Advanced Studies in Pure Mathematics 24, 1996
Progress in Algebraic Combinatorics
pp. 309–321

Spherical Designs and Tensors

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§1. Introduction

The set X of the 12 vertices of a regular icosahedron on the unit sphere Ω in $\mathbf{R}^3$ provides a first example of a spherical 5-design (of strength 5). It satisfies

$$\frac{1}{n}\sum_{x\in X} h(x) = \int_{\Omega} h(u)\, d\sigma(u),$$

short, $\operatorname*{Ave}_X h = \operatorname*{Ave}_{\Omega} h$ for all polynomials h in 3 variables of degree ≤ 5 and $n = 12$. If the defining relation only refers to the homogeneous polynomials of degree q, then we use the term spherical design of index q. Thus strength q means index 1, 2, ..., and q.

The second part of the title refers to symmetric tensors, and to the desire to express symmetric polynomials as the inner products of tensors, for instance

$$\sum_{i,j,k=1}^{d} h_{ijk}a_i a_j a_k = \langle h, a\otimes a\otimes a\rangle,$$

where $a = (a_1, a_2, \dots, a_d) \in \mathbf{R}^d =: V$. The linear space $S^q(V)$ of the symmetric q-tensors on V is spanned by the q-fold tensor powers $\otimes^q a := a\otimes a\otimes\cdots\otimes a$. This space is isomorphic to the space $\mathrm{Hom}_q(V)$ of homogeneous polynomials of degree q in d variables.

Section 2 deals with tensors in $\mathbf{R}^d$, in particular with the distribution q-tensor D, and the Sidelnikov inequality. In Section 3 this leads to the tensor-definition of spherical designs of index q and of strength q. This notion was introduced by Delsarte, Goethals and Seidel, and was further developed by Bannai. We recall that the combinatorial t-(v,k,λ) design can be phrased in analogous terms. Generalization to t-(v,K,λ)

Received October 27, 1994.
Revised February 1, 1995.

designs with unequal block sizes from a set K suggests generalization to multispherical designs as defined in joint papers with Neumaier and with Delsarte. In the present Section 4 this is put into the natural framework of isometric linear maps from real spaces with the 2-norm into real Banach spaces with the q-norm, following Reznick and Lyubich-Vaserstein. This leads to the main Theorem 4.4 on the existence of cubature formulae, isometric embeddings, and (multi-)spherical designs of index q. These all are corollaries of Hilbert's solution of the Waring problem. Their equivalence provides a link between combinatorics and local Banach theory [LV93]

In Section 5 the notions of Eutactic star and Euclidean t-design are phrased in the present terminology. The final Section 6 comes back to spherical designs of strength t, and reviews the main results before and after the omni-existence theorem by Seymour and Zaslavsky. The paper ends with a new proof, by A. Blokhuis and the author, for the construction by Hardin and Sloane of spherical 4-designs in 3-space, by use of tensors.

§2. Symmetric q-tensors

We consider real d-dimensional Euclidean space $V = \mathbf{R}^d$ with standard basis $B = \{e_1, \dots, e_d\}$. Vectors $a \in V$ are indicated by their coordinates $(a_1, \dots, a_d)$ with respect to B, their standard inner product by $\langle a, b \rangle$, and their q-norm by $\|a\|_q$. The coordinates of the q-fold tensor power $\otimes^q a = a \otimes \cdots \otimes a$, are the monomials of degree q in $a_1, \dots, a_d$. The space $S^q(V)$ of the *symmetric q-tensors* over V is spanned by the tensor powers $\otimes^q a$, $a \in V$, and has dimension $\binom{d+q-1}{d-1}$. Each basis B defines an isomorphism of the symmetric algebra $S(V)$ with the polynomial algebra over $\mathbf{R}$ in d variables. [For these elementary facts of tensor algebra see [Sha82], Chapter 10, in particular 10.5.2 and 10.5.1]. For the inner products in $S^q(V)$ it follows that

$$\langle \otimes^q a, \otimes^q b \rangle = \langle a, b \rangle^q, \quad \langle h, \otimes^q x \rangle = h(x) \in \mathrm{Hom}_q(V),$$

for $a, b \in V$, $h \in S^q(V)$, $h(x)$ the corresponding homogeneous polynomial of degree q in d variables. From now on we fix the dimension d and the degree q. As a special symmetric q-tensor we define the *distribution q-tensor D* as follows.

