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Deformation Theory for the Hyperplane Line Bundle on $\mathbb{P}^1$

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Dedicated to Professor M. Kuranishi, on the occasion of his seventieth birthday.

Abstract.

We develop a correspondence between deformations of the standard CR structure on S^3 and deformations of formal neighbourhoods of the hyperplane bundle over $\mathbb{P}^1$; this correspondence leads to a geometric description of obstructions to the embeddability of CR structures.

§1. Introduction

In recent years, much work has been done on the imbeddability of CR structures on S^3. See, for example, [B], [BlEp], [BuEp], [CaLe], [Ep1], [Ep2], [Le1], [Le2]. In [B] and [Le1], a geometric description of sufficient conditions for embeddability was provided; moreover, it follows from a stability result in [Le1] that these conditions are also necessary for CR structures that are sufficiently close to the standard spherical CR structure. However, a geometric interpretation of the obstructions to embeddability was still lacking. In this paper, we look to providing such an interpretation.

Since $S^3 \subset \mathbb{C}^2 \subset \mathbb{P}^2$, we can view S^3 as bounding the complement of the unit ball in $\mathbb{P}^2$; call this complement U. We begin by surveying some results that relate the CR deformation theory of S^3 to the deformation theory for the pseudoconcave manifold U that it bounds, with particular emphasis on the embeddability question. We then show how the analysis of any sufficiently small deformation of the standard CR structure on S^3 can be localized to an analysis of the extended deformation of the complex structure in formal neighbourhoods of the hyperplane at infinity. Moreover, stable embeddability corresponds to the formal

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neighbourhoods of the deformed structure being equivalent to formal neighbourhoods of the undeformed structure. One consequence is a new description of the *obstructions* to the embeddability of CR structures on S^3 in a neighbourhood of the standard CR structure.

Remark 1.1. *We mention here a notational convention. We will usually be working with expansions of various functions and tensors in powers of* ζ. *We will let* $[\phi]_k$ *denote the expansion of* ϕ *truncated at the* k^{th} *term, and let* ϕ_k *denote the coefficient of the* k^{th} *term itself; thus, for example,* $\phi_k\zeta^k = [\phi]_k - [\phi]_{k-1}$.

§2. Embeddability

In this section we will extend deformations of the CR structure on S^3 to deformations of the complex structure of a pseudoconcave manifold U, and indicate how the embeddability question for S^3 is related to certain properties of the deformed pseudoconcave manifold that it bounds.

We begin by introducing the notation and the framework. Let $\mathbf{z} = (\mathbf{z^1}, \mathbf{z^2})$ denote Euclidean coordinates on $\mathbb{C}^2$ with the Euclidean norm $||\mathbf{z}||^2 = |\mathbf{z^1}|^2 + |\mathbf{z^2}|^2$. Recall that $\mathbb{P}^2$ can be obtained from $\mathbb{C}^2$ by attaching a $\mathbb{P}^1$ at infinity, and that points on the hyperplane at infinity naturally correspond to lines through the origin in $\mathbb{C}^2$. We choose local coordinates in a neighbourhood of the hyperplane at infinity by setting $w := z^2/z^1, \zeta := 1/z^1$ for the lines on which $z^1 \neq 0$, and $\hat{w} := z^1/z^2, \hat{\zeta} := 1/z^2$ for the lines on which $z^2 \neq 0$. Let V_1 denote the open set on $\mathbb{P}^1$ on which $z^1 \neq 0$, and let V_2 denote the open set on $\mathbb{P}^1$ on which $z^2 \neq 0$.

Let $\pi : E \to \mathbb{P}^1$ denote the hyperplane bundle over $\mathbb{P}^1$. Recall that the total space E is naturally biholomorphic to the complement of the origin in $\mathbb{P}^2$, with the zero section of E corresponding to the hyperplane at infinity, and the fibres of E corresponding to the lines through the origin in $\mathbb{C}^2 \subset \mathbb{P}^2$. More precisely, we may represent the hyperplane line bundle using local coordinates $\{(w,\zeta) : w \in V_1\}$, $\{(\hat{w},\hat{\zeta}) : \hat{w} \in V_2\}$ with transition functions

$$\hat{w} = \frac{1}{w} \qquad \hat{\zeta} = \frac{\zeta}{w} \qquad \text{on } V_1 \cap V_2 . \tag{2.1}$$

We can obtain a concrete embedding $\iota : E \hookrightarrow \mathbb{P}^2$ by setting

$$z^1 = 1/\zeta \qquad z^2 = w/\zeta \qquad \text{on } \pi^{-1}(V_1) \setminus \mathbb{P}^1 \tag{2.2}$$

and

$$z^2 = 1/\hat{\zeta} \qquad z^1 = \hat{w}/\hat{\zeta} \qquad \text{on } \pi^{-1}(V_2) \setminus \mathbb{P}^1 . \tag{2.3}$$

The inverse of the Euclidean norm $||\mathbf{z}||^{-2}$ restricted to the complement of the origin defines a hermitian metric $h = ||\mathbf{z}||^{-2}$ on E. In the local coordinates above, h is given by the formula

(2.4)
$$h(w,\zeta) = e^{-H(w)}\,|\zeta|^2 \text{ on } \pi^{-1}(V_1) \text{ and } h(\hat{w},\hat{\zeta}) = e^{-H(\hat{w})}\,|\hat{\zeta}|^2 \text{ on } \pi^{-1}(V_2)$$

where $e^{H(w)} = (1+|w|^2)$. Let U denote the total space of the (open) unit disk bundle of (E,h). Notice that U is biholomorphic to the complement of the closed unit ball in $\mathbb{P}^2$, and $\partial U = S^3$. The open pseudoconcave manifold U is covered by two coordinate charts, $U_1 = (\pi^{-1}V_1) \cap U \cap \{|w| < 4\}$, $U_2 = (\pi^{-1}V_2) \cap U \cap \{|\hat{w}| < 4\}$.

