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Towards a Classification of Spin Models in terms of Association Schemes

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§1. Introduction

The *spin models* considered here have been introduced by V. Jones [Jo] (in the symmetric case) and by Kawagoe, Munemasa, Watatani [KMW] (in the general case) as basic data for a certain construction of invariants of links in 3-space. Such a spin model consists in a pair of square matrices satisfying some constraints which we call *invariance equations.* Links are represented by plane diagrams and the matrices of the spin model are used to assign to every such diagram a number (this number is the value of the *partition function*). The invariance equations represent sufficient conditions on the matrices of the spin model which insure that the partition function (multiplied by a suitable normalization factor) is invariant under simple deformations of diagrams called *Reidemeister moves.* These moves describe in terms of diagrams the natural topological equivalence of links, and hence the partition function of every spin model defines a link invariant.

The pioneering work of Jones gave two examples of symmetric spin models and raised the question of finding new ones. It turned out that this question is intimately related with the theory of association schemes. Indeed many subsequent works (in particular [Ja1], [B3], [BB1], [BB2], [BBIK], [BBJ], [BJS], [I1], [I2], [I3], [Ja3], [N1], [N2], [N3]) confirmed the importance of the following situation: the matrices of a spin model belong to the Bose-Mesner algebra of some self-dual association scheme, and can be obtained by solving a certain *modular invariance equation* associated with the (suitably indexed) first eigenmatrix of the scheme (more details can be found in the surveys [B2], [Ja2], [Ja4]). The main purpose of the present paper is to show in the symmetric case that actually this situation is completely general.

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The idea of the proof is to associate with every symmetric spin model a matrix-valued partition function Z defined on plane tangle diagrams with four ends. The required Bose-Mesner algebra will be the linear span of the image of Z (which we call the tangle closure of the spin model). The fact that this is indeed a self-dual Bose-Mesner algebra is a consequence of the following properties: Z converts the "vertical" and "horizontal" products of tangles into the ordinary and Hadamard products of matrices, and converts the rotation through angle $\frac{\pi}{2}$ into a duality map. The modular invariance equation is also an algebraic translation via Z of a simple topological equivalence of tangles.

Shortly after we obtained this "topological" proof, K. Nomura found a simpler and purely algebraic proof [N4]. This proof was then extended in [JMN] to non symmetric spin models. We believe that our topological proof is still of interest, because it gives some topological insight on the relationship between spin models and association schemes. Actually we indicate in Section 5.3 how Nomura's proof could be interpreted topologically. The same thing would be possible for most of the results in [JMN]. However for the sake of simplicity we have chosen to restrict the present study to the case of symmetric spin models.

The paper is organized as follows. Section 2 gives the necessary preliminaries on association schemes and spin models. Section 3 introduces tangles and their relevant properties. Section 4 presents our main results, Theorem A (on the tangle closure of a symmetric spin model), Theorem B (on a Bose-Mesner subalgebra of the tangle closure which has an explicit algebraic description and which we call the algebraic closure of the spin model), and Theorem C which deals with the modular invariance equation. Section 5 gives some additional results: the Bose-Mesner algebras of Theorems A and B satisfy the *planar duality property* introduced in [Ja3]; the link invariants associated with spin models are invariant under *mutation* of links; the algebras of Theorems A and B are contained in Nomura's algebra. Finally we conclude in Section 6 with some remarks and open questions.

This work was done while the author had a visiting research position at RIMS at the invitation of the project "Algebraic Combinatorics 94". I would like to express my warmest gratitude to the organizers of this project and to all the people who made my stay at RIMS enjoyable and fruitful, and in particular to Eiichi Bannai, Kyoji Saito, Bill Kantor, Viakhalatur Sunder, Makoto Matsumoto, Akihiro Munemasa. I would also like to thank R. Bacher, P. de la Harpe and V. Jones for very helpful conversations; their use of diagrams and partition functions to describe some algebraic properties of spin models (see [BHJ]) had a strong influence on the present work.

§2. Association schemes and spin models for link invariants

2.1. Symmetric spin models on link diagrams

A (tame) *link* consists of a finite collection of disjoint simple closed smooth curves (the *components* of the link) embedded in 3-space. If each component has received an orientation, the link is said to be *oriented.* (Oriented) links can be represented by (*oriented*) *diagrams.* A diagram of a link is a generic plane projection (there is only a finite number of multiple points, each of which is a simple crossing), together with a "marking" at each crossing to indicate which part of the link goes over the other. For oriented links, the edges of the diagrams are oriented according to the orientations of the corresponding link components.

Two links are *ambient isotopic* if there exists a (smooth) isotopy of the ambient 3-space which carries one onto the other (for oriented links, this isotopy must preserve the orientations). This natural equivalence of links is described in terms of diagrams by Reidemeister's Theorem, which asserts that two diagrams represent ambient isotopic links if and only if one can be obtained from the other by a finite sequence of elementary local transformations, the *Reidemeister moves.* These moves belong to three basic types described for the unoriented case in Figure 1.

A move is performed by replacing a part of diagram which is one of the configurations of Figure 1 by an equivalent configuration without

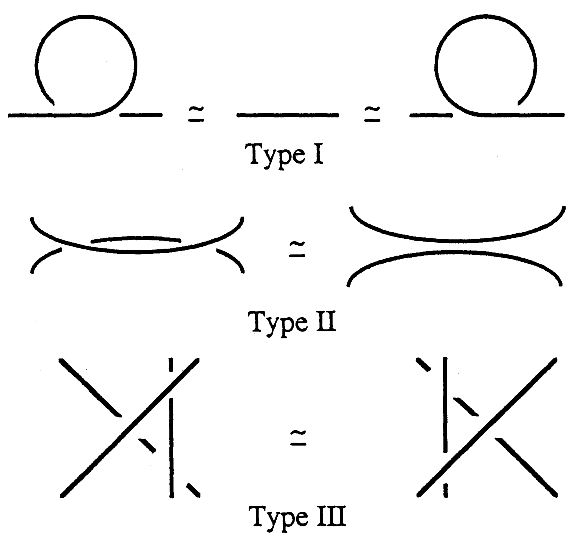
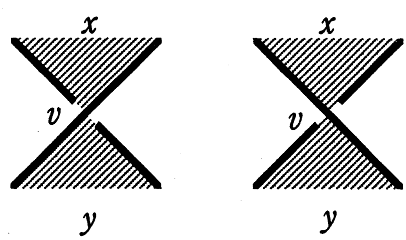


Fig. 1

modifying the remaining part of the diagram. For the oriented case, all local orientations of these pairs of equivalent configurations must be considered. More details can be found for instance in [BZ] or [K].

Reidemeister's Theorem allows the combinatorial definition of a link invariant as an assignment of values to diagrams such that the value of any diagram is preserved by Reidemeister moves performed on this diagram. It is shown in [Jo] that the concept of *spin model*, which plays an important role in statistical mechanics, can be used to obtain such invariants. This idea can be briefly described as follows.

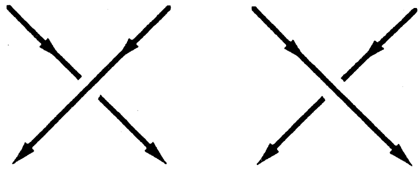
Let X be a finite set of "spins". Given a connected diagram L, we color the faces of the diagram (i.e., the connected components of the complement of L in the plane) with two colors, black and white, in such a way that adjacent faces receive different colors (we then obtain a *face-colored diagram*). Let $B(L)$ be the set of black faces. The mappings from $B(L)$ to X are called *states.* Every state σ determines at every crossing v a *local weight* $< \sigma, v >$. The local weight $< \sigma, v >$ is of the form $W_{\pm}[x, y]$, where x, y are the values of σ on the two black faces incident to v, and W_+, W_- are complex matrices with rows and columns indexed by X. The choice between these two matrices is made as shown on Figure 2. To insure that $< \sigma, v >$ is well defined we assume that W_+, W_- are symmetric (in the general case we should take into account some orientation of the diagram as in [KMW], but this case will not be considered here). The *weight* of the state σ is the product over all crossings v of the local weights $< \sigma, v >$, and the *partition function* is the sum of weights of all states.



