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Vector-Valued Forms and CR Geometry

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Dedicated to Professor M. Kuranishi on his 70th birthday

§1. Introduction.

Vector-valued forms arise in the study of various higher codimensional geometries. This note gives an overview of how the invariant theory of the Levi form (a vector-valued form) can be used to understand higher codimensional CR- structures.

Roughly speaking, the Levi form of a CR- structure of codimension c on a manifold M of dimension $2n+c$ can be interpreted as a map from M to the vector space consisting of c-tuples of $n \times n$ hermitian matrices (a vector space that we denote as Herm). However, this interpretation depends on a prior choice of moving coframesthat is, local sections of the cotangent bundle of M. Fortunately, there is a natural action of the group $G = GL(n, C) \times GL(c, R)$ on Herm that accounts for the effects of these choices. More precisely, there is a natural map from M to the quotient space Herm$/G$. Knowledge about the structure of this quotient space can be used to define canonical objects in higher codimensional CR- geometry. At present, the best developed example (discussed in §5) is a canonical connection for suitably generic CR- structures. The simplest examples, though, are functions defined on Herm$/G$, or equivalently, G-invariant functions defined on Herm : these lead one to explore the invariant theory of vector-valued forms as a tool in the study of CR-geometry. The history of invariant theory suggests two lines of approach. The first, discussed in §3, is to use methods of classical invariant theory to find explicit polynomial functions of vector-valued forms that are (relatively) invariant under the group action. While these techniques are quite old, the ensuing results for vector-valued forms are recent. The second, discussed in §4, is to use modern geometric invariant theory to study the quotient space directly. While the set Herm$/G$ has a standard quotient topology, it does not carry a globally defined differentiable

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structure: to obtain a differentiable structure one must first eliminate certain non-generic points (the "unstable" ones). The conditions defining these points are essentially geometric in concept, but involve a fair amount of technical intricacy in practice intricacies that are rooted in the aforementioned classical techniques!

As we shall note in §4, the invariant theory of vector-valued forms has three technically distinct cases: codimension 1, codimension 2, and higher codimension. Essentially, the codimension 1 case can be understood using nothing more than the standard notions of rank and signature of a hermitian matrix, and is therefore quite easy. The codimension 2 case is considerably more involved, but still fairly elementary: it rests on the analysis of roots of polynomials in one unknown. However, the higher codimensional case involves the zero-locus of polynomials in many unknowns, and consequently shares much of the richness of classical algebraic geometry. In CR-geometry to date, the codimension 1 case is the only one to have received a great deal of attention (see [Bo] and [J] for introductions and bibliographies), so the invariant theory of forms has not been featured in the CR literature. The approach we describe here is carried out in detail in three papers: [M] treats both the invariant theory and the differential geometry for codimension 2, [GM1] develops the invariant theory for higher codimension, and [GM2] treats the corresponding differential geometry. An introduction to higher-codimensional CR-geometry, including the equivalence problem, is contained in [Tu]. Selected references to other approaches to CR- geometry of codimension greater than 1 include [Be], [Ta1] and [Ta2].

Both of the authors are honored to participate in this tribute to Professor Kuranishi, and acknowledge with gratitude the fundamental influence his ideas have had on their work. One of the authors (Mizner) would also like to take this opportunity to express his appreciation for the care and graciousness with which Professor Kuranishi supervised his doctoral work: as time passes, he realizes ever more fully how fortunate he was to have been a student of Kuranishi.

§2. Definitions.

CR- structures arise concretely in connection with real submanifolds of a complex space. For example, let M be the zero locus of c real valued functions $g^\alpha : \mathbb{C}^{2n+c} \to \mathbb{R}$. If the real differentials dg^α are linearly independent, then M is a real submanifold of codimension c and dimension $2n + c$. If the holomorphic differentials ∂g^α are linearly independent as well, then the complex structure of $\mathbb{C}^{2n+c}$ determines a complex rank n subbundle $\mathcal{H}$ of the complexified tangent bundle $\mathbb{C} \otimes TM$. This subbundle is an instance of a CR- structure of codimension c and dimension n. Each function g^α determines a $(2n+c)\times(2n+c)$ hermitian matrix with entries $\frac{\partial^2 g^\alpha}{\partial z^j \partial \bar{z}^k}$; together these matrices constitute a vector-valued hermitian form (i.e., a c-tuple of scalar hermitian forms) on the complexified tangent bundle $\mathbb{C}\otimes TM$. Note that this c-tuple is defined only up to a choice of the defining functions g^α and coordinates on $\mathbb{C}^{2n+c}$. The restriction of this form to the subbundle $\mathcal{H}$ is called the Levi form of the CR- structure.