We next recall some basic facts of the CR deformation theory for $S^3 = \partial U$. Let η denote the connection for the hermitian metric h, and let $H_{(1,0)}U$ denote the space of the horizontal lifts of tangent vectors of type $(1,0)$ on $\mathbb{P}^1$; locally, $H_{(1,0)}U$ is spanned by the horizontal vector field

$$e = \frac{\partial}{\partial w} + \bar{w}\, e^{-H}\,\zeta \frac{\partial}{\partial \zeta}\,.$$

The holomorphic tangent bundle for S^3, $H_{(1,0)}S^3 := (T_{(1,0)}\mathbb{C}^2) \cap (\mathbb{C} \otimes TS^3)$, is simply $H_{(1,0)}U$ restricted to ∂U.

Remark 2.5. *The choice of a different hermitian norm on* $\mathbb{C}^2$ *induces a different hermitian metric* $\tilde{h}$ *on* E*, with the corresponding hermitian connection* $\tilde{\eta}$ *and horizontal* $(1,0)$ *vector field* $\tilde{e}$*. This choice can be interpreted as choosing a different circular domain in* $\mathbb{C}^2$*; note, however, that the circular domain still admits an* S^1 *action which preserves the holomorphic tangent space.*

A result of Kiremidjian [Kir] says that any small deformation of the CR structure on S^3 extends to define an integrable deformation of the complex structure on U. Moreover, in [B], an explicit extension is obtained in which the holomorphic structure on the hyperplane at infinity is left unchanged. For the convenience of the reader, we outline the argument here, and recall the precise statement of the result.

It is well known (see e.g. [B], [CL]) that every small deformation of the standard CR structure on S^3 is equivalent to one whose deformation tensor is of the form $\phi \in \Gamma\left(S^3, \mathrm{Hom}\left(H_{(0,1)}, H_{(1,0)}\right)\right)$. Moreover, by considering the action of the group of contact diffeomorphisms on deformations, one can show (see [B]) that, up to equivalence, ϕ is of the form (locally)

$$\phi = \sum_{k=1}^{\infty} \phi_k(w)\zeta^k d\bar{w} \otimes e\,,$$

where $\phi_k(w)$ are smooth functions. We refer to this as *exterior form.*

In the statement of the next theorem, and throughout the paper, we will use the anisotropic Folland Stein Γ^s norms [FS] to measure smoothness, and to introduce a topology on the various function spaces. These anisotropic norms measure L^2 derivatives only in the CR or conjugate CR directions; after fixing the connection form $\tilde{\eta}$, the span of these directions is precisely the span of the vector fields $\tilde{e}, \bar{\tilde{e}}$, or the distribution which is dual to the connection form.

Remark 2.6. *Throughout the paper, "smooth" objects will refer to objects with an appropriate degree of smoothness in some* Γ^s *norm.*

Theorem 2.7. *(Bland* [B]*) Let* ϕ *be a sufficiently small deformation of the standard CR structure on* $S^3 = \partial U$*, measured in the* Γ^s *norm relative to the standard framing of* S^3*,* $s \geq 6$*. Then* ϕ *is equivalent to a CR structure of the form*

$$\phi = \sum_{k=0}^{\infty} \phi_k(w)\zeta^k d\bar{w} \otimes e\,,$$

where $\phi_k(w)$ *are smooth functions of* w*; moreover, there exists a (possibly different) connection form* $\tilde{\eta}$ *with its corresponding horizontal lift* $\tilde{e}$ *of the basic vector field* $\partial/\partial w$ *such that* ϕ *is equivalent to a CR structure of the form*

$$\phi = \sum_{k=1}^{\infty} \tilde{\phi}_k(w)\zeta^k d\bar{w} \otimes \tilde{e}\,, \tag{2.8}$$

where $\tilde{\phi}_k(w)$ *are smooth functions of* w.

Throughout the remainder of the paper, we will assume that the deformation tensor is normalized according to equation (2.8). Moreover, we will drop the decoration "$\tilde{}$", and refer to the connection form as η and the corresponding horizontal lift of $\partial/\partial w$ as e. The Folland Stein Γ^s norms will be defined relative to the horizontal distribution in the tangent space on S^3 which is defined by η.

Using the ideas of [BD], one can show that ϕ extends in the obvious way to define an integrable deformation of the complex manifold U. Notice that ϕ vanishes along the zero section of E.

The following theorem summarizes the discussion. The extension result is a special case of the theorem of Kiremidjian [Kir], while the normalization procedure was contained in [B] (see also [BD]).

Theorem 2.9. *(Kiremidjian* [Kir], *Bland* [B]*) Let ϕ be a sufficiently small deformation of the standard CR structure on $S^3 = \partial U$, measured in the Γ^s norm relative to the standard framing of S^3, $s \geq 6$. Then ϕ extends to define an integrable deformation of the complex structure on U.*

Moreover, up to equivalence, ϕ can be taken to be of the form

$$\phi = \sum_{k=1}^{\infty} \phi_k(w)\zeta^k d\bar{w} \otimes e\,, \tag{2.10}$$

where $\phi_k(w)$ are smooth functions of w.

As a consequence of Kiremdjian's result and well known results of Harvey–Lawson [HL] and Folland and Kohn [FoKo], we have the following theorem, first obtained by Lempert in [Le1] .

Theorem 2.11. *(Lempert* [Le1]*) Let ϕ denote a sufficiently small deformation of the standard CR structure on S^3, as measure in the Γ^s norm, $s \geq 6$. Then (S^3, ϕ) is C^1 embeddable if and only if there exists a compact complex surface X and an embedding $(S^3, \phi) \hookrightarrow X$ for which (S^3, ϕ) disconnects X into two connected components.*

Proof. Suppose first that (S^3, ϕ) is embeddable. Then by Harvey–Lawson [HL], there is a normal Stein space V for which (S^3, ϕ) is the pseudoconvex boundary; resolve any singularities to obtain a smooth complex manifold $\tilde{V}$ for which (S^3, ϕ) is the pseudoconvex boundary. Kiremidjian's result implies that there is a complex manifold (U, ϕ) for which (S^3, ϕ) is the pseudoconcave boundary. Glue these two pieces along (S^3, ϕ); thus, we obtain a C^1 compact manifold X with an integrable complex structure, and (S^3, ϕ) disconnects X. By the Newlander–Nirenberg theorem, X is a smooth compact complex manifold.

