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Geometry of Matrices

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***In Memory of Professor L. K. Hua* (1910–1985)**

§1. Introduction

The study of the geometry of matrices was initiated by L. K. Hua in the mid forties [5–10]. At first, relating to his study of the theory of functions of several complex variables, he began studying four types of geometry of matrices over the complex field, i.e., geometries of rectangular matrices, symmetric matrices, skew-symmetric matrices, and hermitian matrices. In 1949, he [11] extended his result on the geometry of symmetric matrices over the complex field to any field of characteristic not 2, and in 1951 he [12] extended his result on the geometry of rectangular matrices to any division ring distinct from $\mathbb{F}_2$ and applied it to problems in algebra and geometry. Then the study of the geometry of matrices was succeeded by many mathematicians. In recent years it has also been applied to graph theory.

To explain the problems of the geometry of matrices we are interested in, it is better to start with the Erlangen Program which was formulated by F. Klein in 1872. It says: "A geometry is the set of properties of figures which are invariant under the nonsingular linear transformations of some group". There F. Klein pointed out the intimate relationship between geometry, group, and invariants. Then a fundamental problem in a geometry in the sense of Erlangen Program is to characterize the transformation group of the geometry by as few geometric invariants as possible. The answer to this problem is often called the fundamental theorem of the geometry.

In a geometry of matrices, the points of the associated space are a certain kind of matrices of the same size, and there is a transformation

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group acting on this space. Take the geometry of rectangular matrices as an example. Let D be a division ring, and m and n be integers ≥ 2. The *space* of the geometry of rectangular matrices over D consists of all $m \times n$ matrices over D and is denoted by $\mathcal{M}_{m\times n}(D)$. The elements of $\mathcal{M}_{m\times n}(D)$ are called the *points* of the space. $\mathcal{M}_{m\times n}(D)$ admits transformations of the following form

$$\begin{aligned} \mathcal{M}_{m\times n}(D) &\to \mathcal{M}_{m\times n}(D) \\ X &\mapsto PXQ+R, \end{aligned} \tag{1}$$

where $P \in GL_m(D)$, $Q \in GL_n(D)$, and $R \in \mathcal{M}_{m\times n}(D)$. All these transformations form a *transformation group* of $\mathcal{M}_{m\times n}(D)$, which is denoted by $G_{m\times n}(D)$. Then the geometry of rectangular matrices aims at the study of the invariants of its geometric figures (or subsets) under $G_{m\times n}(D)$. For instance, for the figure formed by two $m\times n$ matrices X_1 and X_2 over D, rank $(X_1 - X_2)$ is an invariant under $G_{m\times n}(D)$. If rank $(X_1 - X_2) = 1$, X_1 and X_2 are called *adjacent*. L. K. Hua proved that the invariant "adjacency" alone is "almost" sufficient to characterize the transformation group $G_{m\times n}(D)$ of $\mathcal{M}_{m\times n}(D)$, which will be explained in detail in the next section.

§2. Geometry of rectangular matrices

Fundamental Theorem of the Geometry of Rectangular Matrices. Let D be a division ring, m and n integers ≥ 2, $\mathcal{A}$ a bijective map from $\mathcal{M}_{m\times n}(D)$ to itself. Assume that both $\mathcal{A}$ and $\mathcal{A}^{-1}$ preserve the adjacency, i.e., for any two points X_1 and X_2 of $\mathcal{M}_{m\times n}(D)$, X_1 and X_2 are adjacent if and only if $\mathcal{A}(X_1)$ and $\mathcal{A}(X_2)$ are adjacent. Then, when $m \neq n$, $\mathcal{A}$ is of the form

$$\mathcal{A}(X) = PX^{\sigma}Q + R \text{ for all } X \in \mathcal{M}_{m\times n}(D), \tag{2}$$

where $P \in GL_m(D)$, $Q \in GL_n(D)$, $R \in \mathcal{M}_{m\times n}(D)$, σ is an automorphism of D, and X^{σ} is the matrix obtained from X by applying σ to all its entries. When $m = n$, besides (1) $\mathcal{A}$ can also be of the form

$$\mathcal{A}(X) = P\,{}^{t}(X^{\tau})\,Q + R \text{ for all } X \in \mathcal{M}_{m\times n}(D), \tag{3}$$

where P, Q, and R have the same meaning as above, and τ is an anti-automorphism of D. Conversely, both maps (2) and (3) are bijections, and they and their inverses preserve the adjacency. Q.E.D.