The abstract notion of a CR- structure and its accompanying Levi form generalize this example.

2.1. Definition. Let M be a smooth (i.e. C^∞) manifold. A CR-structure of dimension n and codimension c is a rank n complex subbundle $\mathcal{H}\subset\mathbb{C}\otimes TM$ with the following properties:

(1) $\mathcal{H}\cap\bar{\mathcal{H}}$ is the zero subbundle;
(2) $[\mathcal{H},\mathcal{H}]\subset\mathcal{H}$. (This condition, called the integrability condition, is an important technicality that is automatically satisfied by CR-structures arising on real submanifolds.)

2.2. Definition. The Levi form of $\mathcal{H}$ is the bundle map

$$L:\mathcal{H}\times\mathcal{H}\to\mathbb{C}\otimes TM/(\mathcal{H}\oplus\bar{\mathcal{H}}),$$

defined by

$$L(X,Y)=i\pi[\bar{X},Y],$$

for all sections X and Y of $\mathcal{H}$, where $\pi:\mathbb{C}\otimes TM\to\mathbb{C}\otimes TM/(\mathcal{H}\oplus\bar{\mathcal{H}})$ is the natural projection. It is easy to verify that L is well-defined, and that L is hermitian: i.e., $L(Y,X)$ and $L(X,Y)$ are conjugate.

As indicated in the introduction, by choosing suitably adapted moving coframes (local sections of the complex cotangent bundle $\mathbb{C}\otimes TM$), one can express the Levi form as a locally defined map from M to the real vector space whose points are c-tuples of $n\times n$ hermitian matrices (denoted $\mathrm{Herm}(n,c)$, or Herm for short). However, this expression of L as a vector-valued hermitian form depends on the choice of sections. In order to escape this dependency, one defines a natural action of the group $G=GL(n,\mathbb{C})\times GL(c,\mathbb{R})$ on Herm , and verifies that the result of composing the locally defined Herm -valued map with the quotient projection $\mathrm{Herm}\to\mathrm{Herm}/G$ is independent of choice of sections. Consequently, the locally defined Herm/G-valued maps piece together to yield a globally defined map $\mathcal{L}:M\to\mathrm{Herm}/G$.

2.3. Definition. The map $\mathcal{L}:M\to\mathrm{Herm}/G$ is called the Levi map .

Again as mentioned in the introduction, since every CR-manifold of dimension n and codimension c is mapped into the same quotient space Herm$/G$, information about Herm$/G$ can be used to define canonical objects in CR- geometry. In §5 we provide examples, but in order to do this, we must first examine Herm$/G$ in some detail.

§3. Classical invariant theory of vector-valued forms.

An absolute invariant of a vector-valued hermitian form is a function f : Herm $\rightarrow \mathbb{C}$ that is constant on each orbit of the action of the group $G = GL(n, \mathbb{C}) \times GL(c, \mathbb{R})$, and hence equivalent to a function on the quotient space Herm$/G$: in symbols, $f(gB) = f(B)$. A relative invariant (of weight χ) is a function that satisfies the weaker condition $f(gB) = \chi(g)f(B)$, where $\chi : G \rightarrow \mathbb{C}^*$ is a homomorphism. (Of course, if χ is the trivial homomorphism, the relative invariant is in fact absolute.) From here on we use the word invariant to cover both cases. Although a relative invariant is not constant on each orbit, it does have the property that if it vanishes at any one point in an orbit then it vanishes at all points in that orbit. Therefore, although a relative invariant does not determine a function on the quotient space, it does nonetheless determine a zero-locus–a fact that has significant geometric repercussions.

The basic procedure of classical invariant theory in situations such as ours is to consider only homogeneous polynomial invariants, to note that these invariants constitute a ring, and to list the generators of this ring in a so-called First Fundamental Theorem. Next, the relations among the generators is given in a Second Fundamental Theorem. Continuing, one then seeks the relations among the relations, the relations among these new relations, etc., which is called computing the syzygies of the ring of invariants.