Conversely, if there exists X and an embedding $(S^3, \phi) \hookrightarrow X$ which disconnects X, then (S^3, ϕ) is the pseudoconcave boundary of one component, and the pseudoconvex boundary of the other component. Let V denote the pseudoconvex component. By the results of Folland and Kohn on the solvability of $\bar{\partial}$ on compact complex manifolds with pseudoconvex boundary [FoKo], one can construct sufficiently many functions which are holomorphic on V and C^1 to the boundary to embed (S^3, ϕ). ∎

Using the analysis of Morrow and Rossi [MR], much more can be said about the manifold X; in fact, we obtain the following stability result of Lempert [Le1].

Theorem 2.12. *For $s \geq 6$, there is a neighbourhood of the standard CR structure on S^3 such that (S^3, ϕ) is embeddable if and only if (S^3, ϕ) is embeddable in $\mathbb{C}^2$.*

Proof. (Lempert [Le1]) In the previous theorem, we showed that if (S^3, ϕ) embedded, then it embedded into a complex surface X as a disconnecting hypersurface. In light of the normal form analysis of Theorem 2.9, we know that we can choose the pseudoconcave component of X to contain a rational curve $\mathbb{P}^1$ with the hyperplane bundle as its normal bundle. In this situation, a rigidity result of Morrow and Rossi [MR] states that X must be birational to $\mathbb{P}^2$, and the rational curve is a standard linear hyperplane; that is, we may choose X to be $\mathbb{P}^2$, and the pseudoconcave component is a neighbourhood of the hyperplane at infinity. ▌

The following corollary is immediate from the construction of the manifold X.

Corollary 2.13. *There is a neighbourhood of the standard CR structure on S^3 such that (S^3, ϕ) is embeddable if and only if (U, ϕ) is biholomorphic to a neighbourhood of the zero section of E.*

This relates the embeddability of (S^3, ϕ) to the deformation theory for the pseudoconcave complex manifold (U, ϕ) which it bounds. Moreover, we can infinitesimalize this result to arbitrarily small neighbourhoods of the rational curve $\mathbb{P}^1$. However, the analysis leads us to questions of convergence, and we will delay this result until the end of the next section.

§3. Formal Embeddability

In this section, we will relate the CR deformation theory for S^3 to the Morrow–Rossi deformation theory for formal neighbourhoods of the hyperplane at infinity.

Throughout this section, (S^3, ϕ) will denote a sufficiently small deformation of the standard CR structure in the Γ^s norm, $s \geq 6$, and (U, ϕ) will denote the extension of the deformation to the pseudoconcave side. We will assume that the deformation tensor has been placed in exterior form; that is, it can be expressed as $\phi = \sum_{k=1}^{\infty} \phi_k(w) \zeta^k d\bar{w} \otimes e$, where

ϕ_k are Γ^s functions which are constant on the fibres. Each graded piece $\phi_k(w)\zeta^k d\bar{w} \otimes e$ has a natural interpretation as a deformation tensor on $\mathbf{P}^1$ twisted by a positive power of the dual of the hyperplane line bundle, i.e. as a section of $\mathrm{Hom}(T_{(0,1)}\mathbf{P}^1, T_{(1,0)}\mathbf{P}^1) \otimes \otimes^k E^*$; moreover, since there are no zeroth order terms, the complex structure on $\mathbf{P}^1$ is left unchanged. We will henceforth refer to this rational curve as the $\mathbf{P}^1$.

Throughout this paper, we will be concerned only with objects which are holomorphic in the ζ variable; thus, we may identify them with the sum of sections of powers of the dual of the hyperplane bundle.

We now describe how to pass from the Dolbeault approach to deformation theory to the Cech approach in this situation. In brief, although the coordinate cover U_i is not a Stein cover, it is still sufficiently nice that we can pass from the deformation tensor to new coordinate functions (ξ, ρ) which are holomorphic in the deformed structure.

We can write down explicit formal expressions for local functions (ξ, ρ) and $(\hat{\xi}, \hat{\rho})$ which are holomorphic in the deformed structure and which converge on the chart $|w|^2 < 4$, (respectively, $|\hat{w}|^2 < 4$) (see, for instance, [B]). Then we look at the transition functions as expressed using the new coordinate systems (ξ, ρ) and $(\hat{\xi}, \hat{\rho})$. The next two propositions analyze these transition functions in a manner which is reminiscent of the formal neighbourhoods of $\mathbf{P}^1$ as studied by Morrow and Rossi [MR]; we will refer to this observation again after the statement and proofs of the propositions.

Proposition 3.1. *Let (S^3, ϕ) be a deformation of the standard CR structure of S^3 which is in exterior form and sufficiently small in the Γ^s norm, $s \geq 6$; let (U, ϕ) be its extension to U. Let U_1, U_2 be the standard coordinate cover of the neighbourhood U with coordinates (w, ζ) and their hatted counterparts.*

Then there exist local coordinates (ξ, ρ) on U_1 and their hatted counterparts on U_2 which are holomorphic to order k for the deformed complex structure.

Moreover, the new coordinates can be taken to be of the form

$$\rho = \zeta(1 + \sum_{j=1}^{k} \rho_j \zeta^j) + O(\zeta^{k+1}) \quad \textit{and} \quad \xi = w + \sum_{j=1}^{k} \xi_j \zeta^j + O(\zeta^{k+1})$$

where ξ_j, ρ_j are smooth functions of w.

Proof. We illustrate the approach in this case, introducing the formalism which we use in solving the $\bar{\partial}$ equation for the deformed structure (that is, $\bar{\partial}_\phi$), and the recursive algorithm.

