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Vertex Operator Algebras and Moonshine: A Survey

Chongying Dong and Geoffrey Mason

§1. Introduction

This survey is, in many regards, a reprise of the paper [Ma1]. That paper reviewed what was known at the time about the connections between finite groups and elliptic modular forms. Shortly after, Borcherds presented the notion of a *vertex (operator) algebra* (VOA), and the work of Frenkel–Lepowsky–Meurman showed that the Monster was the automorphism group of a particular VOA—the *Moonshine module.* Although the Conway–Norton conjectures were not proved till later, it became clear that *in principle*, one can associate modular functions to the elements of a finite group, following the dictates of *Moonshine*, whenever the group operates as automorphisms of a VOA. Indeed, the bulk of the Conway-Norton conjectures, together with Norton's generalization, can be understood in the framework of the representation theory of so-called *holomorphic orbifold models.* One sees that the theory applies to any finite group, and only the *genus zero* condition marks the Moonshine module as special.

In this survey we present the main ideas which led to the framework in which one can understand Moonshine as a general and natural phenomenon as opposed to an apparent miracle. Thus it is about VOAs, their representations, and their automorphisms. There are no proofs—indeed there are no lemmas or theorems either! Rather, we try and explain the main ideas, tell what is known, what remains to be done. Thus the reader will find 26 "problems" in the course of the paper. These may be considered guides to what still needs to be proved in order to make the theory more complete. A few of them are rather specialized, but most are of a general nature and (it seems to us) quite difficult!

There is no pretense at presenting a general survey of algebraic conformal field theory—that would be a vast undertaking. Rather, we have

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included only those topics in VOA theory which impinge directly on the explication of "Moonshine." On the other hand, our notion of Moonshine is broader than that of others. Thus we have included some topics of a more cohomological and topological nature with the expectation (and hope) that equivariant elliptic cohomology—whatever this turns out to really mean—will eventually become closely identified with Moonshine.

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§2. Axioms

We discuss the axioms for a *vertex operator algebra* (VOA). A fuller account may be found in [FLM], [FHL], [DL].

Let us begin with the definition of a VOA. This entails a $\mathbb{Z}$-graded vector space V over the complex numbers $\mathbb{C}$:

$$V = \coprod_{n\in\mathbb{Z}} V_n \tag{2.1}$$

such that $V_n = 0$ for $n \ll 0$ and $\dim V_n$ is finite for all n. If $v \in V_n$ we say that v is *homogeneous of (conformal) weight* n and write $wt(v) = n$. Roughly speaking, V is endowed with a countable infinity of bilinear products $u *_n v$ (u, $v \in V$, $n \in \mathbb{Z}$) which are required to satisfy infinitely many identities.

For fixed $u \in V$ we assemble the endomorphisms $u_n \in \operatorname{End} V$, defined by $u_n v = u *_n v$, into a generating function (or *vertex operator*)

$$Y(u, z) = \sum_{n\in\mathbb{Z}} u_n z^{-n-1} \in \operatorname{End} V[[z, z^{-1}]] \tag{2.2}$$

which provides a linear map $Y\colon V \to \operatorname{End} V[[z, z^{-1}]]$, $u \mapsto Y(u, z)$.

The other VOA axioms are as follows:

(i) For u, $v \in V$, there is $n = n(u, v) \in \mathbb{Z}$ such that $u_m v = 0$ for $m > n$,

This tells us that the sum $Y(u, z)v := \sum_m u_m v z^{-m-1}$ is a Laurent series, i.e., only finitely many non-zero negative powers occur.

(ii) *Existence of vacuum*: There is a distinguished element $1 \in V$ which satisfies

(a) $Y(1, z) = \text{id}$,

(b) $Y(u, z)1 = u + \sum_{n=2}^{\infty} u_{-n} 1 z^{n-1}$.

Part (b) tells us that $u_n 1 = 0$ for $n \geq 0$ and $u_{-1}1 = u$. In particular, the map $u \mapsto Y(u, z)$ is an injection. One shows that in fact $1 \in V_0$.

(iii) *Existence of conformal vector*: There is a distinguished element $\omega \in V$ with generating function $Y(\omega, z) = \sum_n L_n z^{-n-2}$ such that the component operators L_n generate a copy of the *Virasoro algebra* Vir represented on V with *central charge* c. This means that there are relations

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m,-n} c \tag{2.3}$$

for some constant c. It turns out that c is an important invariant associated to V. And inasmuch as V is a module for Vir, the representation theory of the Lie algebra Vir (cf. [KR]) will be valuable for VOA theory. One has $\omega \in V_2$.