When $D \neq \mathbb{F}_2$, the theorem was proved by L. K. Hua [12] in 1951. The proof for the case $D = \mathbb{F}_2$ was supplemented by Z. Wan and Y.

Wang [24] in 1962. The key tool to prove this theorem is the maximal set introduced by L. K. Hua. A *maximal set* in $\mathcal{M}_{m\times n}(D)$ is a maximal set of points such that any two of them are adjacent. Thus the concept of a maximal set is actually the concept of a maximal clique appeared in graph theory twenty years later. Clearly a bijective map $\mathcal{A}$ for which both $\mathcal{A}$ and $\mathcal{A}^{-1}$ preserve the adjacency carries maximal sets into maximal sets. The main steps Hua used to prove the above theorem is as follows: First he determined the normal forms of maximal sets under $G_{m\times n}(D)$. They are

$$\left\{ \left(\begin{array}{cccc} x_{11} & x_{12} & \dots & x_{1n} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{array} \right) \middle| \, x_{11}, x_{12}, \dots, x_{1n} \in D \right\} \tag{4}$$

and

$$\left\{ \left(\begin{array}{cccc} x_{11} & 0 & \dots & 0 \\ x_{21} & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ x_{m1} & 0 & \dots & 0 \end{array} \right) \middle| \, x_{11}, x_{21}, \dots, x_{m1} \in D \right\}. \tag{5}$$

Then by defining the intersection of two maximal sets which contain two adjacent points in common to be a *line* in any one of them, he proved that $\mathcal{A}$ induces bijective maps on maximal sets, which carries lines into lines and that a line in the maximal set (4) is of the form

$$\left\{ \left(\begin{array}{cccc} ta_{11} + b_{11} & ta_{12} + b_{12} & \dots & ta_{1n} + b_{1n} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{array} \right) \middle| \, t \in D \right\}, \tag{6}$$

where $a_{11}, a_{12}, \dots, a_{1n}, b_{11}, b_{12}, \dots, b_{1n} \in D$. When $D \neq \mathbb{F}_2$, by the fundamental theorem of affine geometry, after subjecting $\mathcal{A}$ to a bijective map of the form (2) or (3) (which will be needed only when $m = n$), it can be assumed that $\mathcal{A}$ leaves both the maximal sets (4) and (5) pointwise fixed. Finally it can be proved that $\mathcal{A}$ leaves every point of $\mathcal{M}_{m\times n}(D)$ fixed.

In [12], from the above theorem L. K. Hua deduced the explicit forms of automorphisms, semi-automorphisms, Jordan automorphisms, and Lie automorphisms of the total matrix ring $\mathcal{M}_n(D)(n \geq 2)$ over D. For Jordan automorphisms it is assumed that the characteristic of D is not 2, and for Lie automorphisms it is assumed that the characteristic of D is not 2 and 3. He also deduced the fundamental theorem of the projective geometry of rectangular matrices over D (for detailed proof, cf. [17]). When D is a field, the latter was proved by W. L. Chow [2] in 1949. In 1965, S. Deng and Q. Li [3] deduced the fundamental theorem of the geometry of rectangular matrices over a field from Chow's result.

Call the points of $\mathcal{M}_{m\times n}(D)$ the *vertices* and define two vertices *adjacent* if they are adjacent points. Then a *graph* is obtained. Denote this graph by $\Gamma(\mathcal{M}_{m\times n}(D))$. Naturally, the fundamental theorem of the geometry of rectangular matrices can be interpreted as a theorem on graph automorphisms of $\Gamma(\mathcal{M}_{m\times n}(D))$ [1].