As explained in [B], the $\bar\partial$ operator for the deformed structure is expressed as $\bar\partial_\phi = \bar\partial - \phi\circ\partial$. Let u be a function on U_1 that is holomorphic in the fibre directions; that is, u is a function of the form $u = \sum_j u_j\zeta^j$, where u_j is a function of w. Then

$$\phi\circ\partial(u) := (\sum_{l=1}^{\infty}\phi_l\zeta^l)\sum_j(e^{jH}\frac{\partial(e^{-jH}u_j)}{\partial w})\zeta^j d\bar w\,.$$

If we ask for a function which agrees with w when $\zeta = 0$, and is holomorphic to order k about $\mathbb{P}^1$, then we consider a function of the form $u = w + \sum_{j=1}^k u_j\zeta^j$, and solve inductively:

$$\bar\partial(u) = \phi\circ\partial(u)\,; \tag{3.2}$$

$$\sum_{j=1}^{k}\bar\partial u_j\zeta^j = (\sum_{l=1}^{\infty}\phi_l\zeta^l)\sum_{j=1}^{k}(e^{jH}\frac{\partial(e^{-jH}u_j)}{\partial w})\zeta^j d\bar w\,. \tag{3.3}$$

Since for each power of ζ we are solving a one variable $\bar\partial$ equation for u_k in terms of data which has been previously determined, we can obtain k^{th} order formal solutions for all k. Finally, it is a simple matter to observe that for any order k, we may obtain local functions of the form given in the proposition which are holomorphic to order k in the deformed structure.

∎

This proposition is a form of the statement that deformations of complex structures are locally trivial; moreover, the cover U_i is a "good" cover of U. The Cech data for the deformed complex manifold is given by the transition functions for the cover. We compute these in the next proposition.

Proposition 3.4. *Let (U,ϕ) and U_i be as above, and let*

$$\rho = \zeta(1+\sum_{j=1}^{k}\rho_j\zeta^j) \quad \text{and} \quad \xi = w + \sum_{j=1}^{k}\xi_j\zeta^j$$

be new coordinates which are holomorphic in the deformed complex structure as constructed in the last proposition. Then after possibly choosing new representative holomorphic functions for the deformed structure, which we still denote by (ξ,ρ), holomorphic transition functions for the deformed manifold (U,ϕ) can be taken to be of the form

$$\xi\hat\xi = 1 + \sum_{i=4}^{k}(\sum_{j=2}^{i-2}\frac{a_{ij}}{w^j})\zeta^i + O(\zeta^{k+1}) \tag{3.5}$$

and

$$\hat{\rho}\xi/\rho = 1 + \sum_{i=2}^{k}(\sum_{j=1}^{i-1}\frac{b_{ij}}{w^j})\zeta^i + O(\zeta^{k+1})\,; \tag{3.6}$$

that is, we can find holomorphic functions (ξ, ρ) *on* U_1 *and* $(\hat{\xi}, \hat{\rho})$ *on* U_2 *for the deformed complex structure* $\bar{\partial}_\phi$ *such that they satisfy the relations given above on* $U_1 \cap U_2$.

Remark 3.7. *Equations (3.5), (3.6) take on the more standard form of transition functions if we solve explicitly for* $\hat{\xi}, \hat{\rho}$ *respectively.*

Proof. Let ξ, ρ be of the form given in the previous proposition. Consider the first order expansion. The transition functions are given by

$$\begin{aligned} \xi\hat{\xi} &= (w + \xi_1\zeta)(\hat{w} + \hat{\xi}_1\hat{\zeta}) + O(\zeta^2) \\ &= 1 + \zeta(\hat{w}\xi_1) + \hat{\zeta}(w\hat{\xi}_1) + O(\zeta^2) \\ &= 1 + \zeta(\frac{1}{w}\xi_1 + \hat{\xi}_1) + O(\zeta^2)\,. \end{aligned} \tag{3.8}$$

Since the product $\xi\hat{\xi}$ is holomorphic on the intersection and the zeroth order term is constant, a simple calculation shows that the first order term $(\frac{1}{w}\xi_1 + \hat{\xi}_1)$ is holomorphic in the standard structure. Since $\xi, \hat{\xi}$ are only determined up to the addition of functions that are holomorphic in $w, \hat{w}$ respectively, we easily observe that we can choose these functions in such a way as to normalize the first order term to be zero.

Similarly, we consider the transition function for the fibre variable. In this case, we have

$$\begin{aligned} \frac{\hat{\rho}\xi}{\rho} &= \frac{\hat{\zeta}(1 + \hat{\rho}_1\hat{\zeta})(w + \xi_1\zeta)}{\zeta(1 + \rho_1\zeta)} + O(\zeta^2) \\ &= 1 + \hat{\zeta}(\hat{\rho}_1\hat{\zeta}w + \xi_1\zeta)/\zeta - \hat{\zeta}w\rho_1\zeta/\zeta + O(\zeta^2) \\ &= 1 + \zeta(\hat{\rho}_1/w + \xi_1/w - \rho_1) + O(\zeta^2) \end{aligned} \tag{3.9}$$

where as before, $(\hat{\rho}_1/w + \xi_1/w - \rho_1)$ is holomorphic on $U_1 \cap U_2$, ξ_1 is a smooth function determined by the previous step, and $\rho_1, \hat{\rho}_1$ are determined up to the addition of functions that are analytic in $w, \hat{w} = 1/w$ respectively. It is clear that we can normalize the expression in the brackets to be zero. This completes the first order normalization.

We now proceed to the inductive step. Assume that $\xi, \hat{\xi}, \rho, \hat{\rho}$ have been chosen to order $k-1$ in such a way as to place the transition functions in normal form to order $k-1$. Then

$$
\begin{aligned}
\xi\hat{\xi} &= (w+\sum_{i=1}^{k}\xi_i\zeta^i)(\hat{w}+\sum_{i=1}^{k}\hat{\xi}_i\hat{\zeta}^i)+O(\zeta^{k+1}) \\
&= (w+\sum_{i=1}^{k-1}\xi_i\zeta^i+\xi_k\zeta^k)(\hat{w}+\sum_{i=1}^{k-1}\hat{\xi}_i\hat{\zeta}^i+\hat{\xi}_k\hat{\zeta}^k)+O(\zeta^{k+1}) \\
&= (w+\sum_{i=1}^{k-1}\xi_i\zeta^i)(\hat{w}+\sum_{i=1}^{k-1}\hat{\xi}_i\hat{\zeta}^i)+w\hat{\xi}_k\hat{\zeta}^k+\hat{w}\xi_k\zeta^k+O(\zeta^{k+1}) \qquad (3.10)\\
&= (w+\sum_{i=1}^{k-1}\xi_i\zeta^i)(\hat{w}+\sum_{i=1}^{k-1}\hat{\xi}_i\hat{\zeta}^i)+\hat{\xi}_k\zeta^k/w^{k-1}+\xi_k\zeta^k/w \\
&+ O(\zeta^{k+1}).
\end{aligned}
$$