(iv) V_n is the L_0-eigenspace of V with eigenvalue n. That is, if $v \in V_n$ then $L_0 v = nv$.

(v) $\dfrac{d}{dz} Y(v, z) = Y(L_{-1}v, z)$.

Note from (2.3) that the span of L_0, L_{-1}, L_1 is a Lie subalgebra of Vir isomorphic to $sl(2)$. The last two axioms suggest that L_0 and L_{-1} are particularly important operators on V.

(vi) *Jacobi Identity*: we have

$$\begin{aligned} & z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y(u, z_1)Y(v, z_2) \\ & \qquad - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y(v, z_2)Y(u, z_1) \\ & = z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y(Y(u, z_0)v, z_2). \end{aligned} \tag{2.4}$$

This is the VOA embodiment of commutativity and associativity. With a suitable interpretation of the delta function $\delta(\ )$ (cf. [FLM], [FHL]), when applied to $w \in V$, both the l.h.s. and r.h.s. of (2.4) are expressions of the form $\sum P_{ijk} z_0^i z_1^j z_2^k$ where P_{ijk} is a *finite* sum of vectors in V (recall axiom (i)). In this way, (2.4) represents a threefold infinity of identities which must be satisfied.

We observe that if $v \in V_m$ then

$$v_n \colon V_p \to V_{p+m-n-1}. \tag{2.5}$$

In particular, there is a linear map $V \to \mathrm{End}(V_p)$ such that $v \mapsto v_{wt(v)-1}$ if v is homogeneous.

There are alternate ways to view the Jacobi identity—the so-called *duality* axioms. Not only are they conceptually easier to swallow, but they are important in a number of situations. To set the stage, let $V = \coprod V_n$ be a VOA, let V_n^* be the dual space $\mathrm{Hom}(V_n, \mathbb{C})$, and set $V' = \coprod V_n^*$. Thus there is a canonical pairing $\langle\ ,\ \rangle \colon V' \times V \to \mathbb{C}$ so that $\langle V_m^*, V_n \rangle = 0$ for $m \neq n$.

Next, let R be the subring of $\mathbb{C}(z_1,z_2)$ obtained by localizing $\mathbb{C}[z_1,z_2]$ at the monoid generated by homogeneous polynomials of degree 1. For each of the two orderings (i_1, i_2) of $\{1, 2\}$ there is an injective ring map

$$\iota_{i_1,i_2} \colon R \to \mathbb{C}[[z_1, z_1^{-1}, z_2, z_2^{-1}]] \tag{2.6}$$

by which $(az_1 + bz_2)^{-1}$ is expanded as a power series in non-negative powers of z_{i_2}. Then we have

(vii) *Duality*: let u, v, $w \in V$, $w' \in V'$. There is $f \in R$ of the form $f = h(z_1, z_2)/z_1^r z_2^s (z_1 - z_2)^t$ with $h(z_1, z_2) \in \mathbb{C}[z_1, z_2]$, r, s, $t \in \mathbb{Z}$ such that

(a) *Rationality*: $\langle w', Y(u, z_1) Y(v, z_2) w\rangle = \iota_{12} f(z_1, z_2)$
(b) *Commutativity*: $\langle w', Y(v, z_1) Y(u, z_2) w\rangle = \iota_{21} f(z_1, z_2)$
(c) *Associativity*: $\langle w', Y(Y(u, z_0)v, z_2) w\rangle = \iota_{20} f(z_0 + z_2, z_2)$.

It transpires ([FLM], [FHL]) that the duality axioms are *equivalent* to the Jacobi identity. In fact, there is an even simpler variant:

(viii) If u, $v \in V$ there is $n = n(v, v) \geq 0$ such that

$$(z_1 - z_2)^n [Y(u, z_1), Y(v, z_2)] = 0 \tag{2.7}$$

(cf. [DL, chptr.7]). In conjunction with the equality $[L_{-1}, Y(v, z)] = Y(L_{-1}v, z)$, this is also equivalent to (vi). This formulation is closely related to the approach via *local systems*, which is frequently used in the physics literature (cf. [G], for example). We discuss this further below. To illustrate (2.7) we observe that the Virasoro axioms (2.3) imply that

$$(z_1 - z_2)^4 [Y(\omega, z_1), Y(\omega, z_2)] = 0. \tag{2.8}$$

This concludes our discussion of the VOA axioms, but there is much more to be said. In particular there are a number of more or less obvious variations that are important. For example, it is clear that we can define VOAs over *any* field, or even more general rings of coefficients. Moreover, one often wants to *weaken* the above axioms in various ways. For example one may allow the $\mathbb{Z}$-grading on V to be unbounded below, and/or allow the homogeneous spaces to be of infinite dimension. One may also dispense with the Virasoro algebra and keep just the degree operator L_0 and derivation L_{-1}. This is essentially the original approach of Borcherds in [Bo1], where he studies so-called *vertex algebras.*