$< \sigma, v >= W_+[x, y]$ $\qquad$ $< \sigma, v >= W_-[x, y]$

Fig. 2

Our aim is to obtain an invariant of links from the partition function. For this it is actually necessary to introduce an orientation of L and multiply the partition function by some *normalization factor* which depends in part on this orientation. The *Tait number* (or *writhe*) $T(L)$



+1 −1

Fig. 3

of an oriented diagram L is the sum of signs of its crossings, where the sign of a crossing is defined on Figure 3. Let q be a square root of $|X|$ and let a be some non-zero complex number. The *normalized partition function* is $Z(L) = a^{-T(L)}q^{-|B(L)|}Z'(L)$, where $Z'(L)$ denotes the partition function. Then it is not difficult to derive (from a simple analysis of the effect of Reidemeister moves on the states and their weights) some conditions on W_+, W_- which guarantee that the normalized partition function defines an invariant of oriented links (see [Jo]). We shall call these conditions *invariance equations.*

Before writing down these equations we need the following notations. We denote by I the identity matrix and by J the all-one matrix of appropriate sizes. $\mathcal{M}(X)$ is the set of complex matrices with rows and columns indexed by X. The Hadamard product of two matrices A, B in $\mathcal{M}(X)$ is denoted by $A \circ B$ and given by $(A \circ B)[x, y] = A[x, y]B[x, y]$.

The invariance equations are

(1) $$I \circ W_+ = aI, \qquad I \circ W_- = a^{-1}I;$$

(2) $$JW_+ = W_+J = qa^{-1}J, \qquad JW_- = W_-J = qaJ;$$

(3) $$W_+W_- = |X|I;$$

(4) $$W_+ \circ W_- = J;$$

(5) (Star-triangle equation) for every α, β, γ in X,

$$\sum_{x \in X} W_+[\alpha, x]W_+[\beta, x]W_-[\gamma, x] = qW_+[\alpha, \beta]W_-[\beta, \gamma]W_-[\gamma, \alpha].$$

We shall call a *symmetric spin model* a 5-tuple (X, W_+, W_-, a, q), where $a \neq 0$, $q^2 = |X|$, and W_+, W_- are symmetric matrices in $\mathcal{M}(X)$ which satisfy properties (1) to (5).

We shall need in the sequel a variation of the normalization factor which does not require the link diagram L to be connected. By adding a point at infinity in the unbounded face we now view L as drawn on the sphere. We denote by $\chi_b(L)$ the Euler characteristic of the union of the black faces of L. In general, if a surface is decomposed into v vertices, e edges (each of which is homeomorphic to an open segment) which connect these vertices, and r regions (each of which is homeomorphic to an open disk), its Euler characteristic is equal to $v-e+r$. Note that if L is connected, each black face is an open disk and hence $\chi_b(L) = |B(L)|$. Otherwise some "holes" may appear in the black faces, and each such hole contributes -1 to $\chi_b(L)$. It is easy to see that, if the invariance equations hold, the expression $Z(L) = a^{-T(L)}q^{-\chi_b(L)}Z'(L)$ defines the same link invariant as before.

2.2. Symmetric association schemes

We shall need the following basic facts concerning symmetric association schemes (see [BI], [D], [BCN] for more details).

A *d-class symmetric association scheme* on the finite non-empty set X is a partition of $X\times X$ into $d+1$ non-empty symmetric relations R_i, $i = 0,\dots,d$, where $R_0 = \{(x,x)|\ x\in X\}$, which satisfies the following property:

(6) For x, y in X, the number of elements z which satisfy given scheme relations with x and y only depends on which scheme relation is satisfied by the pair (x,y). That is, for every i,j,k in $0,\dots,d$ there exists an integer p_{ij}^k such that

$$|\{z\in X|\ (z,x)\in R_i, (z,y)\in R_j\}| = p_{ij}^k$$

for every x, y in X with (x,y) in R_k.

Define matrices A_i, $i = 0,\dots,d$, in $\mathcal{M}(X)$ by

(7) $\qquad A_i[x,y]$ equals 1 if $(x,y)\in R_i$, and equals 0 otherwise.

The above properties can then be reformulated as follows:

(8) $\qquad A_i^t = A_i \qquad$ (where A_i^t denotes the transpose of A_i);

(9) $$A_i\neq 0,\ A_i\circ A_j = \delta(i,j)A_i;$$

(10) $$A_0 = I;$$

(11) $$\sum_{i=0,\dots,d} A_i = J;$$

(12) $$A_iA_j = A_jA_i = \sum_{k=0,\dots,d} p_{ij}^k A_k.$$

Let $\mathcal{A}$ be the subspace of the complex vector space $\mathcal{M}(X)$ spanned by the matrices A_i, $i = 0,\dots,d$. By (9) these matrices are linearly independent and hence form a basis of $\mathcal{A}$. Then (9) and (11) imply that, under Hadamard product, $\mathcal{A}$ is an associative commutative algebra with unit J, and $\{A_i,\ i = 0, \ldots, d\}$ is a basis of orthogonal idempotents of this algebra. Moreover by (8) $\mathcal{A}$ consists of symmetric matrices. Finally it follows from (10) and (12) that under ordinary matrix product $\mathcal{A}$ is also an associative commutative algebra with unit I. The subspace $\mathcal{A}$ of $\mathcal{M}(X)$ is called the *Bose-Mesner algebra* [BM] of the association scheme. Conversely, a $(d+1)$-dimensional subspace of $\mathcal{M}(X)$ which contains I, J, consists of symmetric matrices, and is closed under Hadamard product and ordinary matrix product, is the Bose-Mesner algebra of some symmetric d-class association scheme (see [BCN], Th. 2.6.1). Such a subspace will be called here a *symmetric Bose-Mesner algebra* on X.

A *duality* of a symmetric Bose-Mesner algebra $\mathcal{A}$ on X is a linear map Ψ from $\mathcal{A}$ to itself which satisfies the following properties:

(13) For every matrix M in $\mathcal{A}$, $\quad \Psi(\Psi(M)) = |X|M;$

(14) For any two matrices M, N in $\mathcal{A}$, $\quad \Psi(MN) = \Psi(M)\circ\Psi(N).$

It easily follows that $\Psi(I) = J$ and $\Psi(J) = |X|I$.

A symmetric Bose-Mesner algebra will be called *self-dual* if it admits a duality.

§3. Tangles

Tangles and tangle diagrams are classical objects in knot theory (see for instance [C]) which also play an important role in the study of quantum invariants (see for instance [Tur]). Most of the content of this section is well known, but some of our definitions and notations are not standard.

3.1. Tangles and tangle diagrams

In what follows the Euclidean 3-dimensional space $\mathbb{R}^3$ is defined by Cartesian coordinates x, y, z.

Let S be the square $[-1,1]\times[-1,1]$ in the (x,y)-plane and let $B = S\times\mathbb{R}$. We shall call an *S-tangle* any subset T of B which is the disjoint union of a finite collection of simple smooth curves (the *components* of T) and which satisfies the following condition:

(15) $$\partial T = T\cap\partial B = \{\text{NE,NW,SW,SE}\},$$

where NE$=(1,1,0)$, NW$=(-1,1,0)$, SW$=(-1,-1,0)$, SE$=(1,-1,0)$.

Thus exactly two components of T are homeomorphic to a closed interval, while every other component is homeomorphic to a circle.

An S-tangle is *oriented* if each of its components has received an orientation.

The notions and properties which we now introduce for S-tangles are immediate extensions of classical notions and properties for knots and links (see for instance [BZ], [K]).

Two S-tangles T_0, T_1 are *isotopic* if there exists a (smooth) ambient isotopy $H : B\times[0,1]\to B\times[0,1]$, $H(b,t) = (h_t(b),t)$, such that each h_t fixes ∂B pointwise, h_0 is the identity map, and $h_1(T_0) = T_1$. For oriented S-tangles it is also required that H preserves the orientation of each component.