Using the fact that $\xi_k, \hat{\xi}_k$ are determined only up to the addition of a holomorphic function in $w, \hat{w}$ respectively, it is easy to observe that the normal form for the transition function is

$$
\xi\hat{\xi} = 1+\sum_{i=4}^{k}(\sum_{j=2}^{i-2}\frac{a_{ij}}{w^j})\zeta^i+O(\zeta^{k+1}). \qquad (3.11)
$$

A similar argument for the fibre variable shows that

$$
\begin{aligned}
\hat{\rho}\xi/\rho &= \hat{\zeta}(1+\sum_{i=1}^{k}\hat{\rho}_i\hat{\zeta}^i)(w+\sum_{i=1}^{k}\xi_i\zeta^i)/\zeta(1+\sum_{i=1}^{k}\rho_i\zeta^i)+O(\zeta^{k+1}) \\
&= \hat{\zeta}(1+\sum_{i=1}^{k-1}\hat{\rho}_i\hat{\zeta}^i)(w+\sum_{i=1}^{k-1}\xi_i\zeta^i)/\zeta(1+\sum_{i=1}^{k-1}\rho_i\zeta^i)+\hat{\zeta}w\hat{\rho}_k\hat{\zeta}^k/\zeta \\
&\quad +\hat{\zeta}\xi_k\zeta^k/\zeta-\hat{\zeta}w\rho_k\zeta^k/\zeta+O(\zeta^{k+1} \qquad (3.12)\\
&= \hat{\zeta}(1+\sum_{i=1}^{k-1}\hat{\xi}_i\hat{\zeta}^i)(w+\sum_{i=1}^{k-1}\xi_i\zeta^i)/\zeta(1+\sum_{i=1}^{k-1}\xi_i\zeta^i) \\
&\quad +\frac{\hat{\rho}_k}{w^k}\zeta^k+\xi_k\zeta^k/\omega-\rho_k\zeta^k+O(\zeta^{k+1}).
\end{aligned}
$$

Using the fact that $\rho_k, \hat{\rho}_k$ are determined only up to the addition of a holomorphic function in $w, \hat{w}$ respectively, and that ξ_k has been determined above, it is easy to observe that the normal form for the transition functions is

$$\hat{\rho}\xi/\rho = 1 + \sum_{i=2}^{k}(\sum_{j=1}^{i-1} \frac{b_{ij}}{w^j})\zeta^i . \tag{3.13}$$

∎

We now recall the Morrow–Rossi invariants. In [NirSp] (see also [MR]), Nirenberg and Spencer considered the deformation theory for embedded complex submanifolds. Their results in the current context are easy to describe. Two deformations (U, ϕ_1), (U, ϕ_2) are said to be formally k^{th} order equivalent along $\mathbb{P}^1$ if there is a diffeomorphism $(U, \phi_1) \to (U, \phi_2)$ which fixes $\mathbb{P}^1$ and is holomorphic to order $(k+1)$ along $\mathbb{P}^1$. We will call a deformation (U, ϕ) k^{th} *order standard* if (U, ϕ) is k^{th} order equivalent to the undeformed U along P^1. Nirenberg and Spencer showed that the obstruction to extending a k^{th} order formal equivalence to a $(k+1)^{st}$ order equivalence lies in the first cohomology of $\mathbb{P}^1$ with values in the tangent bundle of $\mathbb{P}^2$ restricted to $\mathbb{P}^1$, twisted by the $(k+1)$ power of the dual to the hyperplane line bundle. Since the tangent bundle to $\mathbb{P}^2$ restricted to $\mathbb{P}^1$ is $E^2 \oplus E$, their results in the current context can be stated as follows.

Theorem 3.14. *(Nirenberg–Spencer* [NirSp]*) Let (U, ϕ) be a deformation of U that is $(k-1)^{st}$ order standard along $\mathbb{P}^1$. The obstruction to (U, ϕ) being k^{th} order standard lies in $H^1(\mathbb{P}^1, (E^2 \oplus E) \otimes E^{-k})$.*

We may now cast the results of Proposition 3.4 in terms of the Morrow–Rossi invariants.

Corollary 3.15. *The deformed manifold (U, ϕ) is k^{th} order standard along $\mathbb{P}^1$ if and only if the coefficients a_{ij}, b_{ij} vanish for all $j \leq k$.*

Proof. As in Morrow and Rossi [MR], one can compute the invariants by considering a coordinate cover of $\mathbb{P}^1$ and computing normalized transition functions. A straightforward comparison shows that these are the same invariants as have been calculated in the previous proposition.

∎

The next proposition relates the invariants introduced above to the stable embeddability of the new structure. That such a relationship exists is clear, but it will be convenient to indicate an explicit algorithm for the procedure.

Our approach will be to obtain a deformation of the identity embedding $(z^1, z^2) : U \hookrightarrow \mathbb{P}^2$. Notice that while the functions (z^1, z^2)

are well defined on the complement of $\mathbf{P}^1$, they extend as meromorphic functions only to $\mathbf{P}^2$ minus a point on the hyperplane at infinity. (As a map, $(z^1, z^2) : U \hookrightarrow \mathbf{P}^2$ is well defined, but the components are not defined at the points $\zeta = w = 0$ in U_1 and $\hat{\zeta} = \hat{w} = 0$ in U_2.) We can, however, treat these functions as being defined on S^3. We will look for deformed functions (σ^1, σ^2) that "agree with (z^1, z^2) on $\mathbf{P}^1$".

To explain the meaning of this statement, notice that the circular action on S^3 induces a natural Fourier decomposition on the space of functions on S^3; any function with only negative Fourier components can be extended to U as a function that is holomorphic in the fibre variable ζ; conversely, functions that are holomorphic in the fibre variable restrict to S^3 as functions with only negative Fourier components. Thus, all functions are well defined on S^3, and the vanishing of the first k negative Fourier coefficients on S^3 corresponds to the vanishing to k^{th} order along $\mathbf{P}^1$ of the extended function. Moreover, k^{th} order formal neighbourhoods correspond to functions on S^3 with negative Fourier components up to order k. Thus, whenever the extension to $\mathbf{P}^1$ comes into question, we can view the analysis as taking place on S^3.