One can also add *additional* axioms. In the physics literature, for example, it is often an implicit axiom that V carries the structure of a Hilbert space. In the approach of [BPZ](see also [MS]), even assumptions concerning modular-invariance of characters (see Section 10) are contemplated. We will always stick with our axiom system (i)-(vi).

A *morphism* of VOAs V_1, V_2 is a linear map $f\colon V_1 \to V_2$ which preserves the vacuum, the conformal vector and each of the binary products $*_n$. The latter condition is equivalent to

$$fY_1(u,z) = Y_2(fu,z)f. \tag{2.9}$$

(Here and below, we frequently refer to a VOA via its underlying space, say V_1, rather than through the pair (V_1, Y_1) consisting of V_1 and its Y-map, or even to $(V_1, Y_1, 1_1, \omega_1)$ etc.)

An *automorphism* of the VOA V is an invertible endomorphism of V. The set of these form a group $\mathrm{Aut}(V)$. Since $\mathrm{Aut}(V)$ leaves ω invariant, (2.9) shows that $\mathrm{Aut}(V)$ commutes with each component L_n of $Y(\omega, z)$. Thus $\mathrm{Aut}(V)$ commutes with the Virasoro algebra Vir in their joint action on V. This can have useful consequences (see [DLM], for example).

Note in particular that since $\mathrm{Aut}(V)$ commutes with L_0 then each V_n affords a representation of $\mathrm{Aut}(V)$, thanks to axiom (iv). It is through this mechanism that group representation theory enters into the study of VOAs.

§3. Examples of VOA

One of the difficulties of the subject is that examples are not easy to describe. In our exposition we will generally only discuss how to construct the *Fock space*, that is the underlying graded space V (2.1), and eschew a description of the vertex operators themselves.

(i) *Local systems*: Let M be a $\mathbb{C}$-vector space and let $F(M)$ be the subspace of $\mathrm{End}\, M[[z, z^{-1}]]$ consisting of power series $u(z) = \sum_m u_m z^{-m-1}$ which satisfy (i) of Section 2. Define a relation on $F(M)$ as follows: $u(z)$ and $v(z)$ are related if (2.7) holds, that is $(z_1-z_2)^n[u(z_1),v(z_2)] = 0$ for some $n \geq 0$.

This relation is symmetric, but not necessarily reflexive. One usually says that $u(z)$ and $v(z)$ are *mutually local* if they are related as above. We may call $u(z)$ a local vertex operator if it is mutually local with itself. From Section 2 we know that if V is a VOA then V consists of a set of mutually local power series $u(z) = Y(u, z)$ in which M is V itself. The idea is now that VOAs and local systems are essentially equivalent. This is the point-of-view taken in Goddard's work [G] and in his joint work with Dolan and Montague ([DGM1], [DGM2], for example).

There is an axiomatic treatment due to Li ([Li1]). If we start with a Virasoro module M such that $L(z) = \sum_n L_n z^{-n-2}$ satisfies (i) of Section 2 (we call M *restricted* if this holds), then $L(z)$ is a local vertex operator (2.8). We define a *local system* on M to be a maximal set of mutually local operators $u(z)$ on M which contain $L(z)$. These exist by dint of Zorn's Lemma. Li shows that a local system in this sense is a VA (vertex algebra); that is, the axioms for a VOA are satisfied except that the homogeneous spaces may have infinite dimension.

If S is any set of mutually local operators which includes $L(z)$ we may define $\langle S \rangle$ to be the smallest VOA containing S.

This is a rather abstract way to define VOAs, but it has the advantage that *existence* is not in doubt. We refer to the papers of Li and Goddard et al. for application of these ideas.

(ii) *Virasoro Modules*: For reference here we suggest [KR]. It is natural, given the requirement that a VOA admit an action of Vir, that various Vir-modules carry the structure of a VOA. This is indeed the case.