In [GM1], a first fundamental theorem for vector-valued hermitian forms is obtained by specializing a first fundamental theorem for sesquilinear forms, which in turn is obtained by adapting the proof of a first fundamental theorem for bilinear forms. The basic ideas of the proof stand out most clearly in the bilinear case, which we now describe.

A vector-valued bilinear form is a bilinear map $V \times V \rightarrow W$. For concreteness, we take V and W to be $\mathbb{C}^n$ and $\mathbb{C}^c$ respectively, and denote the vector space of all such forms as $Bil(n, c)$, or simply Bil. We view a point in Bil either as a map or as a c-tuple of $n \times n$ matrices as convenience dictates. The group $GL(n, \mathbb{C}) \times GL(c, \mathbb{C})$ acts on Bil, with the group element $g = (A, P)$ transforming the form $B = (B^1, \cdots, B^c)$ in stages: the matrix A acts on each component matrix of B, yielding an intermediate c-tuple

$$({}^t(A^{-1})B^1A^{-1}, \cdots, {}^t(A^{-1})B^cA^{-1});$$

the components of the final c-tuple are linear combinations of these intermediate components, with coefficients drawn from P. In short, the α-th component of $(A, P)B$ is

$$P_1^\alpha \, {}^t(A^{-1})B^1A^{-1} + \cdots + P_c^\alpha \, {}^t(A^{-1})B^cA^{-1}.$$

This action is natural in that the following diagram commutes:

$$\begin{array}{ccc} V \times V & \xrightarrow{B} & W \\ {\scriptstyle A\times A}\big\downarrow & & \big\downarrow{\scriptstyle P} \\ V \times V & \xrightarrow[gB]{} & W. \end{array}$$

For technical purposes, it is more convenient to use a compact indicial notation, in which each component matrix B^α is represented by its entries B^α_{jk}, and the new form $(A, P)B$ is represented by matrices D^α, where

$$D^\alpha_{jk} = P^\alpha_\beta (A^{-1})^r_j (A^{-1})^s_k B^\beta_{rs}.$$

Here the usual summation convention is in force: whenever an index appears in both a subscript and a superscript a summation is implied.

We note in passing that for both the sesquilinear and hermitian cases, all of these formulas are modified by conjugating each left-hand A. A precise statement of the first fundamental theorem for vector-valued bilinear forms (as given in [GM1]) involves some technical notation that while standard in invariant theory is not immediately transparent to the uninitiated. However, the basic idea can be paraphrased in familiar terms, at the price of succinctness.

3.1. Theorem (First Fundamental Theorem for vector-valued bilinear forms).

Part 1. If r is divisible by c and $2r$ is divisible by n, then the following construction will yield either an invariant homogeneous polynomial of degree r and weight

$$\chi(A, P) = (\det A)^{-4r/n} (\det P)^{r/c}$$

or the zero polynomial.

a) Consider a monomial of degree r in the components of B:

$$B^{j_1}_{i_1 i_2} B^{j_2}_{i_3 i_4} \cdots B^{j_r}_{i_{2r-1} i_{2r}}.$$

b) Select n of the subscript positions, and take an alternating sum, as in the computation of a determinant. That is, consider the $n!$ monomials obtained by successively replacing these n subscripts by each of the $n!$ permutations of the numbers $(1, 2, \cdots, n)$, attach a coefficient of $+1$ if the permutation is even or -1 if the permutation is odd, and sum the terms.

c) Select an additional n subscript positions, and compute a similar alternating sum, thereby obtaining a polynomial with $(n!)^2$ terms.

d) Continue in this way until each literal subscript has been replaced by one of the numbers $1, \cdots, n$. (This is possible because $2r$ is divisible by n.)

e) In an analogous sequence of steps, replace each literal superscript by one of the numbers $1, \cdots, c$. (This is possible because r is divisible by c.)

Part 2. The preceding construction depends on a partition of the $2r$ subscript positions into n-fold blocks and the r superscript positions into c-fold blocks. Each such partition determines either an invariant homogeneous polynomial of weight χ or the zero polynomial. Every non-zero linear combination of these polynomials is also an invariant homogeneous polynomial of weight χ.

Part 3. No other invariant homogeneous polynomials exist.