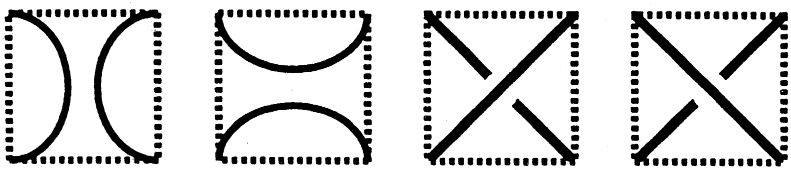
We denote by $\mathcal{T}$ (respectively: $\mathcal{OT}$) the set of isotopy classes of S-tangles (respectively: oriented S-tangles).

Isotopy classes of S-tangles can be represented by diagrams, which we shall call *S-diagrams*, as follows.

Given an element T of $\mathcal{T}$ or $\mathcal{OT}$, we may choose an S-tangle in the isotopy class T which lies in general position with respect to the (x,y)-plane, so that its projection onto that plane is generic. This means that there is only a finite number of multiple points, each of which is a simple crossing. Thus we may consider this projection as a finite graph D embedded in the square S (identified with $S\times 0$). The graph D lies in the interior of S except for four vertices of degree 1 situated at the points NE, NW, SW, and SE. The other vertices have degree 4 and will be provided with a marking indicating in the usual fashion the spatial structure of the corresponding crossings. The resulting marked graph embedded in S will be called an S-diagram of T. If T belongs to $\mathcal{OT}$, each edge of the S-diagram will receive the orientation inherited from the corresponding S-tangle component, and the S-diagram will be said to be oriented. S-diagrams will be considered up to isomorphisms of plane graphs which preserve markings, the above mentioned properties of the embeddings in S, and, in the case of oriented S-diagrams, orientations. Let $\mathcal{D}$ (respectively: $\mathcal{OD}$) be the set of S-diagrams (respectively: oriented S-diagrams). Clearly any S-diagram of T uniquely determines T. Hence we may define a map θ from $\mathcal{D}$ onto $\mathcal{T}$ and a map also denoted by θ from $\mathcal{OD}$ onto $\mathcal{OT}$ such that, for every (possibly oriented) S-diagram D, $\theta(D)$ is the isotopy class of S-tangles represented by D.

Figure 4 displays the basic S-diagrams X_0, X_∞, X_+, X_-. Here and in other pictures of S-diagrams, the x-axis is horizontal and the y-axis is vertical.

Reidemeister moves for S-diagrams are defined exactly as for link diagrams: it is enough to require that all configurations appearing in Figure 1 (or in its oriented versions) lie in the interior of S. Let us say that two S-diagrams (respectively: oriented S-diagrams) D_1, D_2 are



X_0 X_∞ X_+ X_-

Fig. 4

Reidemeister equivalent (this will be written $D_1 =_R D_2$) if it is possible to transform one into the other by a finite sequence of Reidemeister moves (respectively: oriented Reidemeister moves). Then Reidemeister's Theorem for S-tangles (see for instance Section 3.5 of [Tur]) states that two S-diagrams D_1, D_2 represent the same element of $\mathcal{T}$ (or the same element of $\mathcal{OT}$) if and only if they are Reidemeister equivalent. We record this as:

$$(16) \qquad D_1 =_R D_2 \text{ if and only if } \theta(D_1) = \theta(D_2)$$

whenever D_1 and D_2 are two elements of $\mathcal{D}$, or two elements of $\mathcal{OD}$.

In the sequel we shall mainly work with S-diagrams. However the interpretation (16) of Reidemeister equivalence in terms of S-tangles via the map θ will be crucial for the proof of the Main Lemma of Section 3.3.

3.2. Products

Let T_1, T_2 be elements of $\mathcal{T}$. Intuitively speaking, their *vertical product* $T_1 \div T_2$ will be obtained by placing T_1 above T_2 and gluing them together. More precisely, we translate an S-tangle of type T_1 by the vector $(0,1,0)$, we translate an S-tangle of type T_2 by the vector $(0,-1,0)$, and we apply to the union the transformation $(x,y,z) \to (x,y/2,z)$. We may then use an ambient isotopy of B to deform the resulting object into an S-tangle (we only need to move the two points of intersection with the x-axis towards the interior of B and realize smoothness at these points). Clearly the resulting S-tangle only depends on the isotopy classes T_1, T_2 and we denote by $T_1 \div T_2$ its isotopy class.

We define similarly the *horizontal product* $T_1 \# T_2$ which is obtained by placing T_1 to the left of T_2 and gluing them together.

We also define in an analogous way the vertical product $D_1 \div D_2$ and the horizontal product $D_1 \# D_2$ of two S-diagrams D_1, D_2 in $\mathcal{D}$ (see

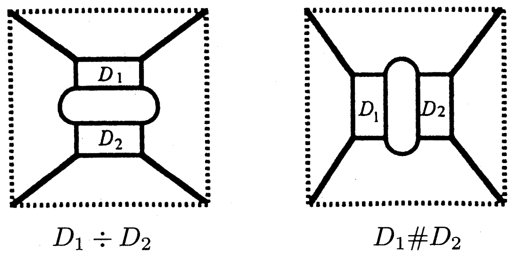


Fig. 5

Figure 5). Then

$$\begin{aligned}&\theta(D_1 \div D_2) = \theta(D_1) \div \theta(D_2) \text{ and}\\ &\theta(D_1 \# D_2) = \theta(D_1) \# \theta(D_2) \text{ for all } D_1,\ D_2 \text{ in } \mathcal{D}.\end{aligned} \tag{17}$$

Remark. It is clear that both products $\div$ and $\#$ on $\mathcal{D}$ are associative. The vertical product has unity element X_0 and the horizontal product has unity element X_∞ (see Figure 4). Similar properties hold for $\mathcal{T}$.

3.3. Transformations and identities

We shall need the following notations for some transformations of $\mathbb{R}^3$: ρ is the rotation about the z-axis through angle $\pi/2$ such that $\rho((1,0,0)) = (0,1,0)$, α is the rotation about the x-axis through angle π, and β is the rotation about the y-axis through angle π.

These smooth transformations preserve B, ∂B and $\{\mathrm{NE}, \mathrm{NW}, \mathrm{SW}, \mathrm{SE}\}$. Hence the image of an S-tangle under one of these transformations is also an S-tangle. Since the images of two isotopic S-tangles are also isotopic, this defines actions of ρ, α, β on $\mathcal{T}$ and on $\mathcal{OT}$.

Similarly, since the transformations ρ, α, β preserve S, they act naturally on S-diagrams. Their action on the underlying (possibly oriented) graphs is clear. Their action on markings will be determined by the following requirements:

(18) for every S-diagram D, $\theta(\rho(D)) = \rho(\theta(D))$ and similarly when ρ is replaced by α or β.

Examples of the action of ρ, α, β on $\mathcal{D}$ are displayed on Figure 6.

The following identities are immediate from Figure 4:

$$\rho(X_0) = X_\infty,\ \rho(X_\infty) = X_0,\ \rho(X_+) = X_-,\ \rho(X_-) = X_+. \tag{19}$$

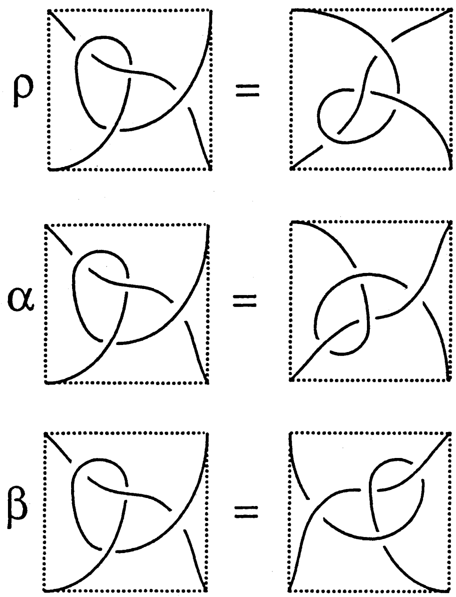


Fig. 6

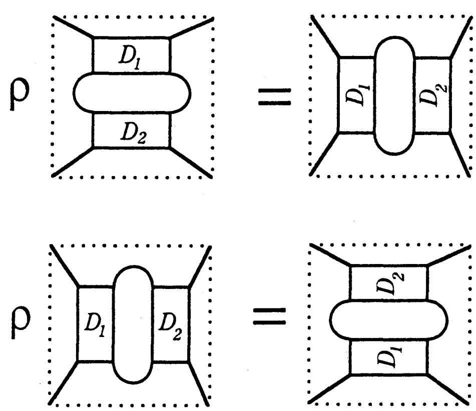


Fig. 7

The following identities are immediate from Figure 7:

(20) $\rho(D_1 \div D_2) = \rho(D_1) \# \rho(D_2)$ for all D_1, D_2 in $\mathcal{D}$,

(21) $\rho(D_1 \# D_2) = \rho(D_2) \div \rho(D_1)$ for all D_1, D_2 in $\mathcal{D}$.