Definition 2.1.

$$D := \int_\Omega \otimes^q u \, d\sigma(u).$$

Here Ω is the unit sphere in V, $d\sigma$ is the normalized standard measure on Ω, and the integral is explained in terms of coordinates. Clearly, $D = 0$ if q is odd.

Lemma 2.2.

$$\langle D, D\rangle = \delta, \quad \langle D, \otimes^q a\rangle = \delta \, \|a\|_2^q,$$

where $\delta = \dfrac{1 \cdot 3 \cdots (q-1)}{d(d+2)\cdots(d+q-2)}$ *if q is even, $\delta = 0$ if q is odd.*

This follows from the properties of the inner product in $S^q(V)$ referred to above, and from the well-known formula

$$\int_\Omega \langle u, v\rangle^q \, d\sigma(u) = \int_\Omega u_1^q \, d\sigma(u) = \delta.$$

Another useful formula is the inequality of *Sidelnikov* [Sid74] for a finite set $U \subset \Omega \subset V$, $|U| = n$.

Lemma 2.3.

$$0 \leq \|D - \frac{1}{n} \sum_{u \in U} \otimes^q u\|_2^2 = -\delta + \frac{1}{n^2} \sum_{u,v \in U} \langle u, v\rangle^q.$$

Proof. The square of the 2-norm of $D - \frac{1}{n}\sum_{u\in U} \otimes^q u$ is the inner product of that tensor with itself. Now evaluate and use Lemma 2.2.
Q.E.D.

§3. Designs in Euclidean space

Definition 3.1. A *spherical design* of index q is a finite subset U of size n of the unit sphere Ω in $V = \mathbf{R}^d$, such that

$$D = \frac{1}{n} \sum_{u \in U} \otimes^q u.$$

Thus, a spherical design represents an extremal case of Sidelnikov's inequality, cf. [GS81], and by Lemma 2.3

$$\frac{1}{n^2} \sum_{u,v \in U} \langle u, v\rangle^q = \delta;$$

is an equivalent definition. Also equivalent is:

$$\frac{1}{n} \sum_{u \in U} h(u) = \int_\Omega h(u) \, d\sigma(u),$$

for any homogeneous polynomial h of degree q; just take tensor inner products of both sides of 3.1 with any q-tensor h. Hence, the present definition 3.1 is equivalent to the original definition of the notion in [DGS77]. For the case of odd q the same definition is accepted with $D = 0$.

Remark. The name design is justified for the following reason. A combinatorial t-(v, k, λ) design consists of a v-set V and a collection X of k-subsets (blocks) of V such that any t-subset of V is in exactly λ blocks. Describing the blocks by v-vectors having coordinates $x_i \in \{0, 1\}$ with $\sum_{i=1}^{v} x_i = k$, we can prove [Sei90] that the definition above is equivalent to requiring that

$$\frac{1}{n} \sum_{x \in X} f(x) = \binom{v}{k}^{-1} \sum_{x \in \binom{V}{k}} f(x),$$

for all square-free monomials

$$f(x) = x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(t)}, \quad \sigma \in \mathrm{Sym}(v).$$

As a consequence, both spherical designs of strength t and combinatorial t-(v, k, λ) designs can be defined by similar formulae of the type

$$\operatorname*{Ave}_{X} h = \operatorname*{Ave}_{\Omega} h$$

the average over the vectors of the finite set X equals the average over all vectors, for given sets of test functions h. The analogy still goes further. The combinatorial t-(v, K, λ) designs with unequal block sizes from the set K serve as a model for the multispherical designs to be introduced in the next section.