Note that by taking a ratio of two of these relatively invariant polynomials of equal weight we obtain absolutely invariant rational functions. Also note that the first fundamental theorem for sesquilinear and hermitian forms (given in [GM1]) is similar, except that r must be divisible by both n and c, only certain partitions of the subscripts are allowed, and the weight χ is replaced by the weight

$$\lambda(A, P) = (\det A)^{-r/n} (\det \bar{A})^{-r/n} (\det P)^{r/c}.$$

Appearances notwithstanding, this theorem has a straightforward proof. It is based on a translation into the language of representation theory, a trick, an invocation of a basic theorem about representations of the general linear group, and translation back into the language of bilinear forms.

The first translation proceeds as follows. The space Bil is isomorphic to the space $V^* \otimes V^* \otimes W$; a homogeneous polynomial of degree r defined on this space is equivalent to a linear function defined on the space $(V^* \otimes V^* \otimes W)^{\otimes r} = (V^*)^{\otimes 2r} \otimes (W)^{\otimes r}$; such a linear function is an element of the dual space $V^{\otimes 2r} \otimes (W^*)^{\otimes r}$. The group $GL(n, \mathbb{C})$ acts on $V = \mathbb{C}^n$ by matrix multiplication; similarly, $GL(c, \mathbb{C})$ acts on $W = \mathbb{C}^c$. These actions determine a standard representation of the group $GL(n, \mathbb{C}) \times GL(c, \mathbb{C})$ on the space $V^{\otimes 2r} \otimes (W^*)^{\otimes r}$. Routine unwinding of the definitions shows that the element of $V^{\otimes 2r} \otimes (W^*)^{\otimes r}$ corresponding to an invariant homogeneous polynomial of degree r is the basis of a 1-dimensional invariant subspace. Therefore the problem shifts to the description of all 1-dimensional invariant subspaces of $V^{\otimes 2r} \otimes (W^*)^{\otimes r}$.

The trick is to show that each such space is isomorphic to the tensor product of a 1-dimensional $GL(n, \mathbb{C})$-invariant subspace of $V^{\otimes 2r}$ with a 1-dimensional $GL(c, \mathbb{C})$-invariant subspace of $(W^*)^{\otimes r}$. The fact that these groups are reductive is essential.

The invocation refers to the classical description of the irreducible representations of the general linear groupa staple of both invariant theory and representation theory described repeatedly throughout the literature. It is at this point that all of the alternating sums described in the theorem make their entrance.

The retranslation basically reverses the first step.

§4. Quotient spaces.

From an algebraic-geometric perspective, the space Herm$/G$ can be understood in terms of the spectrum of the ring of invariant polynomials. Unfortunately, current knowledge of this ring is exhausted by the first fundamental theorem, which is by no means adequate to explicate the structure of its spectrum.

From a differential-geometric perspective, one would like to have a smooth structure on Herm$/G$ with respect to which the Levi map is smooth. Unfortunately, as is typical with such quotient space or moduli problems, certain "unstable" points in Herm get in the way. However, we do have the following theorem (Theorem 2.4 of [GM2]).

4.1. Theorem. *Let $G = GL(n, \mathbb{C}) \times GL(c, \mathbb{R})$, let $K \subset G$ be the subgroup consisting of all pairs $(zI_n, |z|^2 I_c)$ for z in the complex multiplicative group $\mathbb{C}^*$, and suppose that $c > 2$ and $n > c^2$. There exists a non-empty G-invariant open subset $Z \subset$ Herm whose image Z/G by the* *projection* ρ : Herm $\to$ Herm $/G$ *can be given a smooth structure in such a way that* $Z \to Z/G$ *is a principal bundle with structure group* G/K.

The proof of this theorem is much too long and technical to be satisfactorily summarized, but it is not difficult to develop the definition of the open set Z. First we note that every element of K fixes each point in Herm ; in order to be in Z, a form B must be fixed by no other elements of G. Moreover, B must have no non-zero null vectors $x \in \mathbb{C}^n$that is, in order to be in Z, B must have the property that

$$({}^t\bar{x}B^1x, \cdots, {}^t\bar{x}B^cx) = (0, \cdots, 0) \quad \text{implies} \quad x = 0.$$

The statement of the one remaining condition in the definition of Z, which is the most interesting, requires a few preliminaries.