Finally the following result will be essential in the sequel.

Main Lemma.

(i) $(X_+ \div (X_- \# D)) \# X_- =_R \rho(D)$ *for every* D *in* $\mathcal{D}$.

(ii) $(X_- \div (X_+ \# D)) \# X_+ =_R \rho(D)$ *for every* D *in* $\mathcal{D}$.

Proof. (i) By (16), (17), (18), this is equivalent to the equality

$$(\theta(X_+) \div (\theta(X_-) \# T)) \# \theta(X_-) = \rho(T) \qquad \text{for every } T \text{ in } \mathcal{T}.$$

This equality is proved on Figure 8.

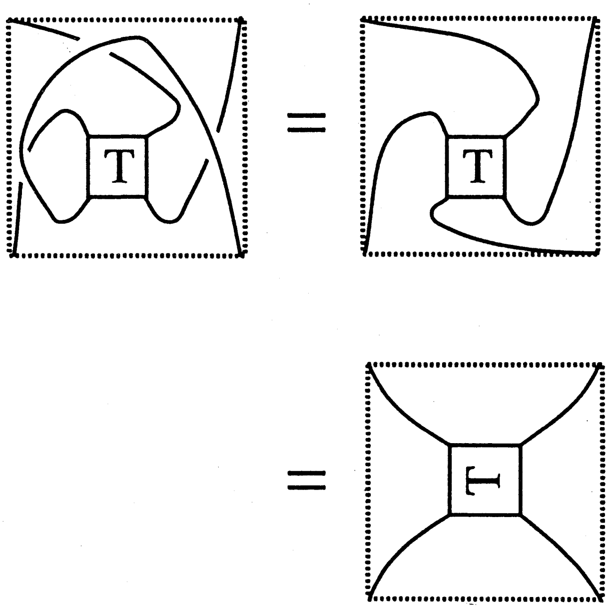


Fig. 8

The proof for (ii) is obtained from the proof for (i) by reflection in the (x, y)-plane (this is the *mirror image operation*). Q.E.D.

§4. Bose-Mesner algebras associated with symmetric spin models

4.1. The matrix-valued partition function Z

We call *faces* of an S-diagram D the connected components of $(S - \partial S) - D$. The *North face* $N(D)$ is the one whose boundary contains $[-1,1]\times\{1\}$ and the *South face* $S(D)$ is the one whose boundary contains $[-1,1]\times\{-1\}$ (these faces can be identical: for instance this is the case for X_0). The faces of D can be (uniquely) colored with two colors, black and white, in such a way that any two faces adjacent along an edge of D receive different colors, and $N(D)$, $S(D)$ are colored black. From now on the faces of every S-diagram D are colored in this way, and we denote by $B(D)$ the set of black faces of D. We call *black set* of D the union of the black faces of D and we denote by $\chi(D)$ the Euler characteristic of this set. Finally, if D is oriented, we denote by $T(D)$ the *Tait number* (or *writhe*) of D, that is, the sum of signs of its crossings, where the sign of a crossing is defined on Figure 3.

Let (X, W_+, W_-, a, q) be a symmetric spin model. We define a mapping Z from $\mathcal{D}$ to $\mathcal{M}(X)$ as follows.

Let D be an S-diagram. For i, j in X we call *state of D of type (i,j)* any mapping σ from $B(D)$ to X such that $\sigma(N(D)) = i$, $\sigma(S(D)) = j$. Then for any such mapping σ and for any vertex v of degree 4 of D, we define a *local weight* $< \sigma, v >$ as in the case of link diagrams (see Figure 2). Now the *weight* $< \sigma, D >$ of the state σ is the product of the $< \sigma, v >$ over all vertices v of degree 4 of D (this product being 1 if there are no such vertices). We define $Z'(D)$ to be the matrix in $\mathcal{M}(X)$ with (i,j) entry given by

$$Z'(D)[i,j] = \sum < \sigma, D >, \tag{22}$$

where the summation runs over all states σ of D of type (i,j) (the sum being 0 if there are no such states).

Finally, we define the mapping Z from $\mathcal{D}$ to $\mathcal{M}(X)$ by

$$Z(D) = q^{2-\chi(D)} Z'(D). \tag{23}$$

Proposition 1. *If D_1', D_2' are Reidemeister equivalent oriented S-diagrams, and if D_i is the S-diagram obtained from D_i' by forgetting its orientation $(i = 1,2)$, $a^{-T(D_1')} Z(D_1) = a^{-T(D_2')} Z(D_2)$.*

Proof. We may proceed in essentially the same way as for link diagrams. There is only one minor difference. We need to show the invariance under Reidemeister moves of $Z(D)[i,j]$ (up to a writhe factor) for each given pair (i, j). But the fact that we restrict the summation in (22) to states of type (i, j) is not significant since, in the analysis of a given Reidemeister move, we consider (as in the case of link diagrams) the state values as fixed for all black faces except possibly one "central face" of the move. Q.E.D.

The proof of the following result is easy (see Figures 2, 4) and is left to the reader.

Proposition 2.
(i) $Z(X_0) = qI$,
(ii) $Z(X_\infty) = J$,
(iii) $Z(X_+) = W_+$, $Z(X_-) = W_-$.

We shall need the following important properties of the mapping Z.

Proposition 3.
(i) $Z(D_1 \div D_2) = q^{-1} Z(D_1) Z(D_2)$ *for all* D_1, D_2 *in* $\mathcal{D}$,
(ii) $Z(D_1 \# D_2) = Z(D_1) \circ Z(D_2)$ *for all* D_1, D_2 *in* $\mathcal{D}$,
(iii) $Z(\rho^2(D)) = (Z(D))^t$ *for all* D *in* $\mathcal{D}$.

Proof. (i) Figure 9 (i) shows that there is a bijective correspondence between the set of states of type (i, j) of $D_1 \div D_2$ and the set of triples (k, σ_1, σ_2), where k is an element of X, σ_1 is a state of type (i, k) of D_1, and σ_2 is a state of type (k, j) of D_2, such that $< \sigma, D_1 \div D_2 > = < \sigma_1, D_1 >< \sigma_2, D_2 >$ whenever σ corresponds to (k, σ_1, σ_2), for some k in X. Hence

$$Z'(D_1 \div D_2)[i, j] = \sum_{k \in X} Z'(D_1)[i, k] Z'(D_2)[k, j].$$

Moreover the black set of $D_1 \div D_2$ can be obtained from the disjoint union of the black sets of D_1 and D_2 by the attachment of a single band. Hence

$$\chi(D_1 \div D_2) = \chi(D_1) + \chi(D_2) - 1.$$

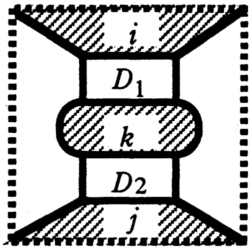
The two above equalities together yield (i).

(ii) The proof is quite similar to that of (i). Figure 9 (ii) shows that

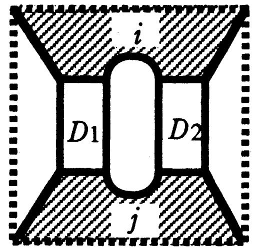
$$Z'(D_1 \# D_2)[i, j] = Z'(D_1)[i, j] Z'(D_2)[i, j].$$

Now the black set of $D_1 \# D_2$ is obtained from the disjoint union of the black sets of D_1 and D_2 by the attachment of two bands, and hence

$$\chi(D_1 \# D_2) = \chi(D_1) + \chi(D_2) - 2.$$



(i)



(ii)

Fig. 9

The result follows immediately.