§4. Linear maps subject to condition (D)

Spherical designs are often produced by linear maps of Euclidean spaces. Let

$$F : \mathbf{R}^d \to \mathbf{R}^N \qquad x \mapsto y = Fx$$

be a linear map with standard $N \times d$ matrix F. We denote again by F the set of the vectors in $\mathbf{R}^d$ which correspond to the rows of this standard matrix. We take q even and consider the following condition (D).

Definition 4.1. A linear map F from $\mathbf{R}^d$ into $\mathbf{R}^N$ satisfies *condition* (D) whenever

$$D = \delta \sum_{f \in F} \otimes^q f,$$

where D is the distribution q-tensor for $\mathbf{R}^d$.

If all row vectors f of the standard matrix F have equal length, then condition (D) says that F is proportional to a spherical design of index q in $\mathbf{R}^d$. However, if the row vectors of F are not requested to have equal length, then they are arranged in $\mathbf{R}^d$ on various concentric spheres; we then say that these vectors form a *multispherical design* of index q in $\mathbf{R}^d$, cf. [NS88, Sei90]. The tensor formula in Definition 4.1 can be phrased in different terms, yielding equivalent notions.

Theorem 4.2. *The existence of a linear map F satisfying condition* (D) *is equivalent to the existence of any of the following formulae:*

$$\text{Cubature:} \qquad \int_\Omega h(u)\, d\sigma(u) = \delta \sum_{f \in F} h(f), \quad h \in \operatorname{Hom}_q(\mathbf{R}^d).$$

$$\text{Waring:} \qquad \langle x, x\rangle^{q/2} = \|x\|_2^q = \sum_{f \in F} \langle f, x\rangle^q, \quad x \in \mathbf{R}^d.$$

$$\text{Isometry:} \qquad \|x\|_2 = \|y\|_q := \Big(\sum_{\nu=1}^{N} y_\nu^q\Big)^{1/q}, \quad x \in \mathbf{R}^d.$$

Proof. Take tensor inner products of both sides of (4.1), first with any symmetric q-tensor h, and then with any tensor power $\otimes^q x$. This gives the cubature and the Waring formulas. The last formula is a rewriting of its predecessor since the coordinates of $y = Fx$ are the inner products of x with the row vectors f of the matrix F.

Cubature formulae for homogeneous polynomials, as above, can also be expressed by use of weights for a finite number of points on the unit sphere $\Omega \subset \mathbf{R}^d$.

Waring's formulae refers to the classical problem of expressing integers as sums of q-th powers of integers. This problem was solved by Hilbert, cf. [Ell71, Rie53, Rez92].

Isometry refers to the linear map F from $\mathbf{R}^d$ with 2-norms, into $\mathbf{R}^N$ with equal q-norms, so to isometric embedding of $\mathbf{R}^d$ into $\mathbf{R}^N$. The existence of such an isometry is the $\varepsilon = 0$ case of Dvoretzky's theorem for finite dimensional Banach space, cf. [LV93, p.329], [Sei94]. Q.E.D.

The solution of Waring's problem by Hilbert (1909) was later simplified by Stridsberg (1916) and others. In the formulation of G. J. Rieger the essential *Lemma of Hilbert* reads as follows; its proof in [Rie53] covers less than 2 pagesc.

Lemma 4.3. *Given $d \in \mathbf{N}$, $q \in 2\mathbf{N}$, there exist $N \in \mathbf{N}$ and an identity*

$$\langle x, x\rangle^{q/2} = \sum_{\nu=1}^{N} r_\nu \langle a_\nu, x\rangle^q, \qquad \textit{for } x \in \mathbf{R}^d,$$

with positive rational r_ν and nonzero integral $a_{\nu,i}$.

In addition, for $N = N(d,q)$ we have, [LV93],

$$\binom{d+\frac{1}{2}q-1}{d-1} \leq N(d,q) \leq \binom{d+q-1}{d-1}.$$

As a consequence of Hilbert's Lemma 4.3, and of Theorem 4.2, we now have the following existence theorem.