Proposition 3.16. *Let (S^3, ϕ), (U, ϕ) be as above.*

Suppose that there exist deformations (σ^1, σ^2) of (z^1, z^2) that are meromorphic to order $(k-1)$ along $\mathbf{P}^1$ in the deformed structure defined by ϕ and agree with (z^1, z^2) along $\mathbf{P}^1$; then the Morrow Rossi invariants vanish for all $j \leq k$.

Conversely, suppose that the Morrow Rossi invariants vanish for all $j \leq k$; then there exist deformations (σ^1, σ^2) of (z^1, z^2) that are meromorphic to order $(k-1)$ along $\mathbf{P}^1$ in the deformed structure defined by ϕ and agree with (z^1, z^2) along $\mathbf{P}^1$.

Proof. Let σ^1, σ^2 be the deformations of z^1, z^2 respectively that are holomorphic to order $(k-1)$ along $\mathbf{P}^1$ relative to the deformed complex structure. That is,

$$\sigma^1 = z^1 + O(\zeta^0) \qquad \sigma^2 = z^2 + O(\zeta^0)$$

and

$$\bar{\partial}_\phi \sigma^1 = O(\zeta^k) \qquad \bar{\partial}_\phi \sigma^2 = O(\zeta^k)$$

where the orders refer to the order of vanishing along $\mathbf{P}^1$. We can define local holomorphic coordinates by (notice that, while the functions may be defined to all orders, they are only holomorphic to the indicated order along $\mathbf{P}^1$):

$$\begin{aligned} \rho &= 1/\sigma^1 + O(\zeta^{k+2}) & \xi &= \sigma^2/\sigma^1 + O(\zeta^{k+1}) & &\text{on } U_1 \\ \hat{\rho} &= 1/\sigma^2 + O(\zeta^{k+2}) & \hat{\xi} &= \sigma^1/\sigma^2 + O(\zeta^{k+1}) & &\text{on } U_2 \end{aligned}$$

and it follows automatically that

$$\xi\hat{\xi} = 1 + O(\zeta^{k+1}) \qquad \hat{\rho}\xi/\rho = 1 + O(\zeta^{k+1}) \qquad \text{on } U_1 \cap U_2 \,.$$

In particular, the Morrow Rossi invariants vanish to order k.

Conversely, suppose that

$$\rho = \zeta(1 + \sum_{i=1}^{k} \rho_i \zeta^i) + O(\zeta^{k+1}) \qquad \xi = w + \sum_{i=1}^{k} \xi_i \zeta^i + O(\zeta^{k+1}) \,.$$

Then we can define sections σ^1, σ^2 by

$$\sigma^1 = 1/\rho \qquad \sigma^2 = \xi/\rho \qquad \text{on } U_1 \tag{3.17}$$

$$\sigma^2 = 1/\hat{\rho} \qquad \sigma^1 = \hat{\xi}/\hat{\rho} \qquad \text{on } U_2 \,, \tag{3.18}$$

where σ^1 is well defined to order $n, n \leq (k-1)$ if and only if

$$\begin{aligned} & 1/\rho - \hat{\xi}/\hat{\rho} = O(\zeta^{n+1}) & &\text{on } U_1 \cap U_2 \\ \iff\quad & \rho\hat{\xi}/\hat{\rho} = 1 + O(\zeta^{n+2}) & &\text{on } U_1 \cap U_2 \,. \end{aligned} \tag{3.19}$$

Similarly, σ^2 is well defined to order n if and only if

$$\begin{aligned} & \hat{\rho}\xi/\rho = 1 + O(\zeta^{n+2}) & &\text{on } U_1 \cap U_2 \,, \\ \iff\quad & \rho/(\hat{\rho}\xi) = 1 + O(\zeta^{n+2}) & &\text{on } U_1 \cap U_2 \\ \iff\quad & (\rho\hat{\xi}/\hat{\rho}) \cdot (1/\xi\hat{\xi}) = 1 + O(\zeta^{n+2}) & &\text{on } U_1 \cap U_2 \,. \end{aligned} \tag{3.20}$$

Therefore, the pair of sections σ^1, σ^2 are well defined to order n if

$$\hat{\rho}\xi/\rho = 1 + O(\zeta^{n+2}) \qquad \xi\hat{\xi} = 1 + O(\zeta^{n+2}) \qquad \text{on } U_1 \cap U_2 \,; \tag{3.21}$$

that is, if $n \leq (k-1)$, and if the Morrow Rossi invariants vanish to order $(n+1)$. ▌

Remark 3.22. *We can expand σ_1, σ_2 in powers of ζ, and obtain*

$$\sigma^1 = 1/\rho = \frac{1}{\zeta} \frac{1}{(1 + \sum_{i=1}^{k} \rho_i \zeta^i)} = \frac{1}{\zeta}\left(1 + \sum_{i=1}^{k} \sigma_i \zeta^i\right) + O(\zeta^k) \tag{3.23}$$

and

$$(3.24)\qquad \sigma^2 = \xi/\rho = (w + \sum_{i=1}^{k} \xi_i \zeta^i)(1 + \sum_{i=1}^{k} \sigma_i \zeta^i)/\zeta + O(\zeta^k)\,.$$

Before stating the next theorem, we introduce the canonical solution operator for the $\bar\partial_b$ equation on (S^3, η). We denote it by $G\partial$, where $G\partial$ is defined by the properties (1) $G\partial\bar\partial u = u$ for all u orthogonal to the space of CR functions; (2) $G\partial\alpha = 0$ for all α orthogonal to the range of $\bar\partial$. Notice that $G\partial\bar\partial$ preserves the grading induced by the Fourier decomposition, and that the operator $G\partial$ gains one anisotropic derivative; that is, the operator $G\partial$ satisfies the regularity estimate in the Γ^s norms, $||G\partial u||_{k+1} \leq c||u||_k$ for some constant c.