To describe the Fock space, recall (*loc. cit.*) that for a pair of complex numbers (c, h) there is a *Verma module* $M(c, h)$ which is the universal highest weight module for the Lie algebra Vir of central charge c, and such that $L_0 v = hv$ for maximal vector $0 \neq v \in M(c, h)$. Moreover $M(c, h)$ has a unique maximal submodule with quotient denoted by $L(c, h)$. Thus $L(c, h)$ is the unique simple Vir-module of highest weight (c, h).

To construct a VOA from $M(c, h)$ in which v is the vacuum we must take $h = 0$ (by axiom (iv) of Section 2) and we need $L_{-1}v \equiv 0$ (by axioms (ii)(a) and (v)). Therefore, let M_c be the quotient of $M(c, 0)$ by the Vir-submodule generated by $L_{-1}v$. Then indeed M_c has the structure of a VOA, as shown by Frenkel–Zhu [FZ]. There is also a proof due to Li [Li1] using the technique of local systems.

$L(c, 0)$ also gets the structure of VOA, being a VOA quotient of M_c. These are the physicist's *minimal models* [BPZ].

(iii) *Kac–Moody modules*: Let $\mathfrak{J}$ be a finite-dimensional, complex, simple Lie algebra with $\hat{\mathfrak{J}} = \mathfrak{J} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$ the corresponding affine Lie algebra [K]. There is a theory parallel to (ii) in which Vir is replaced by $\hat{\mathfrak{J}}$, and again there are mathematical descriptions due to Frenkel-Zhu [FZ] and Li [Li1].

Briefly, let $L(\lambda)$ be the simple $\mathfrak{J}$-module with highest weight λ and let* $M(c, \lambda) = \mathcal{U}(\hat{\mathfrak{J}}) \otimes_B L(\lambda)$ where $B = \mathfrak{J} \otimes \mathbb{C}[t] \oplus \mathbb{C}c$, c acts on $L(\lambda)$ as multiplication by c (the *level*) and $g \otimes t^n$ acts as 0 if $n > 0$ and as g if $n = 0$ ($g \in \mathfrak{J}$). Then $M(c, 0)$ has the structure of VOA if $c + g_0 \neq 0$, where g_0 is the dual Coxeter number—this condition is necessary for the existence of a conformal vector. Again as in Example (ii), $M(c, \lambda)$ has a unique maximal $\hat{\mathfrak{J}}$-submodule with quotient $L(c, \lambda)$, and $L(c, 0)$ is a quotient VOA of $M(c, 0)$ *as long as* $c + g_0 \neq 0$ and $c \neq 0$.

When c is a positive integer, $L(c, 0)$ is a (highest weight) integrable $\hat{\mathfrak{J}}$-module and the theory in this case is essentially the physicist's *WZW-model.*

(iv) *Lattice Theories*: Let L be a *positive-definite even* lattice; that is, $L \cong \mathbb{Z}^d$ is a free abelian group equipped with an integral, positive-definite, symmetric, bilinear form $(\ ,\ ) \colon L \times L \to \mathbb{Z}$ such that $(x, x) \in 2\mathbb{Z}$ for $x \in L$. Let $H = \mathbb{C} \otimes_{\mathbb{Z}} L$ and set

$$V_L = S(H_1 \oplus H_2 \oplus \cdots) \otimes \mathbb{C}[L]. \tag{3.1}$$

Here, each H_i is linearly isomorphic to H, $S(\ )$ denotes symmetric algebra, and $\mathbb{C}[L]$ is the group algebra of L with basis e^α, $\alpha \in L$. Then we can give V_L the structure of VOA in which the central charge c is equal to $d/24$. This result was announced in [Bo1], with a complete discussion provided in [FLM].

If L is the root lattice of a simple Lie algebra (of type ADE) then we recover the level 1 WZW theory of (iii) above. The advantage of the present approach is that it is quite explicit and computable, and because of this the lattice theories are of the utmost importance.

(v) *Spinor Constructions*: The theory of (iv) is often called a *bosonic* theory, being based on symmetric algebras. Spinor constructions, on the other hand, are called *fermionic theories* and are based on exterior and Clifford algebras. When available, spinor constructions are very useful and often very amenable to calculation. We give just a sample of the idea, based on [FFR] and [DM1].

Let $A \cong \mathbb{C}^{2\ell}$ be a complex linear space equipped with a non-degenerate, symmetric, bilinear form $(\ ,\ )$ and with a polarization

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*$\mathcal{U}$ denotes universal enveloping algebra.