§3. Geometry of alternate matrices

In this section we assume that F is a field and n is an integer ≥ 2. Let A be an $n\times n$ matrix over F. If ${}^tA = -A$ and all entries along the main diagonal of A are 0's, then A is called an $n\times n$ *alternate matrix* over F. Denote by $\mathcal{K}_n(F)$ the set of all $n\times n$ alternate matrices over F, and call it the *space* of the geometry of $n\times n$ alternate matrices and its elements the *points*. Transformations of $\mathcal{K}_n(F)$ to itself of the following form

$$\begin{array}{rcl}\mathcal{K}_n(F) & \to & \mathcal{K}_n(F)\\ X & \mapsto & {}^tPXP+K,\end{array}\tag{7}$$

where $P\in GL_n(F)$ and $K\in\mathcal{K}_n(F)$, form a *transformation group* of $\mathcal{K}_n(F)$, denoted by $GK_n(F)$. Let X_1 and $X_2\in\mathcal{K}_n(F)$. If rank $(X_1-X_2)=2$, then X_1 and X_2 are said to be *adjacent*. Clearly, the adjacency is an invariant under $GK_n(F)$. Conversely, we have

Fundamental Theorem of the Geometry of Alternate Matrices. Let F be a field of any characteristic, n an integer ≥ 4, and $\mathcal{A}$ a bijective map from $\mathcal{K}_n(F)$ to itself. Assume that both $\mathcal{A}$ and $\mathcal{A}^{-1}$ preserve the adjacency. Then, when $n>4$, $\mathcal{A}$ is of the form

$$\mathcal{A}(X) = a\,{}^tPX^{\sigma}P+K \quad\text{for all } X\in\mathcal{K}_n(F),\tag{8}$$

where $a\in F^*$, $P\in GL_n(F)$, $K\in\mathcal{K}_n(F)$, and σ is an automorphism of F. When $n=4$, $\mathcal{A}$ is of the form

$$\mathcal{A}(X) = a\,{}^tP(X^*)^{\sigma}P+K \quad\text{for all } X\in\mathcal{K}_4(F),\tag{9}$$

where a, P, K, and σ have the same meaning as above, and $X \to X^*$ is either the identity map of $\mathcal{K}_4(F)$ or the following map

$$(10) \quad \begin{pmatrix} 0 & x_{12} & x_{13} & x_{14} \\ -x_{12} & 0 & x_{23} & x_{24} \\ -x_{13} & -x_{23} & 0 & x_{34} \\ -x_{14} & -x_{24} & -x_{34} & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & x_{12} & x_{13} & x_{23} \\ -x_{12} & 0 & x_{14} & x_{24} \\ -x_{13} & -x_{14} & 0 & x_{34} \\ -x_{23} & -x_{24} & -x_{34} & 0 \end{pmatrix}.$$

Conversely, both maps (8) and (9) are bijective, and they and their inverses preserve the adjacency. Q.E.D.

The above theorem was proved by M. Liu [16] in 1966, the proof relies also on the concept of *maximal sets.* When $F = \mathbb{C}$ and $\mathcal{A}$ satisfies further conditions, it was proved by L. K. Hua [5] in 1945. The map (10) was also discovered by L. K. Hua [5] in 1945.

This theorem has also applications to algebra and geometry [16], and can also be interpreted as a theorem on graph automorphisms [1].

§4. Geometry of symmetric matrices

In this section we assume again that F is a field and n is an integer ≥ 2. An $n \times n$ matrix S over F is called *symmetric* if ${}^tS = S$. Denote by $\mathcal{S}_n(F)$ the set of all $n \times n$ symmetric matrices over F, and call it the *space* of the geometry of $n \times n$ symmetric matrices and its elements the *points.* The set of all transformations of $\mathcal{S}_n(F)$ to itself of the form

$$(11) \quad \begin{array}{rcl} \mathcal{S}_n(F) & \to & \mathcal{S}_n(F) \\ X & \mapsto & {}^tPXP + S, \end{array}$$

where $P \in GL_n(F)$ and $S \in \mathcal{S}_n(F)$, forms a *transformation group* of $\mathcal{S}_n(F)$, denoted by $GS_n(F)$. Let $X_1, X_2 \in \mathcal{S}_n(F)$. When rank $(X_1 - X_2) = 1$, then X_1 and X_2 are said to be *adjacent.* Clearly, the adjacency of two points in $\mathcal{S}_n(F)$ is an invariant under $GS_n(F)$. Conversely, we have