In the paragraph above, a formal replacement of the operators $\bar\partial, \partial$ by $\bar\partial_b, \partial_b$ respectively expresses everything in terms of the boundary $\bar\partial$ operators; however, we choose this notation to emphasize the fact that formally, we may think in terms of the extended operators on the manifold U. Notice that while there is a formal means of passing between the two approaches, we have chosen a 'mixed' notation; that is, we compute L^2 inner products using the spherical volume form and the restriction of the functions to $S^3 = \partial U$, while we use the notation that is naturally associated with solving the $\bar\partial$ equation on U.

Theorem 3.25. *Let (S^3, ϕ), (U, ϕ) be as above. Then the deformed manifold (U, ϕ) is k^{th} order standard if and only if the functions $[\phi\partial(\sum_{j=0}^{n+1}(G\partial\phi\partial)^j z^i)]_n$, defined on S^3, are in the range of $\bar\partial_b$ for all $0 \leq n \leq k$ and $i = 1, 2$.*

Corollary 3.26. *Let (S^3, ϕ) , (U, ϕ) be as above. Then the deformed manifold (U, ϕ) is k^{th} order standard if and only if the functions $[\phi\partial(\sum_{j=0}^{k}(G\partial\phi\partial)^j z^i)]_k$, defined on S^3, are in the range of $\bar\partial_b$ for $i = 1, 2$.*

We will establish two preliminary lemmas before proving the theorem.

Lemma 3.27. *Suppose that $u = \sum_{j=-1}^{k} u_j \zeta^j$ and $v = \sum_{j=-1}^{k} v_j \zeta^j$ satisfy $\bar\partial_\phi u = O(\zeta^{k+1})$, $\bar\partial_\phi v = O(\zeta^{k+1})$, and $u_{-1} = v_{-1}$, $u_0 = v_0$; then*

$$(3.28)\qquad 1)\quad \bar\partial u_n \zeta^n = -[\bar\partial_\phi [u]_{(n-1)}]_n \quad \textit{for} \quad 0 \leq n \leq k$$

$$(3.29)\qquad 2)\quad u = v + O(\zeta^{k+1})\,.$$

Proof. The first observation follows directly from expanding the equation $[\bar\partial_\phi(u)]_k = 0$ in powers of ζ. Notice that the solution to equation 1, if it exists, is unique for $n \geq 1$, and unique up to a constant for $n = 0$.

The second statement follows by induction. It is true for $n = 0$. Assume it is true for $n-1$, $1 \le n \le k$; then for n, we have $\bar\partial(u_n\zeta^n) = -[\bar\partial_\phi[u]_{n-1}]_n = -[\bar\partial_\phi[v]_{n-1}]_n = \bar\partial(v_n\zeta^n)$. Thus, $\bar\partial(u_n\zeta^n) = \bar\partial(v_n\zeta^n)$ with $n \ge 1$ and uniqueness implies that $u_n = v_n$. ▌

Lemma 3.30. *Suppose that* $u = \sum_{j=-1}^{k} u_j\zeta^j$, $\bar\partial_\phi u = O(\zeta^{k+1})$, $u_{-1}\zeta^{-1} = z^1$ *and* $u_0 = [G\partial\phi\partial(z^1)]_0$*; then*

$$u_n\zeta^n = [\sum_{j=0}^{n+1}(G\partial\phi\partial)^j z^1]_n - [\sum_{j=0}^{n+1}(G\partial\phi\partial)^j z^1]_{n-1} \tag{3.31}$$

$$[u]_n = [\sum_{j=0}^{n+1}(G\partial\phi\partial)^j z^1]_n \tag{3.32}$$

for $1 \le n \le k$.

Proof. This also follows by induction. Suppose that the result is true for $n-1$, where $0 < n \le k$. Then

$$\begin{aligned}
\bar\partial u_n\zeta^n &= -\left[\bar\partial_\phi(\sum_{j=-1}^{n-1} u_j\zeta^j)\right]_n \\
&= -\left[(\bar\partial - \phi\partial)(\sum_{j=-1}^{n-1} u_j\zeta^j)\right]_n + \left[(\bar\partial - \phi\partial)(\sum_{j=-1}^{n-1} u_j\zeta^j)\right]_{(n-1)} \\
&= \left[\phi\partial(\sum_{j=-1}^{n-1} u_j\zeta^j)\right]_n - \left[\phi\partial(\sum_{j=-1}^{n-1} u_j\zeta^j)\right]_{(n-1)} \\
&= \left[\phi\partial(\sum_{j=0}^{n}(G\partial\phi\partial)^j z^1)\right]_n - \left[\phi\partial(\sum_{j=0}^{n}(G\partial\phi\partial)^j z^1)\right]_{n-1},
\end{aligned}$$

where we have taken advantage of the fact that $\left[\bar\partial_\phi(\sum_{j=-1}^{n-1} u_j\zeta^j)\right]_{n-1} = 0$ by adding it onto the second line.

Therefore,

$$\begin{aligned}
u_n\zeta^n &= \left[(\sum_{j=0}^{n}(G\partial\phi\partial)^{j+1} z^1)\right]_n - \left[(\sum_{j=0}^{n}(G\partial\phi\partial)^{j+1} z^1)\right]_{n-1} \\
&= \left[(\sum_{j=0}^{n+1}(G\partial\phi\partial)^{j} z^1)\right]_n - \left[(\sum_{j=0}^{n+1}(G\partial\phi\partial)^{j} z^1)\right]_{n-1}.
\end{aligned}$$

▌

We now prove the theorem.

Proof. Suppose that (σ^1, σ^2) defines a formal k^{th} order equivalence. Then $(\lambda\sigma^1 + a, \lambda\sigma^2 + b)$ also defines a k^{th} order equivalence for any constants λ, a, b. Choose a constant a such that

$$\sigma^1 + a = z^1 + G\partial\phi\partial(z^1) + O(\zeta^1);$$

then $\sigma^1 + a$ satisfies the conditions in the last lemma, and

$$[\sigma^1 + a]_k = [\sum_{j=0}^{k+1}(G\partial\phi\partial)^j z^1]_k\,.$$

Moreover, since $\sigma^1 + a$ is meromorphic to order k, it follows that for all $0 \le n \le k$, $[\bar\partial_\phi([\sigma^1 + a]_k)]_n = [\bar\partial_\phi([\sigma^1 + a]_n)]_n = 0$, whence

$$\begin{aligned} \bar\partial([\sigma^1 + a]_n) &= [\phi\partial([\sigma^1 + a]_n)]_n \\ &= [\phi\partial([\sum_{j=0}^{n+1}(G\partial\phi\partial)^j z^1]_n)]_n\,; \end{aligned}$$

in particular, $[\phi\partial([\sum_{j=0}^{n+1}(G\partial\phi\partial)^j z^1]_n)]_n$ is in the range of $\bar\partial_b$ for all $0 \le n \le k$.