Theorem 4.4. *Given $d \in \mathbf{N}$ and $q \in 2\mathbf{N}$, there exist $N \in \mathbf{N}$ and a linear $F : \mathbf{R}^d \to \mathbf{R}^N$ satisfying* (D), *hence satisfying Cubature, Waring, Isometry, and forming a multispherical design of index q.*

§5. Eutactic stars

Let G, of size n, denote a symmetric idempotent matrix, that is,

$$G^t = G = G^2.$$

Then G has the eigenvalues 1 and 0, of multiplicities d and $n-d$, say. We can write $G = HH^t$ with $H^tH = I$ of size d. Hence G is the Gram matrix of n vectors $y \in Y \subset \mathbf{R}^d$, called a *Eutactic star* Y, since its vectors are the projections into $\mathbf{R}^d$ of an orthonormal frame of a space $\mathbf{R}^n$ which contains $\mathbf{R}^d$ as a subspace, cf. [Sei76]. We show that the set Y is a multispherical design of index 2 (and of strength 2 if in addition its vectors add up to zero). Indeed,

$$\begin{aligned}\sum_{x,y\in Y} \langle x, y\rangle^2 &= \operatorname{tr} G^2 = \operatorname{tr} G = \sum_{y \in Y} \langle y, y\rangle = d = \\ &= d^2 \cdot \frac{1}{d} = (\sum_{y\in Y} \|y\|^2)^2 \int_\Omega \langle u, v\rangle^2 \, d\sigma(u),\end{aligned}$$

equivalently,

$$\sum_{y\in Y} y \otimes y = (\sum_{y\in Y} \|y\|^2) \int_\Omega u \otimes u \, d\sigma(u).$$

If G has a constant diagonal, then Y is a spherical design of index 2. If the diagonal of G is not constant, but consists of p distinct numbers, say, then Y is a multispherical design of index 2, also called a Euclidean design of index 2. Then Y has the same regularity condition as a spherical design, but its vectors are distributed over p concentric spheres. The existence of Eutactic stars is implied by the following theorem by Sofman [Sof69].

Theorem 5.1. *Symmetric idempotent matrices G exist iff*

$$\operatorname{trace} G = \operatorname{rank} G, \quad \operatorname{diag} G \geq 0.$$

In other words, stars consisting of n vectors in $\mathbf{R}^d$ at lengths $\xi_1, \ldots, \xi_n$ are eutactic iff

$$\xi_1^2 + \cdots + \xi_n^2 = d, \quad 0 \leq \xi_k \leq 1.$$

Examples of spherical 2-designs are provided by *strongly regular graphs*, since the defining equations for their adjacency matrix A define a symmetric idempotent Gram matrix G with zero row sums as follows:

$$(A - rI)(A - sI) = \mu J, \ AJ = kJ, \ G := \frac{1}{r-s}(A - sI - \frac{k-s}{n}J).$$

Likewise, the minimal idempotents of any association scheme yield spherical 2-designs, cf. [God93], Chapter 13. Some strongly regular graphs yield spherical t-designs with $t > 2$. For example, Smith graphs on 16, 112 and 162 vertices yield spherical 3-designs, and those on 27 and 275 vertices yield spherical 4-designs. In fact, any distance regular graph is represented by a spherical 3-design X in an eigenspace if and only if $\sum_{a \in X} a \otimes a \otimes a = 0$, see [CGS78], [God93, p. 275].

A straightforward generalization of a balanced eutactic star is provided by the following notion [NS88].