Fundamental Theorem of the Geometry of Symmetric Matrices. Let F be a field of any characteristic and n be an integer ≥ 2; when F is of characteristic two and $F \neq \mathbb{F}_2$ we assume further that $n \geq 3$. Let $\mathcal{A}$ be a bijective map from $\mathcal{S}_n(F)$ to itself and assume that both $\mathcal{A}$ and $\mathcal{A}^{-1}$ preserve the adjacency. Then unless $n = 3$ and $F = \mathbb{F}_2$, $\mathcal{A}$ is of the form

$$(12) \quad \mathcal{A}(X) = a\,{}^tPX^\sigma P + S \ \text{ for all } X \in \mathcal{S}_n(F),$$

where $a \in F^*$, $P \in GL_n(F)$, $S \in \mathcal{S}_n(F)$, and σ is an automorphism of F. When $n = 3$ and $F = \mathbb{F}_2$, the bijective map

$$(13)\begin{cases} \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{12} & x_{22} & 0 \\ x_{13} & 0 & x_{33} \end{pmatrix} \mapsto \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{12} & x_{22} & 0 \\ x_{13} & 0 & x_{33} \end{pmatrix} \\ \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{12} & x_{22} & 1 \\ x_{13} & 1 & x_{33} \end{pmatrix} \mapsto \begin{pmatrix} x_{11}+1 & x_{12}+1 & x_{13}+1 \\ x_{12}+1 & x_{22} & 1 \\ x_{13}+1 & 1 & x_{33} \end{pmatrix} \end{cases}$$

from $\mathcal{S}_3(\mathbb{F}_2)$ to itself preserves also the adjacency and $\mathcal{A}$ is a product of maps of the form (12) or (13). Q.E.D.

When $F = \mathbb{C}$ and $\mathcal{A}$ satisfies further conditions, the above theorem was first proved by L. K. Hua [5] in 1945. In 1949 he [11] proved the theorem for any field of characteristic not two by the method of constructing involutions. But there are some gaps in his paper [11] which the author could not fill in. Without any restriction on the characteristic of F the author [18,19] proved the above theorem. In the proof, besides the maximal sets which were defined in the same way as in the geometry of rectangular matrices and were called the *maximal sets of rank* 1 by the author, the *maximal sets of rank* 2 were also introduced. At first, the *distance* $d(X,Y)$ between two points X and Y of $\mathcal{S}_n(F)$ is defined to be the least integer d such that there is a sequence of $d+1$ points

$$X_0 = X, X_1, X_2, \dots, X_d = Y$$

of $\mathcal{S}_n(F)$ for which any pair of consecutive points X_i and X_{i+1} $(i = 0,1,2,\dots,d-1)$ are adjacent. Assume that F is of characteristic not two. Then a subset $\mathcal{L}$ of $\mathcal{S}_n(F)$ is called a *maximal set of rank* 2 if (i) $\mathcal{L}$ contains a maximal set of rank 1, denoted by $\mathcal{M}$, (ii) for any $S \in \mathcal{L} \setminus \mathcal{M}$ and $M \in \mathcal{M}$, $d(S,M) = 2$, and (iii) for any $T \in \mathcal{S}_n(F)$, $d(T,M) = 2$ for all $M \in \mathcal{M}$ implies $T \in \mathcal{L}$. When F is characteristic two, the definition of maximal sets of rank 2 should be modified [19]. Clearly, if $\mathcal{A}$ is a bijective map of $\mathcal{S}_n(F)$ for which both $\mathcal{A}$ and $\mathcal{A}^{-1}$ preserve the adjacency, then $\mathcal{A}$ carries maximal sets of rank 1 into maximal sets of rank 1 and maximal sets of rank 2 into maximal sets of rank 2. The normal form of maximal sets of rank 1 under $GS_n(F)$ is

$$(14)\qquad \left\{ \begin{pmatrix} x & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \middle| \, x \in F \right\},$$

and the normal form of maximal sets of rank 2 under $GS_n(F)$ is

$$\left\{ \left(\begin{matrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{12} & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ x_{1n} & 0 & \dots & 0 \end{matrix} \right) \middle| \, x_{11}, x_{12}, \dots, x_{1n} \in F \right\}. \tag{15}$$