Each hermitian form B determines a polynomial, namely

$$\det(x_1B^1 + \cdots + x_cB^c).$$

For some forms this polynomial vanishes identically, but for generic forms it is homogeneous of degree n and therefore has a zero-locus in the projective space $\mathbb{P}^{c-1}$, which we call the *associated hypersurface.* Thus, there is a map

$$\text{(4.1)} \qquad \text{Herm} - -- > (\text{degree } n \text{ hypersurfaces in } \mathbb{P}^{c-1}),$$

where the dotted arrow signifies that the map is densely, but not globally, defined. Let Y denote the set of those hypersurfaces that satisfy the natural geometric condition of having no non-trivial projective automorphism and no points of multiplicity greater than $c$that is, no points at which the defining polynomial vanishes along with all of its partial derivatives of order c or less. The final condition defining Z is that B is in Z only if its associated hypersurface is in Y.

Given an element (A, P) of G, one can use the matrix P to change coordinates in$\mathbb{P}^{c-1}$; that is, one can view P as an element of the projective linear group PGL and let it act on $\mathbb{P}^{c-1}$ accordingly. It is easy to show that the hypersurface associated to the form $(A, P)B$ differs from the hypersurface associated to B only by the action of the change of coordinates determined by P. Therefore, the map (4.1) determines a densely defined map of quotient spaces

$$\text{Herm}/G - -- > (\text{degree } n \text{ hypersurfaces in } \mathbb{P}^{c-1})/PGL$$

which restricts to a globally defined map

$$Z/G \to Y/PGL.$$

A major step in proving Theorem 4.1 is to show that Y/PGL can be given a smooth structure in such a way that $Y \to Y/PGL$ is a principal bundle with structure group PGL (see Theorem 5.1 of [GM2]). Basically, in studying Z/G by way of Y/PGL we are studying the action of the product group $GL(n,\mathbb{C}) \times GL(c,\mathbb{R})$ one factor at a time, which, in light of the two-step definition of the group action, is a natural approach.

Before going on to apply Theorem 4.1 to CR-geometry, we note that the polynomial $\det(x_1B^1 + \cdots + x_cB^c)$ explains the trichotomy mentioned in §2. If $c = 1$, it simply distinguishes the singular hermitian matrices from the non-singular; if $c = 2$, it is a homogeneous polynomial in two unknowns, which is essentially equivalent to an inhomogeneous polynomial in one unknown; if $c > 2$, algebraic geometry is clearly involved.

§5. CR geometry.

Recall that a CR-structure $\mathcal{H}$ of dimension n and codimension c on the smooth $2n + c$ dimensional manifold M determines the Levi map $\mathcal{L} : M \to \text{Herm}/G$, where Herm is the vector space whose points are c-tuples of $n \times n$ matrices, $G = GL(n,\mathbb{C}) \times GL(c,\mathbb{R})$, and the action of G on Herm is defined so that the α-th component matrix of the c-tuple $(A,P)B$ is

$$P_1^{\alpha t}(\bar{A}^{-1})B^1A^{-1} + \cdots + P_c^{\alpha t}(\bar{A}^{-1})B^cA^{-1}.$$

As noted earlier, the Levi map furnishes a fundamental link between CR-geometry and the invariant theory of vector-valued forms, since it can be used to pull back any specified "structure" on Herm/G to produce a canonical CR- geometric object on M. For instance, we have already seen that each invariant function on Herm determines a zero-locus in Herm/G. The Levi map pulls this back to a subset of M, canonical in the sense that if $F : M \to M'$ is an isomorphism of CR manifolds, then F maps the specified subset of M bijectively to the specified subset of M'. In order to construct richer geometric objects, it seems necessary to restrict attention to those CR-structures that enjoy some type of homogeneity. As part of his general treatment of differential systems Tanaka [Ta1,Ta2] develops a full theory of CR-structures whose Levi maps are constant. Here we take a different approach and consider CR-structures whose Levi maps are valued in a specified open subset $\mathcal{U}$ of Herm/G. If the Levi map is constant, only one orbit of forms in Herm is connected with the CR-structure, and this orbit can be represented by a chosen canonical form. We would like to proceed similarly and choose a canonical form from each of the orbits corresponding to points in $\mathcal{U}$. Of course, in order to be useful for differential-geometric purposes, these choices must be smooth. Thus we need a smooth local section of Herm $\to$ Herm$/G$ defined on the open set $\mathcal{U}$.

