

For a stable bundle $\tilde{V} \to M$, let $\mathcal{M}_{st,\ \tilde{V}}$ be the coarse moduli space of $\tilde{V}$. This exists by virtue of **[20]** and is non–reduced and non–separated in general. Then

$$T\mathcal{M}_{st,\ \tilde{V}}|V' \cong \mathrm{Ext}^1(V',V').$$

Theorem 6.3. *The Kuranishi space of a hyperholomorphic vector bundle over a hyperkähler manifold is isomorphic as a complex space with the intersection of an open ball in* $\mathrm{Ext}^1(V,V)$ *with the quadric cone*

$$\{\varrho \in \mathrm{Ext}^1(V,V)\ : \lambda(\varrho,\varrho) = 0\}.$$

The complete details of this result will appear elsewhere.

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J.F.Glazebrook
Department of Mathematics
Eastern Illinois University
Charleston, IL 61920,
U.S.A.
cfjfg@ux1.cts.eiu.edu

D.Sundararaman
Department of Mathematics
Howard University
Washington, D.C. 20059,
U.S.A.
alama41@humain.bitnet