Then maximal sets of rank 2 are used in the proof of the above theorem instead of the maximal sets used in the proof of the fundamental theorem of the geometry of rectangular matrices. The case when $n = 2$, F is of characteristic two, and $F \neq \mathbb{F}_2$ still remains open.

When F is of characteristic not two, from the above theorem we can deduce the explicit form of the automorphisms of the Jordan ring of $n \times n$ symmetric matrices over F [18] and the fundamental theorem of the dual polar space of type C_n due to W. L. Chow [2] (cf. [15],[23]).

Call the points of $\mathcal{S}_n(F)$ *vertices*. Two vertices are said to be *adjacent* if they are adjacent as points. Then we obtain a *graph*, denoted by $\Gamma(\mathcal{S}_n(F))$. The fundamental theorem of the geometry of symmetric matrices can naturally be interpreted as a theorem on graph automorphisms of the graph $\Gamma(\mathcal{S}_n(F))$[18, 19].

It is interesting that when F is a finite field of characteristic not two and $n \geq 2$, and when F is a finite field of characteristic two and $n \geq 3$, besides $\Gamma(\mathcal{S}_3(\mathbb{F}_2))$, all $\Gamma(\mathcal{S}_n(F))$ are not distance-transitive. But the author [20] proved that $\Gamma(\mathcal{S}_3(\mathbb{F}_2))$ is distance-transitive, hence, distance-regular, and isomorphic to the graph of the folded 7-cube.

Now assume that F is of characteristic not two. Let $X_1, X_2 \in \mathcal{S}_n(F)$. When rank $(X_1 - X_2) = 1$ or 2, we say that X_1 and X_2 are adjacent. Then we obtain also a *graph*, denoted by $\Gamma^*(\mathcal{S}_n(F))$. From the fundamental theorem of the geometry of symmetric matrices we can deduce that the graph automorphisms of $\Gamma^*(\mathcal{S}_n(F))$ are of the form (12) (cf. [18]). When $F = \mathbb{F}_q$, the graph $\Gamma^*(\mathcal{S}_n(\mathbb{F}_q))$ was defined by Y. Egawa [4], who proved that it is distance-regular and computed its parameters.

§5. Geometry of Hermitian matrices

Let D be a division ring which possesses an involution. Denote the involution of D by $-$, i.e.,

$$\begin{aligned} - : D &\rightarrow D \\ a &\mapsto \bar{a}, \end{aligned} \tag{16}$$

is a bijective map which has the following properties: for any $a, b \in D$ we have

$$\overline{a+b} = \bar{a}+\bar{b}, \tag{17}$$
$$\overline{ab} = \bar{b}\bar{a}, \tag{18}$$

and

$$\bar{\bar{a}} = a. \tag{19}$$

Let

$$F = \{a \in D \mid \bar{a} = a\}. \tag{20}$$

Define the trace map

$$\begin{aligned} Tr : D &\rightarrow F \\ a &\mapsto a+\bar{a} \end{aligned} \tag{21}$$

and the norm map

$$\begin{aligned} N : D &\rightarrow F \\ a &\mapsto a\bar{a}. \end{aligned} \tag{22}$$

We make the following assumptions:

Assumption I F is a proper subfield of D and is contained in the center of D.

Assumption II The map Tr is surjective.

We remark that Assumption I excludes the case when D is a field and $-$ is the identity map.