Conversely, suppose that $[\phi\partial([\sum_{j=0}^{n+1}(G\partial\phi\partial)^j z^1]_n)]_n$ is in the range of $\bar\partial_b$ for all $0 \le n \le k$; then define $\sigma^1 = ([\sum_{j=0}^{k+1}(G\partial\phi\partial)^j z^1]_k)$. We calculate

$$\begin{aligned} &(3.33)\quad \bar\partial_\phi(\sigma^1) \\ &= \bar\partial([\sum_{j=0}^{k+1}(G\partial\phi\partial)^j z^1]_k) - \phi\partial([\sum_{j=0}^{k+1}(G\partial\phi\partial)^j z^1]_k) \\ &= \bar\partial G\partial[\phi\partial\sum_{j=0}^{k}(G\partial\phi\partial)^j z^1]_k - \left[\phi\partial([\sum_{j=0}^{k+1}(G\partial\phi\partial)^j z^1]_k)\right]_k + O(\zeta^{k+1}) \\ &= [\phi\partial\sum_{j=0}^{k}(G\partial\phi\partial)^j z^1]_k - [\phi\partial\sum_{j=0}^{k+1}(G\partial\phi\partial)^j z^1]_k + O(\zeta^{k+1}) \\ &= O(\zeta^{k+1})\,. \end{aligned}$$

∎

Finally, we are able to state the main theorem.

Theorem 3.34. *Let (S^3, ϕ), and (U, ϕ) be as above. Then (S^3, ϕ) is embeddable if and only (U, ϕ) is formally standard; that is, if and only if all Morrow Rossi invariants vanish.*

Proof. First notice that if (S^3, ϕ) is embeddable, then (U, ϕ) is formally standard, and hence it is formally standard for all orders k.

Conversely, suppose that (U, ϕ) is formally standard for all k. Then by the last theorem, the functions $[\phi\partial(\sum_{j=0}^{n+1}(G\partial\phi\partial)^j z^i)]_n$, defined on S^3, are in the range of $\bar{\partial}_b$ for all $0 \leq n \leq k < \infty$ and $i = 1, 2$. The formal series $\sum_{j=0}^{\infty}(G\partial\phi\partial)^j z^i$ converges by standard operator estimates. Thus, these formal series define an actual smooth equivalence that is holomorphic to all orders along $\mathbf{P}^1$; by its construction, it is a holomorphic equivalence, and it restricts to S^3 as an embedding of the deformed structure. ▌

As an immediate consequence of the last theorem, we obtain explicit obstructions to embeddability. Recall that in [B], we showed that any sufficiently small deformation tensor can be put in interior normal form—that is, it is equivalent of one of the form $\phi = \mu\bar{\omega} \otimes Z$, where $Z = \bar{z}^2\partial/\partial z^1 - \bar{z}^1\partial/\partial z^2$, $\bar{\omega} = \bar{z}^2 d\bar{z}^1 - \bar{z}^1 d\bar{z}^2$, and $\mu = \mu_- + \mu_+$ where μ_+ corresponds to the part of μ with positive Fourier components, and μ_- is of the form $\mu_- = z^1\bar{h}_1 + z^2\bar{h}_2$ for CR functions h_1, h_2. Moreover, we showed that ϕ is embeddable if and only if $\mu_- = 0$, with a rather direct construction of the embeddability in this case. The results in this paper allow us to give a direct interpretation of the obstructions to embeddability as well.

Corollary 3.35. *If in the notation above,* $\phi = \mu_-\bar{\omega} \otimes Z$*, then* ϕ *is embeddable if and only if* $\mu_- = 0$.

Proof. In the statement, the deformation tensor ϕ is in exterior form. Consequently, Theorem 3.25 applies. Suppose ϕ is embeddable. If $[\phi\partial z^i]_n = 0$ for $i = 1, 2$, then $[\phi\partial z^i]_{n+1} = [(\mu_- Z(z^i)\bar{\omega})]_{n+1}$ is in the range of $\bar{\partial}_b$ for $i = 1, 2$. In particular, for any holomorphic function H

$$\int_{S^3} [(\mu_- Z(z^i))]_{n+1} H \, d\,\mathrm{vol} = 0$$

for $i = 1, 2$. Choosing the specific holomorphic functions h_1, h_2 for two separate choices for H, we find that a necessary condition for embeddability is

$$\int_{S^3} [(\mu_- Z(z^1))]_{n+1} h_2 - [(\mu_- Z(z^2))]_{n+1} h_1 \, d\,\mathrm{vol} = 0$$

or

$$\int_{S^3} [\mu_-]_n (\bar{z}^2 h_2 + \bar{z}^1 h_1) \, d\,\mathrm{vol} = 0 \,.$$

Since $\mu_- = z^1\bar{h}_1 + z^2\bar{h}_2$, this implies that $[\mu_-]_n = 0$, and hence $[\phi\partial z^i]_{n+1} = [(\mu_- Z(z^i)\bar{\omega})]_{n+1} = 0$ for $i = 1, 2$. Thus, we are done by induction.

Conversely, if $\mu_- = 0$, then the structure is spherical, and hence embeddable. ▌

In fact, tracing through the arguments in this paper, one can identify the various terms in μ_- with nonvanishing Morrow–Rossi invariants; the non-embeddability of (S^3, ϕ) corresponds to a nontrivial twisting of the complex structure near the hyperplane at infinity.

In conclusion, we state the following infinitesimal version of the embedding result from the last section.

Theorem 3.36. *For $s \geq 6$, there is a Γ^s–neighbourhood of the standard CR structure on S^3 such that (S^3, ϕ) is embeddable if and only if some neighbourhood of $\mathbf{P}^1 \subset (U, \phi)$ is biholomorphic to a neighbourhood of the zero section of E.*

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