In particular, we need $\mathcal{U}$ to have a smooth structure, so the set Z/G described in §4 is an obvious candidate. Moreover, Z has a sort of homogeneity since each of its points is fixed by the elements of K alone. Regretably, we do not know if $Z \to Z/G$ admits a smooth section. However, since $Z \to Z/G$ is a principal bundle, every point of Z/G has a neighborhood admitting a smooth section. Therefore, we take $\mathcal{U}$ to be such a neighborhood, and fix some section σ.

5.1. Definition. A CR-structure is *tractable of type* $\mathcal{U}$ if its Levi map $\mathcal{L} : M \to$ Herm$/G$ is valued in $\mathcal{U}$.

In standard differential-geometric fashion, every CR-structure, tractable or not, determines a subbundle of the coframe bundle of M, consisting of suitably "adapted" coframes. The structure group of this subbundle is unwieldy, but in the tractable case the subbundle can be reduced dramatically (Theorem 3.1 of [GM2]), yielding a subbundle of "better adapted" coframes with structure group K (which, we recall, is isomorphic to $\mathbb{C}^*$). The proof of this theorem uses a detailed analysis of the structure equations of moving coframes, but the core idea is simple. From the coefficients of these structure equations one can extract a c-tuple of hermitian matrices that represents the Levi form; the assumption of tractability allows one to single out those coframes whose structure equations give rise to canonical c-tuplesthat is, c-tuples in the image of the section σ.

Analysis of the structure equations of the reduced principal bundle associated to a tractable CR-structure leads to the construction of a canonical connection on this bundle (Theorem 4.1 of [GM2]). One immediate corollary (4.2 of [GM2]) is that the automorphisms of a tractable CR-structure constitute a Lie group. Another (4.3 of [GM2]) is that this connection can be canonically extended to an affine connection, thereby introducing an operation of covariant differentiation into the study of CR geometry. A third corollary (4.4 of [GM2]) is a canonical decomposition of the complexified tangent bundle of a tractable CR manifold as a direct sum of $2n + c$ complex line bundles, and a corresponding decomposition of the real tangent bundle as the direct sum of c real line bundle and n real plane bundles with complex structure.

§6. Conclusion.

The study of higher-codimensional CR-structures by way of the invariant theory of vector-valued hermitian forms has barely begun, and open questions abound.

As already noted in §3, the first fundamental theorem is just the first step in the classical approach to describing the ring of invariant polynomials. A second stepthe development of a second fundamental theorem and a description of the syzygiesis in progress ([G]). There remain numerous commutative-algebraic questions, along with the ultimate goal of a thorough understanding of the spectrum, and hence an algebraic-geometric understanding of the quotient space Herm$/G$. From a more practical point of view, there is the problem of computing invariants. The procedure described in Theorem 3.1 is constructive in principle, but hardly efficient, and significant improvements should be possible.

The invariant theory of vector-valued forms can be applied to branches of differential geometry apart from CR geometry. The second fundamental form of a Riemannian submanifold and the holomorphic second fundamental form of a complex submanifold are geometrically important vector-valued forms that are algebraically similar to the Levi form. Additionally, the geometry of a manifold with distribution involves a skew-symmetric bilinear form. Theorem 3.1 applies directly to the latter two cases. For Riemannian geometry, where one needs to consider the action of a product of orthogonal groups rather than general linear groups, the same methods apply, but the resulting formulas are more complex.

In CR geometry, there is the central issue of tractability. Is Z/G itself tractable? Are there tractable subsets with sections that can be explicitly described? Such a description would amount to a procedure for converting a given c-tuple of hermitian matrices to a specified canonical forma sort of "super Gram-Schmidt" process. The decomposition of the tangent bundle of a tractable CR manifold described in §5 shows that there are global obstructions to tractability. Can the invariant theory of forms elucidate any other aspects of global CR geometry?

Finally, to conclude on a note of sheer wishful thinking, might it be possible to use the approach described here, rooted in the teaching of Professor Kuranishi, to illuminate (or indeed, since this is wishful thinking, to solve) the embedding problem for higher codimensional CR structures?

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