$A = A^+ \oplus A^-$ into maximal isotropic subspaces. Let $A(\mathbb{Z}) = A^+ \oplus A_1 \oplus A_2 \oplus \cdots$ and $A(\mathbb{Z} + 1/2) = A_{1/2} \oplus A_{3/2} \oplus \cdots$ where each A_i, $A_{i/2}$ is linearly isomorphic to A. (As in (3.1), the indexing will be useful for later discussions.) Then one can endow both $\Lambda^{\text{even}}(A(\mathbb{Z}+1/2))$ and $\Lambda^{\text{even}}(A(\mathbb{Z} + 1/2)) \oplus \Lambda^{\text{even}}(A(\mathbb{Z}))$ with the structure of VOA in a quite canonical way which makes use of the fact that both $\Lambda(A(\mathbb{Z}))$ and $\Lambda(A(\mathbb{Z}+1/2))$ may be viewed as modules for certain infinite-dimensional Clifford algebras. The *boson-fermion correspondence* ([F], [DM1]) says that in fact $\Lambda^{\text{even}}(A(\mathbb{Z} + 1/2))$ is VOA-isomorphic to the level 1 WZW model of type D_ℓ (alternatively, to V_L in which L is the D_ℓ-root lattice). Similarly, $\Lambda^{\text{even}}(A(\mathbb{Z} + 1/2)) \oplus \Lambda^{\text{even}}(A(\mathbb{Z}))$ is VOA-isomorphic to $V_{L'}$ where L' is the spin lattice of rank ℓ.

These constructions are also important with regard to the so-called Witten-genus. See [Ta].

(vi) *Moonshine Module*: Perhaps the most famous VOA, its existence was announced by Borcherds [Bo1] and established in [FLM]. It is somewhat harder to construct than previous examples, and we will return to this point later. It is usually denoted $V^\natural$, and of course we have $\mathrm{Aut}(V^\natural)$ equal to the Fischer–Griess Monster group $\mathbb{M}$. $V^\natural$ has central charge 24.

(vii) *Tensor Products*: Given VOAs $V_1, \ldots, V_n$, the tensor product $V_1 \otimes \cdots \otimes V_n$ can be given the structure of a VOA in a canonical way. See [FHL] for details. We just observe here that the central charge of the tensor product is the *sum* of the central charges of the V_i.

(viii) For a different sort of example we go back to the Moonshine module $V^\natural$ and note that its grading (2.1) is such that $V_0^\natural = 0$ for $n < 0$ or $n = 1$, while $V_0^\natural = \mathbb{C}1$ is 1-dimensional and $V_2^\natural$ is of dimension 196,884 (and contains the conformal vector ω). $V_2^\natural$ is often denoted by B, and called the *Griess algebra* after its originator ([Gr]). B is a commutative, non-associative algebra with identity $1/2\omega$, the product being given by $u \circ v = u_1 v$ for u_1, $v \in B$ (cf. (2.5)).

There are idempotents e in B where the corresponding vertex operator $Y(e, z) = \sum_n K_n z^{-n-2}$ is such that the components K_n span a Virasoro algebra of central charge $1/2$, namely $L(1/2, 0)$ (see Example (ii)). In fact, one can find 48 linearly independent e's which span a 48-dimensional associative subalgebra of B and such that the corresponding vertex operators have components which commute. They generate a vertex operator subalgebra L of $V^\natural$ isomorphic to $L(1/2, 0)^{\otimes 48}$ (and of central charge 24). See [DMZ], [MN], [Mi] for more information. Frequently one can answer questions about $V^\natural$ by studying L. For examples of this technique see [DLiM3], [Hu].

Note that the idempotent e is associated with a certain involution of type $2A$ in the Monster $\mathbb{M}$ ([C], [MN], [Mi]). Similarly, there are idempotents f in B associated with elements of type $3A$ and such that the components of $Y(f, z)$ span a copy of $L(4/5, 0)$.

Problem 1. *Is there a theory of f's analogous to that for e's? In particular, can one find* 30 *f's spanning a* 30*-dimensional associative subalgebra of B, with a corresponding sub VOA of $V^\natural$ isomorphic to $L(4/5, 0)^{\otimes 30}$?*

§4. Representations of VOAs

To a great extent, the study of VOAs is the study of their representations. So in this regard, VOA theory is similar to more classical algebraic systems such as groups, associative algebras and Lie algebras.

We let V be a VOA, and adopt the notation of Section 2. Roughly speaking, a *module* for V is a $\mathbb{C}$-linear space M such that the operators v_n ($v \in V$, $n \in \mathbb{Z}$) operate on M in such a way that the relations among the v_n implicit in the Jacobi identity are preserved. More precisely, in contrast to (2.1) we require M to possess a $\mathbb{C}$-grading

$$M = \coprod_{x \in \mathbb{C}} M_x \tag{4.1}$$

such that for any $z \in \mathbb{C}$ we have $M_{z+n} = 0$ for integral $n \ll 0$, and $\dim M_x < \infty$ for each x.