(iii) Clearly ρ^2 defines a bijective correspondence between $B(D)$ and $B(\rho^2(D))$ which shows that $\chi(\rho^2(D)) = \chi(D)$, and also yields a weight-preserving bijective correspondence between states of type (i,j) of D and states of type (j,i) of $\rho^2(D)$. Q.E.D.

4.2. The tangle closure of a symmetric spin model

Let us consider a symmetric spin model (X, W_+, W_-, a, q) and the associated map Z: $\mathcal{D} \to \mathcal{M}(X)$ defined in the previous section. Let us denote by $< Z(\mathcal{D}) >$ the linear span of $Z(\mathcal{D})$, which we shall call the *tangle closure* of the spin model.

Theorem A. *$< Z(\mathcal{D}) >$ is a symmetric Bose-Mesner algebra which contains the spin model matrices W_+, W_-. Moreover the map Ψ: $\mathcal{M}(X) \to \mathcal{M}(X)$ defined by $\Psi(M) = aW_-\circ(W_+(W_-\circ M))$ induces a duality on $< Z(\mathcal{D}) >$.*

Proof. By Proposition 2, $< Z(\mathcal{D}) >$ contains W_+,W_-, I and J. By definition $< Z(\mathcal{D}) >$ is a linear subspace of $\mathcal{M}(X)$, and it follows immediately from Proposition 3 (i), (ii) that $< Z(\mathcal{D}) >$ is closed under ordinary matrix product and under Hadamard product. Hence Ψ defines a linear map from $< Z(\mathcal{D}) >$ to $< Z(\mathcal{D}) >$. By properties (3) and (4) of spin models, Ψ is invertible and $\Psi^{-1}(M) = a^{-1}|X|^{-1}W_+\circ(W_-(W_+\circ M))$ for every M in $< Z(\mathcal{D}) >$.

Let D be any element of $\mathcal{D}$. By (i) of the Main Lemma and Proposition 1, $Z((X_+ \div (X_-\#D))\#X_-) = a^k Z(\rho(D))$, where k is the Tait number of some orientation of $(X_+ \div(X_-\#D))\#X_-$ minus the Tait number of the corresponding orientation of $\rho(D)$. One easily obtains from Figure 8 that $k = -1$ for each of its six oriented versions. Hence

$$Z(\rho(D)) = aZ((X_+\div(X_-\#D))\#X_-).$$

Applying Proposition 2 (iii) and Proposition 3 (i), (ii), and using the commutativity of the Hadamard product, we obtain

$$Z(\rho(D)) = aq^{-1}W_{-}\circ(W_{+}(W_{-}\circ Z(D)))$$

or equivalently

$$\Psi(Z(D)) = qZ(\rho(D)). \tag{24}$$

Similarly, using (ii) of the Main Lemma, and considering the mirror images of the six oriented versions of Figure 8, we obtain

$$Z(\rho(D)) = a^{-1}Z((X_{-} \div (X_{+}\#D))\#X_{+})$$

and hence

$$Z(\rho(D)) = a^{-1}q^{-1}W_{+}\circ(W_{-}(W_{+}\circ Z(D))).$$

This is equivalent to

$$\Psi^{-1}(Z(D)) = q^{-1}Z(\rho(D)). \tag{25}$$

It follows from (24), (25) that $\Psi^{-1}(Z(D)) = q^{-2}\Psi(Z(D))$ and hence

$$\Psi^{2}(A) = |X|A \text{ for every } A \text{ in } < Z(\mathcal{D}) >. \tag{26}$$

Moreover, for any two elements D_1, D_2 of $\mathcal{D}$ we have:

$$\begin{aligned}
\Psi(Z(D_1)Z(D_2)) &= q\Psi(Z(D_1 \div D_2)) && \text{(by Proposition 3 (i))}\\
&= q^2 Z(\rho(D_1 \div D_2)) && \text{(by (24))}\\
&= q^2 Z(\rho(D_1)\#\rho(D_2)) && \text{(by (20))}\\
&= q^2 Z(\rho(D_1))\circ Z(\rho(D_2)) && \text{(by Proposition 3 (ii))}\\
&= \Psi(Z(D_1))\circ\Psi(Z(D_2)) && \text{(by (24))}
\end{aligned}$$

Hence $\Psi(AB) = \Psi(A)\circ\Psi(B)$ for every A, B in $< Z(\mathcal{D}) >$. This together with (26) shows that Ψ induces a duality on $< Z(\mathcal{D}) >$.

Finally, by applying (24) twice to the equality $(Z(D))^t = Z(\rho^2(D))$ of Proposition 3(iii), and using (26), we obtain $(Z(D))^t = Z(D)$. Hence all matrices in $< Z(\mathcal{D}) >$ are symmetric. Q.E.D.

Remark. It follows from (24) and Proposition 2 (iii), or from properties (2), (4) of spin models and the expression of Ψ given in Theorem A, that $\Psi(W_{+}) = qW_{-}$.

4.3. The algebraic closure of a symmetric spin model

So far the symmetric Bose-Mesner algebra $< Z(\mathcal{D}) >$ is related to the corresponding spin model in an abstract way via S-tangles or S-diagrams, and we are not able to describe this Bose-Mesner algebra explicitly. We now propose a way to overcome this difficulty.

Let us say that a subset of $\mathcal{M}(X)$ is *weakly Bose-Mesner* (WBM for short) if it is closed under complex linear combinations, ordinary matrix product, Hadamard product, and contains I, J. Thus symmetric Bose-Mesner algebras on X are exactly the WBM subsets of $\mathcal{M}(X)$ consisting only of symmetric matrices.

Clearly $\mathcal{M}(X)$ is WBM, and the intersection of a family of WBM subsets is again WBM. Thus every subset F of $\mathcal{M}(X)$ has a unique *WBM-closure* $\mathcal{C}(F)$ which is the smallest WBM subset of $\mathcal{M}(X)$ containing it. More precisely we may define $\mathcal{C}(F)$ in the following two equivalent ways:

(i) abstract definition: $\mathcal{C}(F)$ is the intersection of all WBM subsets of $\mathcal{M}(X)$ containing F.

(ii) algorithmic definition: $\mathcal{C}(F)$ is obtained by iterating the following process involving a finite current set F', and taking as the initial instance of F' a maximal linearly independent subset of $F \cup \{I, J\}$. Compute ordinary or Hadamard products of two elements of F', checking for each resulting matrix if it belongs to the linear span of F'. If not, the matrix is incorporated into F'. If no such incorporation is possible, the linear span of F' is $\mathcal{C}(F)$.

Theorem B. *Let (X, W_+, W_-, a, q) be a symmetric spin model. $\mathcal{C}(\{W_+\}) = \mathcal{C}(\{W_-\})$ is a symmetric Bose-Mesner algebra. Moreover the map $\Psi : \mathcal{M}(X) \to \mathcal{M}(X)$ defined by $\Psi(M) = aW_- \circ (W_+(W_- \circ M))$ induces a duality on this Bose-Mesner algebra.*

Proof. By Theorem A, W_+ belongs to the symmetric Bose-Mesner algebra $< Z(\mathcal{D}) >$. Since $< Z(\mathcal{D}) >$ is WBM, it contains $\mathcal{C}(\{W_+\})$ and hence all matrices in $\mathcal{C}(\{W_+\})$ are symmetric. The equality $\mathcal{C}(\{W_+\}) = \mathcal{C}(\{W_-\})$ easily follows from property (3) of spin models. Indeed the inverse of W_+ is a linear combination of powers of W_+ and hence W_- belongs to $\mathcal{C}(\{W_+\})$. This shows that $\mathcal{C}(\{W_+\}) \supset \mathcal{C}(\{W_-\})$ and the reverse inclusion is proved similarly. Finally the definition of Ψ shows immediately that $\Psi(\mathcal{C}(\{W_+\})) = \mathcal{C}(\{W_+\})$ and the result follows from Theorem A. Q.E.D.