Definition 5.2. A *Euclidean t-design* of strength t is a finite subset Y of $\mathbf{R}^d$ subject to the conditions, for $k = 1, 2, \ldots, t$,

$$\sum_{y \in Y} \otimes^k y = (\sum_{y \in Y} \|y\|^k) \int_\Omega \otimes^k u \, d\sigma(u).$$

Similar definitions can be given for finite weighted sets (Y, w) in $\mathbf{R}^d$ and, more general, for *measures ξ of strength t* in $\mathbf{R}^d$. This last notion is defined by

$$\int_{R\Omega} p(y) \, d\xi(y) = \sum_{k=0}^{t} \int_{R\Omega} \|y\|^k \, d\xi(y) \int_\Omega p_k(x) \, d\sigma(x),$$

equivalently, see [NS88, Sei90], by

$$\int_{R\Omega} p\, d\xi = \int_{R\Omega} p\, d\xi \circ \gamma\,; \quad \gamma \in O(d).$$

Here $O(d)$ is the orthogonal group in $\mathbf{R}^d$, and

$$R\Omega := \cup_{r \in R} rS\,,\ rS = \{y \in \mathbf{R} : \langle y, y\rangle = r^2\}\,,\ r \in R \subset \mathbf{R}^+,$$

denotes *any* union of concentric spheres in $\mathbf{R}^d$, and the condition should hold for all polynomials p of degree $\leq t$, restricted to $R\Omega$:

$$p = \sum_{k=0}^{t} p_k \in \mathrm{Pol}_t(R\Omega)\,,\ p_k \in \mathrm{Hom}_k(\mathbf{R}^d).$$

Thus, a Euclidean t-design is a measure of strength t having finite support. A spherical t-design has finite support on the unit sphere, with equal weights.

It is interesting to interpret the strength t conditions

$$\int_{r\Omega} f(y)g(y)\, d\xi(y) = \int_{R\Omega} f(x)g(x)\, d\mu(x), \quad \mu = \mu(R\Omega),$$

as the equality of inner products:

$$\langle f, g\rangle_\xi = \langle f, g\rangle_{R\Omega},$$

for all polynomials f and g of degree $\leq e \leq t/2$. This is related to rotatable designs, and to Kiefer's theorem on optimal experimental designs [NS92]. This is also related to Fisher-type inequalities for Euclidean t-designs Y, [DS89]. Indeed, $\langle f, g\rangle_Y = \langle f, g\rangle_{R\Omega}$ implies the isometry

$$\mathrm{Pol}_e(Y) \cong \mathrm{Pol}_e(R\Omega),$$

and for the dimensions of these linear spaces it follows that

$$\begin{aligned} |Y| \geq \dim \mathrm{Pol}_e(R\Omega) &= \sum_{i=0}^{2p-1} \dim \mathrm{Hom}_{e-i}(\mathbf{R}^d) \\ &= \sum_{i=0}^{2p-1} \binom{d+e-i-1}{d-1}. \end{aligned}$$

For $2p \geq e+1$, and for $p = 1$, this formula reads $|Y| \geq \binom{d+e}{e}$, and $|Y| \geq \binom{d+e-1}{d-1} + \binom{d+e-2}{d-1}$ respectively. For the second case, a *tight spherical t-design* Y in $\mathbf{R}^d$ is defined by $|Y| = \binom{d+e-1}{d-1} + \binom{d+e-2}{d-1}$, for $t = 2e$, and $|Y| = 2\binom{d+e-1}{d-1}$, for $t = 2e+1$, antipodal case. We shall come back to tight spherical t-designs in the next Section 6.

§6. Spherical designs

The early constructions of spherical designs by Delsarte, Goethals, Seidel [DGS77, GS79, GS81], by Conway and Sloane [CS77], and by Bannai [Ban84b, Ban84a, Ban86] have their basis in combinatorics and in finite group theory, and use the harmonic analysis of the unit sphere. Highlights are the sphere version of Delsarte's linear programming bound (Theorem 5.10 of [DGS77]), and the proof, by analytic and number-theoretic methods by Bannai and Damerell of

Theorem 6.1. *There exist no tight spherical* $2e$*-designs in* $\mathbf{R}^d$ *for* $d \geq 3$*,* $e \geq 3$*. There exist no tight antipodal* $(2e+1)$*-design in* $\mathbf{R}^d$*, except for* $d = 24$*,* $t = 11$*.*

Tight spherical t-designs do exist for $t = 2, 3, 4, 5, 7, 11$. For details we refer to [DGS77, BD80] and [God93, Chapter 16].