Note that this extends the definition of [FLM], say, who require that the grading (4.1) be rational. Under suitable circumstance one expects that in fact V-modules are rationally graded, but it seem better not to assume this at the outset, but rather try to *prove* it (cf. Problem 5). Furthermore, there is a linear map $Y_M : V \to \operatorname{End} M[[z, z^{-1}]]$, $Y_M : v \mapsto Y_M(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1}$. Once again, various axioms are imposed analogous to axioms (i)-(vi) of Section 2. The analogues of (i), (ii) are still required, though there is no analogue of the vacuum vector. The conformal vector ω is such that the components of $Y_M(\omega, z)$ generate a representation of the Virasoro algebra (2.3) on M with the *same* central charge c. Again, M_x is required to be the L_0-eigenspace with eigenvalue

x, and (v) still holds. The analogue of Jacobi now reads

$$\begin{aligned} & z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y_M(u,z_1)Y_M(v,z_2) \\ & \qquad - z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)Y_M(v,z_2)Y_M(u,z_1) \\ & = z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)Y_M(Y(u,z_0)v,z_2). \end{aligned} \tag{4.2}$$

Now suppose that $g \in \mathrm{Aut}(V)$ has finite order N. There is the notion of a *g-twisted V-module*, or *g-twisted sector*, which, as the name suggests, is some sort of "twisted" module for V. But it would be incorrect to compare it to the notion of twisted module in group representation theory; the correct analogy is with representations of twisted Kac–Moody algebras [K]. The details are as follows: M is a g-twisted V-module if $M = \coprod_{n\in\mathbb{C}} M_n$ is a $\mathbb{C}$-grade $\mathbb{C}$-linear space with the usual conditions on the grading. We also have a linear map $Y_g \colon V \to \operatorname{End} M[[z^{1/N}, z^{-1/N}]]$, $v \mapsto Y_g(v,z)$ satisfying the following: if $v \in V$ is such that $gv = e^{2\pi ij/N}v$ then

$$Y_g(v,z) = \sum_{n\in j/N+\mathbb{Z}} v_n z^{-n-1}, \tag{4.3}$$

and

$$\begin{aligned} & z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y_g(v,z_1)Y_g(w,z_2) \\ & \qquad - z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)Y_g(w,z_2)Y_g(v,z_1) \\ & = z_2^{-1}\left(\frac{z_1-z_0}{z_2}\right)^{-j/N}\delta\left(\frac{z_1-z_0}{z_2}\right)Y_g(Y(v,z_0)w,z_2). \end{aligned} \tag{4.4}$$

Note that if $g = 1$, a g-twisted V-module is precisely a V-module in the earlier sense. One can define twisted sectors even for automorphisms of infinite order, but we will not be concerned with them in this paper.

If (M, Y_g), (M', Y_g') are two g-twisted V-modules, a map $f\colon (M, Y_g) \to (M', Y_g')$ is a linear map $f\colon M \to M'$ which intertwines the Y-maps, i.e., $Y_g'(v,z)f = fY_g(v,z)$, $v \in V$. It is then apparent that there is a *category g-V-Mod* of g-twisted V-modules with objects being g-twisted V-modules and with morphisms as above.

A (g-twisted) submodule of the g-twisted V-module M is, rather obviously, a linear subspace $N \subseteq M$ such that for each $v \in V$ and

$Y_g(v,x) = \sum v_n z^{-n-1}$, each v_n leaves N invariant. There are then the usual notions of *simple* module, *semi-simple* (= completely reducible) module, *indecomposable* module, etc. As in Lie theory, it seems that the issue of complete reducibility is quite crucial. But it is fair to say that it has not been studied much in the present context. So we suggest

Problem 2. *Investigate the notion of complete reducibility in the category* g-V-Mod.

It turns out that the notion of *weak* (g-twisted) V-module is relevant (mainly because of the theory of Zhu algebras—see Section 6). We obtain a definition of a weak (g-twisted) V-module if we allow the homogeneous subspaces M_x ((4.1)) to be *infinite-dimensional*, but otherwise retain the previous definition. There is a corresponding category W-g-V-Mod of weak g-twisted V-modules.