Let n be an integer ≥ 2. An $n \times n$ matrix H over D is called *hermitian* if ${}^t\overline{H} = H$. The *space* of the geometry of hermitian matrices over D, denoted by $\mathcal{H}_n(D)$, is the set of all $n \times n$ hermitian matrices over D, whose elements are called the *points*. The set of transformations of $\mathcal{H}_n(D)$ to itself of the form

$$\begin{aligned} \mathcal{H}_n(D) &\rightarrow \mathcal{H}_n(D) \\ X &\mapsto {}^t\overline{P}XP+H, \end{aligned} \tag{23}$$

where $P \in GL_n(D)$ and $H \in \mathcal{H}_n(D)$, forms a *transformation group* of the space $\mathcal{H}_n(D)$, which is denoted by $GH_n(D)$. Let $X_1, X_2 \in \mathcal{H}_n(D)$. When rank $(X_1 - X_2) = 1$ then X_1 and X_2 are said to be *adjacent.* Clearly, the adjacency of two points is an invariant under $GH_n(D)$. Conversely, we have

Fundamental Theorem of the Geometry of Hermitian Matrices. Let D be a division ring which possesses an involution and assume that Assumptions I and II hold. Let n be an integer ≥ 2 and when $n = 2$ we assume that D is a field. Let $\mathcal{A}$ be a bijective map from $\mathcal{H}_n(D)$ to itself and assume that both $\mathcal{A}$ and $\mathcal{A}^{-1}$ preserve the adjacency. Then $\mathcal{A}$ is of the form

$$\mathcal{A}(X) = \alpha {}^t\overline{P} X^\sigma P + H \text{ for all } X \in \mathcal{H}_n(D), \tag{24}$$

where $\alpha \in F^*$, $P \in GL_n(D)$, $H \in \mathcal{H}_n(D)$, and σ is an automorphism of D which commutes with the involution $-$ of D. If we assume further that the norm map N is bijective, then we can assume that $\alpha = 1$. Q.E.D.

The above theorem was proved by the author [21,22] recently. In the proof, besides the maximal sets of rank 1 and rank 2, which were defined in a similar way as those in the geometry of symmetric matrices, the reduced maximal sets of rank 2 are also introduced. The normal form of maximal sets of rank 1 under $GH_n(D)$ is

$$\left\{ \left. \begin{pmatrix} x_{11} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \right| x_{11} \in F \right\}, \tag{25}$$

and the normal form of maximal sets of rank 2 under $GH_n(D)$ is

$$\left\{ \left. \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ \overline{x_{12}} & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ \overline{x_{1n}} & 0 & \dots & 0 \end{pmatrix} \right| x_{11} \in F, x_{12}, \dots, x_{1n} \in D \right\}. \tag{26}$$

If $\mathcal{M}$ is a maximal set of rank 1, then there is a unique maximal set $\mathcal{L}$ of rank 2 containing $\mathcal{M}$. For any $\mathcal{M}$ containing the zero matrix 0, $\mathcal{L}$ has an additive group structure with respect to matrix addition, $\mathcal{M}$ is its subgroup, and the set of cosets of $\mathcal{L}$ relative to $\mathcal{M}$ is called a *reduced maximal set of rank* 2. Clearly, the reduced maximal set of rank 2 from $\mathcal{L}$ are all the maximal sets of rank 1 contained in $\mathcal{L}$. Hence, if we assume that $\mathcal{A}(0) = 0$, then $\mathcal{A}$ carries reduced maximal sets of rank 2 to reduced maximal sets of rank 2. The reduced maximal sets of rank 2 are used in the proof of the above theorem when $n \geq 3$ as the maximal sets in the proof of the fundamental theorem of the geometry of rectangular matrices. When $n = 2$ and D is a field, the theorem can be proved by studying three maximal sets of rank 1 which have a nonempty intersection [22]. The case when $n = 2$ and D is not a field still remains open.

When $D = \mathbb{C}$ and $\mathcal{A}$ satisfies some other conditions, the above theorem was proved by L. K. Hua [5] in 1945. When $D = \mathbb{F}_q$, it was proved by A. A. Ivanov and S. V. Shpectorov [14] in 1991.

The above theorem has also applications to algebra [22] and geometry [23], and can also be interpreted as a theorem on graph automorphisms [22].

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