In the *group case*, originated by Sobolev [Sob62], and rediscovered in the present context by Conway and Sloane [CS77], the designs are the orbits of a finite subgroup of the orthogonal group $O(d)$ in d-space. Bannai [Ban84b] introduced the notion of a *t-homogeneous* subgroup Γ of $O(d)$ by the property that for each point x of the unit sphere Ω the orbit x^Γ is a spherical t-design. This works if the coefficients $h_1, h_2 \ldots, h_t$ vanish in the Molien series of Γ:

$$\sum_{\gamma\in\Gamma} \frac{1-\lambda^2}{\det(I-\lambda\gamma)} = h_0 + h_1\lambda + \cdots + h_t\lambda^t + h_{t+1}\lambda^{t+1} + \cdots .$$

Here h_k is the dimension of the linear space of the Γ-invariant harmonic polynomials of degree k. This applies to finite reflection groups and beyond, [GS81].

Example. The icosahedral group A_5 has the Molien series

$$1 + \lambda^6 + \lambda^{10} + \cdots ,$$

hence there is a spherical 5-design for every orbit of the icosahedral group: the icosahedron itself, the dodecahedron, the icosidodecahedron, the football, etc. It is interesting to observe that a spherical 9-design is obtained as the orbit of any zero of the polynomial that spans the A_5-invariant harmonic polynomial of degree 6.

Bannai [Ban86] also introduced the notion of *rigidity*. A spherical

t-design in d-space is rigid whenever all sufficiently close spherical t-designs are equivalent under the orthogonal group $O(d)$. Bannai conjectures that for given d and t there are finitely many rigid t-designs mod $O(d)$. For rigidity and reflection groups cf. [Sal94]

Finally we mention the relations between representations of a finite group Γ and the spherical designs generated by Γ. A *representation* of Γ on a real vector space V is a homomorphism $\rho : \Gamma \to GL(V)$. The representation is irreducible if V contains no proper Γ-invariant subspace. So irreducibility means real irreducibility. The following theorem [GS79] deals with representations into the space Harm_k of the harmonic polynomials of degree k in d variables.

Theorem 6.2. *If the representations ρ_k of Γ on Harm_k are irreducible for $k \leq s$, then Γ is $2s$-homogeneous.*

The converse of this theorem was also claimed and proved in Theorem 6.7 of [GS79]. However, Bannai [Ban84b, Ban84a] convincingly demonstrated the falsity of both the statement and its proof, by counterexamples involving the unitary subgroup $U(d)$ of $O(2d)$.

In 1984 Seymour and Zaslavsky [SZ84] proved

Theorem 6.3. $\forall_d \forall_t \exists_{n_0} \forall_{n \geq n_0}$ *there exists a spherical design in $\mathbf{R}^d$ of strength t and size n.*

The proof of this existence theorem is not constructive, and the function $n_0(d,t)$ is important. It would be interesting to have a proof of this theorem that relates to Hilbert's lemma.

Meanwhile, several results have been obtained in this area about new constructions, as well as about the value of n_0, by Mimura, Wagner, Bajnok, Rabau-Bajnok, Grabner-Tichy, Korevaar-Meyers, Hardin-Sloane, and others. The first explicit construction of spherical t-designs for arbitrary t was given by Wagner [Wag91], cf. [GT91]. The explicit construction by Bajnok [Baj92] uses

$$n_0 = C(d)t^{O(d^3)}.$$

Korevaar and Meyers [KM93] show that in 3-space there exist spherical t-designs consisting of $O(t^3)$ points and conjecture $O(t^2)$ points. They think that for $\mathbf{R}^d$ the number $O(t^{d-1})$ should be possible, cf. also [Kor94], Remark 3.3.4.