We will be mainly concerned with certain classes of VOAs for which the category V-Mod (plain V-modules: we habitually drop the "g" in case $g=1$) satisfies certain finiteness conditions. We call V *rational* in case V satisfies the following conditions:

(4.5)

(a) V has only a finite number of inequivalent simple modules.

(b) Every V-module is completely reducible.

(c) Every weak simple V module is an (ordinary) V-module.

Similarly, V is called *g-rational* in case V satisfies (a)–(c) of (4.5) with "g-twisted modules" in place of "modules."

It seems likely that (a)–(c) are not independent conditions, and an affirmative solution of the next problem would be very powerful.

Problem 3. *Does* (4.5)(a) *imply the other two conditions? Same problem for g-twisted modules.*

We call V *holomorphic* in case it satisfies the following conditions:

(4.6)

(a) V has a unique simple V-module, namely itself.

(b) Every V-module is complete reducible.

To clarify the meaning of (4.6)(a), it is clear that any VOA V is itself a V-module, which we will call the *adjoint module* for V. To say that the adjoint module is simple is to say that V is a *simple VOA*, that is, it has no non-trivial ideals.

The group $\mathrm{Aut}(V)$ induces equivalences among the various categories $(W\text{-})g\text{-}V\text{-Mod}$, $g \in \mathrm{Aut}(V)$. Namely, suppose that (M, Y_M) is an object in $(W\text{-})g\text{-}V\text{-Mod}$ and that $h \in \mathrm{Aut}(V)$. Then h induces a categorical equivalence $\varepsilon(h) = \varepsilon_g(h)$:

$$g\text{-}V\text{-Mod} \xrightarrow{\varepsilon(h)} h^{-1}gh\text{-}V\text{-Mod} \tag{4.7}$$

as follows: the image of (M, Y_M) is the pair $(M, Y_M \circ h)$ where by definition $Y_M \circ h(v, z) = Y_M(hv, z)$, $v \in V$; and $\varepsilon(h)f = f$ for a morphism f. This assertion follows from the definitions: see [DM2] for more information.

For a group G and $g \in G$ we define the *centralizer* of g in G via $C_G(g) = \{x \in G | gx = xg\}$. If we set $C(g) = C_{\mathrm{Aut}(V)}(g)$ for $g \in \mathrm{Aut}(V)$ then it is clear from (4.7) that $C(g)$ induces automorphisms of the category $g\text{-}V\text{-Mod}$, and in particular $\mathrm{Aut}(V)$ acts on V-mod.

For $g \in \mathrm{Aut}\, V$ of finite order, let $V\text{-Mod}^g$ denote the full subcategory of g-invariants of V-Mod, so that objects of $V\text{-Mod}^g$ are V-modules (M, Y_M) satisfying $\varepsilon(g)(M, Y_M) \cong (M, Y_M)$. We would like to believe that the number of (isomorphism classes of) simple g-invariant V-modules is equal to the number of (isomorphism classes of) simple g-twisted V-modules—at least under some finiteness assumptions. We state a sharp form of this conjecture:

Problem 4. *Let $g \in \mathrm{Aut}(V)$ have finite order, and assume that V has only a finite number of isomorphism classes of simple modules. Show that there is a $C(g)$-equivariant categorical equivalence*

$$V\text{-Mod}^g \xrightarrow{\cong} g\text{-}V\text{-Mod}. \tag{4.8}$$

This conjecture contains a lot of information (assuming it is true!), and we regard it as one of the basic problems concerning twisted sectors. In particular, if (4.8) holds then being g-equivariant, we may expect that $\varepsilon_g(g)$ leaves invariant (up to isomorphism) each object of g-V-Mod. It is useful to observe that this is indeed the case for simple g-twisted V-modules (M, Y_M). It follows readily from the definition that such M have a grading of the form

$$M = \coprod_{n=0}^{\infty} M_{c+\frac{n}{N}}. \tag{4.9}$$

Here, N is the order $o(g)$ of g, $M_c \neq 0$, and c is some constant.* Then

*Not to be confused with central charge.

by defining $\phi(g)\colon M \to M$ via

$$\phi(g)|M_{c+\frac{n}{N}} = e^{-2\pi in/N}$$

we find that $\phi(g)$ induces an isomorphism $\phi(g)\colon (M, Y_M) \xmapsto{\cong} (M, Y_M \circ g)$ in g-V-Mod.

We note here that the constant c of (4.9), which we call the *top weight* (of M) is an important invariant. So too is the space M_c itself—the *top level* of M—and its dimension.

Problem 5. *Suppose that V has only finitely many simple modules. Prove that the top weights of all simple (twisted) V-modules are rational.*

We refer to [AM] for further information on this subject.