The Bose-Mesner algebra $\mathcal{C}(\{W_+\})$ will be called the *algebraic closure* of the spin model (X, W_+, W_-, a, q). This Bose-Mesner algebra can also be understood in terms of S-diagrams as follows. Call an S-diagram *algebraic* if it can be generated using only vertical and horizontal products from the basic S-diagrams X_0, X_∞, X_+, X_- (the corresponding S-tangles are called algebraic in [C]). Let $\mathcal{AD}$ be the set of algebraic S-diagrams. It is clear from Propositions 2 and 3 (i), (ii) that $Z(\mathcal{AD})$ is contained in $\mathcal{C}(\{W_+, W_-\}) = \mathcal{C}(\{W_+\})$. Conversely, it is easy to see that $\mathcal{C}(\{W_+\})$ is linearly spanned by elements in $Z(\mathcal{AD})$. Thus $\mathcal{C}(\{W_+\})$ is the linear span of $Z(\mathcal{AD})$, which we denote by $< Z(\mathcal{AD}) >$, and clearly we could give a "topological" proof of Theorem B exactly similar to that of Theorem A.

Theorem B generalizes previous results of [Ja1]. In that paper two algebras $\mathcal{M}$ and $\mathcal{H}$ are associated with any symmetric spin model (X, W_+, W_-, a, q): $\mathcal{M}$ is the algebra under matrix product generated by J, W_+, and $\mathcal{H}$ is the algebra under Hadamard product generated by I, W_-. Clearly $\mathcal{M}$ and $\mathcal{H}$ are contained in $\mathcal{C}(\{W_+\})$, and, since $\Psi(W_+) = qW_-$ and $\Psi(J) = |X|I$, $\Psi(\mathcal{M}) = \mathcal{H}$. This equality is the content of Proposition 3 of [Ja1], with however different expressions for Ψ (which are easily obtained from the expression of Theorem B using the star-triangle equation (5)). Assume now that $\mathcal{M}$ is closed under Hadamard product. Then $\mathcal{M}$ is WBM (because it contains I by (3)) and hence equals $\mathcal{C}(\{W_+\})$. This implies Proposition 4 of [Ja1].

We do not know any example of a symmetric spin model for which the tangle closure $< Z(\mathcal{D}) >$ and the algebraic closure $< Z(\mathcal{AD}) >$ are different, but we believe that such an example should exist.

4.4. Modular invariance

Let (X, W_+, W_-, a, q) be a symmetric spin model. We have introduced two symmetric Bose-Mesner algebras $< Z(\mathcal{D}) >$ and $< Z(\mathcal{AD}) >$, each of which contains W_+, W_- and is provided with a duality Ψ given by the expression $\Psi(M) = aW_- \circ (W_+(W_- \circ M))$. Now, given a symmetric Bose-Mesner algebra with duality Ψ, we would like to know if it is associated in this way with some symmetric spin model, and classify such spin models. A key concept here is that of *modular invariance*, formulated by Eiichi Bannai and his coworkers in their study of relations between fusion algebras of conformal field theories and association schemes (see [B1]), and used by them to construct new spin models ([BB2], [BBIK])

We have seen that the Bose-Mesner algebra $\mathcal{A}$ of a d-class symmetric association scheme on a set X has a basis $\{A_i, i = 0, \ldots, d\}$ of orthogonal idempotents for the Hadamard product. It is well known that it has also a basis $\{E_i, i = 0, \ldots, d\}$ of orthogonal idempotents for the ordinary matrix product, where $E_0 = |X|^{-1}J$ (see [BI], Section II.3). The *first* *eigenmatrix* P is defined by

$$A_j = \sum_{i=0,\ldots,d} P[i,j]E_i. \tag{27}$$

Thus P is the transition matrix from the basis $\{A_i, i = 0, \ldots, d\}$ to the basis $\{E_i, i = 0, \ldots, d\}$, and only depends on the indexing of the A_i and the E_i. If Ψ is a duality, the matrices $\Psi(E_i)$ are the A_i in some order. In particular, by (10), $\Psi(E_0) = |X|^{-1}\Psi(J) = I = A_0$. Thus we may choose the indices in such a way that $\Psi(E_i) = A_i$ for $i = 0, \ldots, d$. Then by (27) P is the matrix of Ψ with respect to the basis $\{E_i, i = 0, \ldots, d\}$, and $P^2 = |X|I$ by (13).

The modular invariance property asserts that there exists a diagonal matrix Δ such that $(P\Delta)^3$ is a non-zero multiple of the identity. The following relationship between this property and spin models is presented in [BBJ].

Let us write $W_- = \sum_{i=0,\ldots,d} t_i A_i$ and define W_+ by $\Psi(W_+) = qW_-$. Thus $W_+ = q^{-1}\Psi(W_-) = q^{-1}\sum_{i=0,\ldots,d} t_i \Psi(A_i) = q\sum_{i=0,\ldots,d} t_i E_i$.

Assume that $\Psi(M) = aW_- \circ (W_+(W_- \circ M))$ for every M in the Bose-Mesner algebra $\mathcal{A}$. We write the corresponding equality in terms of matrices with respect to the basis $\{A_i, i = 0, \ldots, d\}$.

The matrix of the linear map $M \to W_- \circ M$ is the diagonal matrix Δ with $\Delta(i,i) = t_i$ for $i = 0, \ldots, d$.

The matrix of the linear map $M \to W_+ M$ is $P^{-1}(q\Delta)P = q^{-1}P\Delta P$. The matrix of Ψ is $P^{-1}(P)P = P$. We obtain the equality $P = q^{-1}a\Delta P\Delta P\Delta$, or equivalently $(P\Delta)^3 = q^3 a^{-1} I$.

We have proved the following result.

Theorem C. *Every symmetric spin model (X, W_+, W_-, a, q) is characterized by some solution Δ to the modular invariance equation $(P\Delta)^3 = q^3 a^{-1} I$, where P is the first eigenmatrix of some self-dual symmetric Bose-Mesner algebra (with indices chosen such that $P^2 = |X|I$).*

In general, a solution to the above modular invariance equation needs not correspond to some spin model (see [BBJ]). However, Theorem C should be a very useful tool for the classification of spin models (see for instance [B3], [CS]).

§5. Complements

5.1. Planar duality

We have just seen that the modular invariance property gives a necessary condition for a symmetric Bose-Mesner algebra to correspond to a symmetric spin model as in Theorems A or B. Another necessary condition is given by the *planar duality property* introduced in [Ja3]. This property is defined in terms of spin models on plane graphs, and some special cases are well known in physics (see also [Bi1], [Bi2]).

Let G be a finite undirected graph (loops and multiple edges are allowed). Its vertex-set and edge-set will be denoted by $V(G)$ and $E(G)$ respectively. Let X be a non-empty finite set and let w be a mapping from $E(G)$ to $\mathcal{M}(X)$ whose values are symmetric matrices. This mapping defines a *spin model on the graph* G. Let us call *state* of G any mapping σ from $V(G)$ to X. If the edge e has ends v_1, v_2, let $w(e|\sigma)$ be the $(\sigma(v_1), \sigma(v_2))$-entry of the matrix $w(e)$. The *weight* $w(\sigma)$ *of the state* σ is then the product of the $w(e|\sigma)$ over all edges e (this will be set to 1 if G has no edge). Finally, the *partition function* $Z(G, w)$ is the sum of weights of all states.

Let $\mathcal{A}$ be a symmetric Bose-Mesner algebra on X and let Ψ be a duality of $\mathcal{A}$. The pair $(\mathcal{A}, \Psi)$ is said to satisfy the planar duality property if for every connected plane graph G and mapping w from $E(G)$ to $\mathcal{A}$

$$Z(G, w) = |X|^{1-|V(G^*)|} Z(G^*, \Psi w \phi), \tag{28}$$

where G^* is the dual plane graph of G and $\phi : E(G^*) \to E(G)$ associates to every edge of G^* its dual edge.

Proposition 4. *Let (X, W_+, W_-, a, q) be a symmetric spin model. Each of the pairs $(< Z(\mathcal{D}) >, \Psi)$, defined in Theorem A, and $(\mathcal{C}(\{W_+\}), \Psi)$, defined in Theorem B, satisfies the planar duality property.*

Proof. The following argument is essentially a symmetric version of the one used for Proposition 12 of [Ja3] (see also the associated remark).