Hardin and Sloane [HS92] used their computer program Gosset to find many new spherical 4-designs. This was achieved by minimizing the socalled average prediction variance $I(D)$ for designs D fitting a quadratic model in the unit ball B in d-space. Here $I(D) := \mathrm{trace}\, M_B M_D^{-1}$

for the moment matrices

$$M_D = \frac{1}{n} \sum_{x \in D} p_\mu(x) p_\nu(x)\,,\ M_B = \frac{1}{|B|} \int_B p_\mu(x) p_\nu(x)\, d\omega(x)$$

of the $\frac{1}{2}(d+1)(d+2)$ basic polynomials p_μ of degree ≤ 2 in d variables. Restricting to designs D with b points on the unit sphere Ω and c coinciding points in the origin ($b + c = n$), they found that in extremal situations for $I(D)$ the b points on Ω form a spherical 4-design. They conjecture that, for $d \leq 8$, the list of their findings is complete. Thus, even in ordinary 3-space many new spherical t-designs have been discovered, cf. also [Rez95] and [HS95].

Hardin and Sloane found infinitely many distinct spherical 4-designs on 12 points in 3-space by rotating the northern hemisphere of a regular icosahedron about a diameter NS over an arbitrary angle, cf. also [Sal94]. A. Blokhuis and the author have the following independent proof by tensors: Let $X = N \cup P \cup Q \cup S$, with poles N, S and planar regular pentagons P and Q, denote the vertex set of a regular icosahedron. Let $P(\phi)$ denote the pentagon obtained from P by a rotation ϕ about NS. Decompose its vectors in and orthogonal to the equator plane following $x_i(\phi) = p_i(\phi) + u$, and observe that

$$\otimes^s (p_i(\phi) + u) = \sum_{a+b=s} \binom{s}{a} (\otimes^a p_i(\phi)) \otimes (\otimes^b u).$$

Since $p_i(\phi)$ form a spherical 4-design we know that $\sum_{i=1}^5 \otimes^a p_i(\phi)$ is independent of $\phi \in O(2)$, for $a = 1, 2, 3, 4$. Therefore, the half-rotated $N \cup P(\phi) \cup Q \cup S$ has sums of the a-th tensor products independent of ϕ, hence equal to the a-th tensor products of the icosahedron. Since the regular icosahedron is a spherical 5-design, the half-rotated icosahedron is a spherical 4-design, for any $\phi \in O(2)$.

References

[Baj92] B. Bajnok, Construction of spherical t-designs, Geom. Dedicata, **43** (1992), 167–179.

[Ban84a] E. Bannai, Spherical designs and group representations, Contemp. Math., **34** (1984), 95–107.

[Ban84b] E. Bannai, Spherical t-designs which are orbits of finite groups, J. Math. Soc. Japan, **36** (1984), 341–354.

[Ban86] E. Bannai, On extremal finite sets in the sphere and other metric spaces, London Math. Soc. Lecture Note Ser., **131** (1986), 13–38.

[BD80] E. Bannai and R. M. Damerell, Tight spherical designs II, J. London Math. Soc., **21** (1980), 13–30.

[CGS78] P. J. Cameron, J.-M. Goethals, and J. J. Seidel, The Krein condition, spherical designs, Norton algebras and permutation groups, Proc. Konink. Nederl. Akad. Wetensch. A, **81** (= Indag. Math., **40**) (1978), 196–206.

[CS77] J. H. Conway and N. J. A. Sloane, personal communication, 1977.

[DGS77] P. Delsarte, J.-M. Goethals, and J. J. Seidel, Spherical codes and designs, Geom. Dedicata, **6** (1977), 363–388.

[DS89] P. Delsarte and J. J. Seidel, Fisher-type inequalities for Euclidean t-designs, Linear Algebra Appl., **114/115** (1989), 213–230.

[Ell71] W. J. Ellison, Waring's problem, Amer. Math. Monthly, **78** (1971), 10–36.

[God93] C. D. Godsil, "Algebraic Combinatorics", Chapman-Hall, 1993.

[GS79] J.-M. Goethals and J. J. Seidel, Spherical designs, Proc. Sympos. Pure Math. (AMS), **34** (1979), 255–272.

[GS81] J.-M. Goethals and J. J. Seidel, Cubature formulae, polytopes and spherical designs, in "The Geometric Vein, Coxeter Festschrift", (B. Grünbaum C. Davis and F. A. Sherk, ed.), Springer, 1981, pp. 203–218.

[GT91] P. J. Grabner and R. F. Tichy, Spherical designs, discrepancy and numerical integration, Math. Comp., **60** (1991), 327–336.

[HS92] R. H. Hardin and N. J. A. Sloane, New spherical 4-designs, Discrete Math., **106/107** (1992), 255–264.

[HS95] R. H. Hardin and N. J. A. Sloane, An improved snub cube and other new spherical designs in three dimensions, Discrete Comput. Geom., 1995, to appear.

[KM93] J. Korevaar and J. L. H. Meyers, Spherical Faraday cage for the case of equal point charges and Chebyshev-type quadrature, J. of Integral Transforms and Special Functions, **1** (1993), 105–117.

[Kor94] J. Korevaar, Chebyshev-type quadratures: use of complex analysis and potential theory, notes by A. B. J. Kuijlaars, in "Complex Potential Theory" (P. M. Gauthier, ed.), Kluwer, 1994, pp. 325–364.

[LV93] Y. I. Lyubich and L. N. Vaserstein, Isometric embeddings between classical Banach spaces, cubature formulas and spherical designs, Geom. Dedicata, **47** (1993), 327–362.

[NS88] A. Neumaier and J. J. Seidel, Discrete measures for spherical designs, eutactic stars and lattices, Proc. Konink. Nederl. Akad. Wetensch. A, **91** (= Indag. Math. **50**) (1988), 321–334.

[NS92] A. Neumaier and J. J. Seidel, Measures of strength $2e$ and optimal designs of degree e, Sankyā, **54** (1991), 299–309.

[Rez92] B. Reznick, Sums of even powers of real linear forms, Mem. Amer. Math. Soc., **96** (1992), no. 463.

[Rez95] B. Reznick, Some constructions of spherical 5-designs, Linear Algebra Appl., **226-228** (1995), 163–196.

[Rie53] G. J. Rieger, Zur Hilbertschen Lösung des Waringschen Problems: Abschätzung von $g(n)$. Arch. Math., **4** (1953), 275–281.

[Sal94] A. Sali, On the rigidity of spherical t-designs that are orbits of finite reflection groups, Des. Codes Cryptogr., **4** (1994), 157–170.

[Sei76] J. J. Seidel, Eutactic stars, in "Combinatorics", (Hajnal A. and V. T. Sós, ed.), North-Holland, 1976, pp. 983–999.

[Sei90] J. J. Seidel, Designs and approximation, Contemp. Math. (AMS), **111** (1990), 179–186.

[Sei94] J. J. Seidel, Isometric embeddings and geometric designs, Discrete Math., **136** (1994), 281–293.

[Sha82] R. Shaw, "Linear Algebra and Group Representations, volume II", Academic Press, 1982.

[Sid74] V. M. Sidelnikov, New bounds for the density of sphere packings in an n-dimensional euclidean space, Mat. Sb., **95** (1974); English transl. Math. USSR Sbornik, **24** (1974), 147–157.

[Sob62] S. L. Sobolev, The formulas of mechanical cubature on a sphere, Sibirsk Mat. Zh., **3** (1962), 769–796.

[Sof69] L. B. Sofman, Diameters of octahedra, Math. Notes, **5** (1969), 258–262.

[SZ84] P. D. Seymour and T. Zaslavsky, Averaging sets, Adv. Math., **52** (1984), 213–240.

[Wag91] G. Wagner, On averaging sets, Monatsh. Math., **111** (1991), 69–78.

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