We consider a connected plane graph G and we want to check (28) for every mapping w from $E(G)$ to $\mathcal{A}$, where $\mathcal{A}$ is either $< Z(\mathcal{D}) >$ or $< Z(\mathcal{AD}) > = \mathcal{C}(\{W_+\})$. The set of these mappings can be identified with a product of m copies of the vector space $\mathcal{A}$, where $m = |E(G)|$. Then each side of (28), considered as a function of w, is clearly a multilinear form on this product of vector spaces. It follows that it is enough to check (28) for mappings w which take their values in some fixed basis of $\mathcal{A}$. Since $\mathcal{A}$ is either $< Z(\mathcal{D}) >$ or $< Z(\mathcal{AD}) >$, we may choose a basis contained in $Z'(\mathcal{D})$ (see (22), (23)). Hence it is enough to check (28) when, for every edge e of G, $w(e) = Z'(D_e)$ for some S-diagram D_e.

We shall need the following classical description of the pair (G, G^*). Let H be a connected 4-regular plane graph. We color its faces in black and white in such a way that adjacent faces receive different colors. The *graph of black faces* of H is a connected plane graph defined as follows. It has one vertex f° for each black face f of H, placed in the interior of that face, and one edge v° for each vertex v of H. If the vertex v of H is incident to the black faces f, g, the edge v° is embedded as a simple curve joining the vertices f°, g° across v. The *graph of white faces* of H is defined similarly, and it is easily seen that it is the dual of the graph of black faces. Moreover, every pair of dual connected plane graphs arises in this way (see [O] p.47).

Thus we may assume that G is the graph of black faces and G^* is the graph of white faces of some face-colored connected 4-regular plane graph H. We now associate with w (satisfying $w(e) = Z'(D_e)$ for every edge e of G) a face-colored connected link diagram L. To construct L, for every edge e of G we locally replace as shown on Figure 10 the corresponding vertex of H by the S-diagram D_e. There is some ambiguity in this construction since we do not specify which black face of H incident to this vertex corresponds to the North face of D_e, but we shall see that this will not matter, due to the fact that $Z'(D_e)$ is a symmetric matrix.

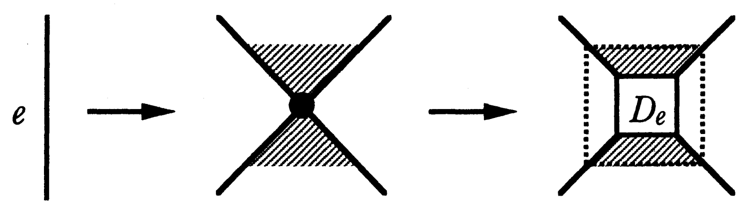


Fig. 10

It is clear from the definitions of the partition functions $Z'(L)$ (see Section 2.1), $Z'(D_e)$ (Section 4.1), and $Z(G, w)$, that

$$Z'(L) = Z(G, w). \tag{29}$$

Let us now exchange black and white in the face-coloring of L. The same argument (and examination of Figure 10) shows that the partition function $Z''(L)$ computed with respect to the new face-coloring satisfies

$$Z''(L) = Z(G^*, w^*), \tag{30}$$

where w^*: $E(G^*) \to \mathcal{A}$ is defined by $w^*(e^*) = Z'(\rho(D_e))$ for all e^* in $E(G^*)$ with dual edge e in $E(G)$. Using (23), (24) we see that

$$\begin{aligned} w^*(e^*) &= q^{\chi(\rho(D_e))-2} Z(\rho(D_e)) \\ &= q^{\chi(\rho(D_e))-3} \Psi(Z(D_e)) \\ &= q^{\chi(\rho(D_e))-3} \Psi(q^{2-\chi(D_e)} Z'(D_e)) \\ &= q^{\chi(\rho(D_e))-\chi(D_e)-1} \Psi(w(e)) \\ &= q^{\chi(\rho(D_e))-\chi(D_e)-1} (\Psi w \phi)(e^*). \end{aligned}$$

Hence (28) is equivalent to

$$Z'(L) = |X|^{1-|V(G^*)|} q^{\mu} Z''(L), \tag{31}$$

where $\mu = \sum_{e \in E(G)} (-\chi(\rho(D_e)) + \chi(D_e) + 1)$.

Recall from Section 2.1 that $Z(L) = a^{-T(L)} q^{-\chi_b(L)} Z'(L)$ defines a link invariant. It is easy to show that the exchange of black and white in the face-coloring of L does not modify the value of this invariant (see [Jo], Proposition 2.14). Hence $q^{-\chi_b(L)} Z'(L) = q^{-\chi_w(L)} Z''(L)$, where $\chi_w(L)$ is the Euler characteristic of the union of the white faces of L. It follows that (31) reduces to the equality

$$\chi_b(L) - \chi_w(L) = 2(1 - |V(G^*)|) + \mu.$$

Clearly $\chi_b(L) = |V(G)| + \sum_{e \in E(G)} (\chi(D_e) - 2)$, and similarly

$$\chi_w(L) = |V(G^*)| + \sum_{e \in E(G)} (\chi(\rho(D_e)) - 2).$$

Hence $\chi_b(L) - \chi_w(L) = |V(G)| - |V(G^*)| + \mu - |E(G)|$ and the result follows from Euler's formula applied to the connected plane graph G. Q.E.D.

The efficiency of Proposition 4 for the classification of spin models is still unclear, mainly because it is computationally difficult to verify the planar duality property (or even its restriction to the case where the plane graph G is the 4-clique, which is of special significance as shown in [Ja3]) .

5.2. Mutation and link invariants described by spin models

Let D be an S-diagram. We denote by $K(D)$ the link diagram obtained from D by joining together the ends of D as shown on Figure 11 (with a face-coloring extending the face-coloring of D). Clearly, every

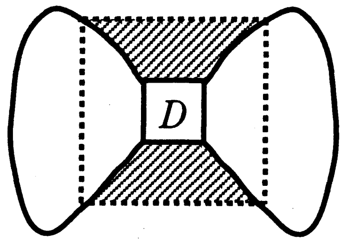


Fig. 11. $K(D)$

link diagram is of the form $K(D)$ for some S-diagram D. Moreover, given any symmetric spin model,

$$Z'(K(D)) = \mathrm{Trace}(Z'(D)). \tag{32}$$

A link diagram L' is obtained by *mutation* of the link diagram L if L is of the form $K(D_1 \div D_2)$ and L' is of the form $K(D_1' \div D_2)$, where D_1' is obtained from D_1 by a rotation about one of the three coordinate axes through angle π, that is, by one of the transformations ρ^2, α, β (see Section 3.3 and Figure 6). For oriented link diagrams, the orientation on D_1' is fully reversed if necessary to match the orientation of D_2. A link is said to be obtained from another by mutation if the same statement holds for some diagrams of these links, and this property can also be defined in a 3-dimensional setting (see for instance [LM]).

If $D_1' = \rho^2(D_1)$, $Z(D_1') = Z(D_1)$ by Proposition 3(iii) and Theorem A.

If $D_1' = \beta(D_1)$, there is an easy weight-preserving bijection between the sets of states of a given type (i,j) of D_1' and D_1. Hence $Z'(D_1') = Z'(D_1)$, and also $Z(D_1') = Z(D_1)$ since D_1' and D_1 have homeomorphic black sets.

Since $\alpha = \beta\rho^2$, we have also $Z(D_1') = Z(D_1)$ if $D_1' = \alpha(D_1)$. Then $Z(D_1' \div D_2) = Z(D_1 \div D_2)$ by Proposition 3(i), and consequently $Z'(D_1' \div D_2) = Z'(D_1 \div D_2)$. It now follows from (32) that $Z'(L') = Z'(L)$. Noting that the normalization factors for these two link diagrams are the same (if some orientation of D_1' is fully reversed the corresponding sum of signs of crossings is not modified) we obtain the following result.

Proposition 5. *If an invariant of oriented links is defined by a symmetric spin model it is invariant under mutation.*

5.3. Nomura's algebra

Immediately after the announcement of Theorem A above, K. Nomura found a simpler and purely algebraic proof of the fact that the matrices of a symmetric spin model belong to some symmetric Bose-Mesner algebra [N4].

Let W_+, W_- be two symmetric matrices in $\mathcal{M}(X)$ which satisfy (3), (4). For every b, c in X, let $\mathbf{u}_{b,c}$ be the column vector indexed by X with x-entry $\mathbf{u}_{b,c}(x) = W_+[b,x]W_-[c,x]$. Let $\mathcal{N}(W_+)$ be the set of symmetric matrices A in $\mathcal{M}(X)$ such that $\mathbf{u}_{b,c}$ is an eigenvector of A for every b, c in X. It is shown in [N4] that $\mathcal{N}(W_+)$ is a symmetric Bose-Mesner algebra. Moreover if (5) holds for some square root q of $|X|$, W_+ and W_- belong to $\mathcal{N}(W_+)$. This Bose-Mesner algebra is related as follows with the Bose-Mesner algebra of Theorem A.

Proposition 6. *Let (X, W_+, W_-, a, q) be a symmetric spin model. The corresponding algebra $< Z(\mathcal{D}) >$ is contained in the Nomura algebra $\mathcal{N}(W_+)$.*

Proof. (sketch): We introduce "hexagonal" tangles with six ends, which we call H-tangles, and corresponding H-diagrams such as the one depicted on Figure 12. It is easy to formalize these notions as we did for S-tangles and S-diagrams. We introduce a standard face-coloring (with the "North", "Southwest" and "Southeast" faces black) and define a partition function Z' and normalized partition function Z for H-diagrams as we did for S-diagrams (but now Z' takes its values in a space of tensors with 3 indices).

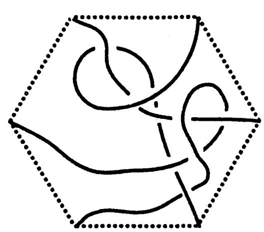


Fig. 12

Evaluation of Z on the two Reidemeister equivalent H-diagrams of Figure 13, where D is any S-diagram, shows that, for every i, b, c in X:

$$\sum_{x\in X} Z(D)[i,x]W_+[b,x]W_-[c,x] = qZ(\rho(D))[c,b]W_+[b,i]W_-[c,i]$$

or equivalently, for every b, c in X:

$$Z(D)\mathbf{u}_{b,c} = qZ(\rho(D))[c,b]\mathbf{u}_{b,c}. \tag{33}$$

Since $Z(D)$ is symmetric by Theorem A, it belongs to $\mathcal{N}(W_+)$. Q.E.D.

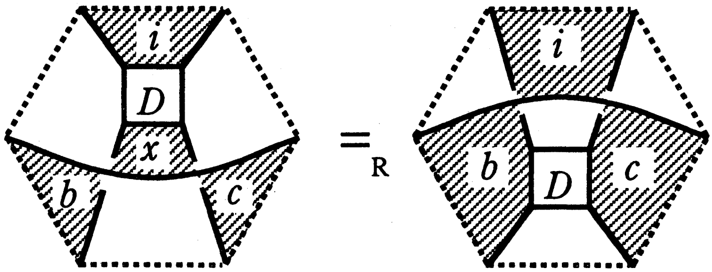


Fig. 13

Using (24), (33) becomes $Z(D)\mathbf{u}_{b,c} = \Psi(Z(D))[c,b]\mathbf{u}_{b,c}$, and hence $A\mathbf{u}_{b,c} = \Psi(A)[c,b]\mathbf{u}_{b,c}$ for every b, c in X and A in $< Z(\mathcal{D}) >$.

We may use this formula for every A in $\mathcal{N}(W_+)$ to define a linear map Ψ: $\mathcal{N}(W_+) \to \mathcal{M}(X)$. It is shown in [JMN] that Ψ is a duality of $\mathcal{N}(W_+)$ which is given by the expression $\Psi(M) = aW_-\circ(W_+(W_-\circ M))$ (i.e., the modular invariance property holds). Thus Nomura's algebra $\mathcal{N}(W_+)$ can play the same role for symmetric spin models as the algebras $< Z(\mathcal{D}) >$ of Theorem A or $< Z(\mathcal{AD}) >$ of Theorem B.

It would not be difficult to develop a "topological" approach to the results on the algebra $\mathcal{N}(W_+)$ given in [N4] and [JMN]. This could be a way to obtain some geometric intuition on these algebraic results (the starting point would be a "picture" of the definition of $N(W_+)$ similar to Figure 13), or even more, a way to actually prove the results (this would require of course an extension of the notion of S-diagram). As an example, we give in Figure 14 a diagrammatic proof (which the reader is invited to convert into an algebraic proof) of the fact that $\mathcal{N}(W_+)$ is closed under Hadamard product.

It is natural to ask whether or not $< Z(\mathcal{D}) >$ and $\mathcal{N}(W_+)$ can be different. Consider a symmetric spin model which satisfies $W_+ = W_-$ (hence W_+ is a Hadamard matrix). Then, for any S-diagram D, $Z(D)$ will be invariant under modification of the spatial structure of crossings. This, together with Proposition 1, shows that for every S-diagram D, $Z(D)$ is a scalar multiple of $Z(X_0)$, $Z(X_\infty)$, or $Z(X_+) = Z(X_-)$. Hence $< Z(\mathcal{D}) >$ has dimension at most 3 (actually, this dimension is 3 except when $|X| = 4$ and $W_+ = 2I - J$). On the other hand, it is shown in [JMN] that there exist symmetric spin models with $W_+ = W_-$ and $\dim \mathcal{N}(W_+) = |X|$ whenever $|X|$ is an even power of 2. Thus the gap between the dimensions of $< Z(\mathcal{D}) >$ and $\mathcal{N}(W_+)$ can be arbitrarily large.

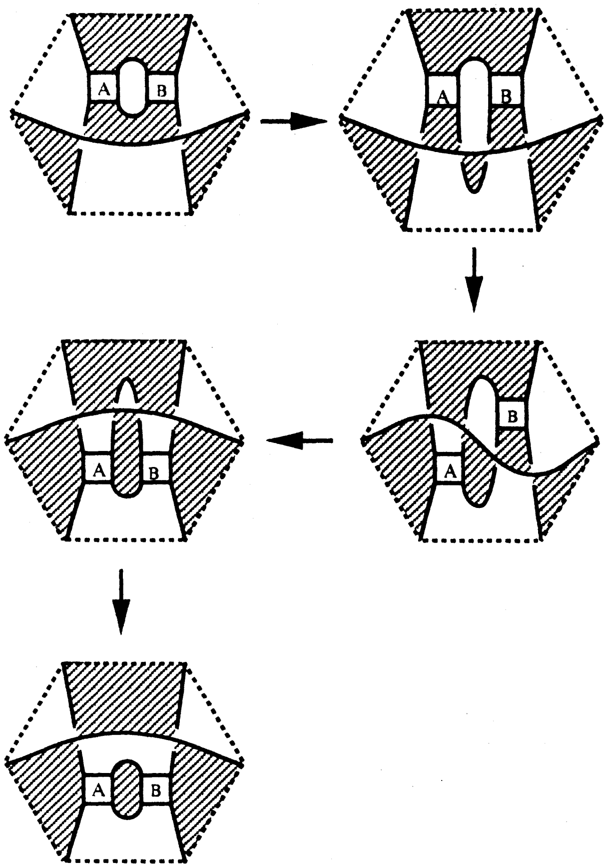


Fig. 14

§6. Conclusion

We believe that our results represent a significant step towards the classification of symmetric spin models: we can restrict our attention to the solutions of the modular invariance equations for self-dual symmetric Bose-Mesner algebras, and then check these solutions to retain those which actually yield spin models. We have proposed the planar duality property as an additional criterion, but it would be very interesting to find other criteria. Ideally we would like to have an intrinsic characterization of symmetric Bose-Mesner algebras which contain some spin model matrices. This question is solved for the 3-dimensional case in [Ja1]. This leads us to concentrate our study on the next and simplest open case of 4-dimensional symmetric Bose-Mesner algebras.

Theorem C is generalized in [JMN] to the non-symmetric spin models of [KMW] using the algebraic approach initiated in [N4]. Here the topological approach does not work well, because to define products for oriented S-tangles one must introduce compatibility conditions on the orientations. However, using the topological approach, we have obtained some preliminary results on the 4-weight spin models of [BB3], which generalize some results in [BB1], but the overall picture is still very unclear.

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