To complete this section we introduce the notion of *extended automorphisms* of g-twisted V-modules. To keep the exposition simple we assume that (V, Y) is a simple VOA and that (M, Y_M) is a simple g-twisted V-module.

Then an extended automorphism of M is a pair $(x, \alpha(x))$ where x, $\alpha(x)$ are invertible linear maps of M, V respectively, and which satisfy

$$Y_M(\alpha(x)v, k) = xY_M(v, z)x^{-1} \tag{4.11}$$

for $v \in V$. We also require that $\alpha(x)\omega = \omega$ and that $\alpha(x)$ commutes with g.

One can show that $x \mapsto \alpha(x)$ is a group homomorphism into the group of $C(g)$. The kernel is the group $\mathbb{C}^*$ of invertible scalar operators on M. Let $\mathrm{Aut}^e(M)$ denote the group of extended automorphisms, where $(x, \alpha(x))(y, \alpha(y)) = (xy, \alpha(xy))$. Generally there is no canonical complement to $\mathbb{C}^*$ in $\mathrm{Aut}^e(M)$—indeed there may be no complement at all. This is why we refer to "extended automorphism."

As an example, with $\phi(g)$ as in (4.10) the assertion that $\phi(g)$ induces an isomorphism from (M, Y_M) to $(M, Y_M \circ g)$ can alternately be viewed as saying that $(\phi(g), g)$ is an extended automorphism of M.

Similarly, if we set $C(M) = \{h \in C(g) | \varepsilon(h)(M, Y_M) \cong (M, Y_M)\}$ (cf. (4.7)), then $C(M)$ is a group (which contains g). And if $\phi(h)\colon M \to M$ is the morphism which realizes the equivalence of (M, Y_M) and its image under $\varepsilon(h)$, then $(\phi(h), h) \in \mathrm{Aut}^e(M)$.

Thus $\mathrm{Aut}^e(M)$ fits into a short exact sequence $1 \to \mathbb{C}^* \to \mathrm{Aut}^e(M) \to C(M) \to 1$ and hence defines an element of $C^2(C(M), \mathbb{C}^*)$.

§5. Examples of V-modules

We give examples of (twisted) V-modules which by-and-large track the examples of VOAs given in Section 3. Before proceeding, however, we want to emphasize that there are some serious gaps in our understanding of (twisted) modules insofar as *existence* is concerned. Here is a sample problem:

Problem 6. *Let V be a VOA with $g \in \mathrm{Aut}(V)$ of finite order. Prove that V has a (non-zero) weak g-twisted V-module. Better, prove that V has a (non-zero) g-twisted V-module.*

Now we turn to some examples.

(i) Let the notation be as in Example (i) of Section 3. Then the vector space M is a module for the VOA $\langle S\rangle$ generated by a set of mutually local operators S.

(ii) Let the notation be as in Example (ii) of Section 3. Then both $M(c,h)$ and $L(c,h)$ are modules for the VOA $L(c,0)$. It is in fact known when $L(c,0)$ is rational in the sense of (4.5), [W], [DMZ]. Namely, $L(c,0)$ is rational precisely when c has the form $c = 1-6(p-q)^2/pq$ for a pair of coprime integers p, q each at least 2. In this case, the simple modules for $L(c,0)$ are the Fock spaces $L(c,h)$ for $h = (np-mq)^2-(p-q)^2/4pq$, $0<m<p$, $0<n<q$.

Again this result is obtained by studying the Zhu algebra associated to $L(c,0)$ [W]. For more information on these remarkable families see, for example, [KR], [GKO], [FQS].

(iii) Let the notation be as in Example (iii) of Section 3. We limit our discussion to the *WZW*-models $L(c,0)$, so that c here is a positive integer. Then $L(c,0)$ is indeed rational, and the simple modules are those $L(c,\lambda)$ for which λ is an integrable weight satisfying $\langle\lambda,\mu\rangle \le c$ for μ the root of maximal height. For a proof, see [FZ] or [Li1].

(iv) Let V_L be the Fock space of the VOA associated to an even lattice L as in Example (iv) of Section 3. The rationality of V_L is established by Dong [Do1], where it is shown that the simple modules are precisely the Fock spaces $V_{L+\lambda}$ for λ in the dual lattice $L^0 = \mathrm{Hom}_{\mathbb{Z}}(L,\mathbb{Z})$. Here $V_{L+\lambda}$ is defined as in (3.1), except that the group algebra $\mathbb{C}[L]$ is now replaced by the "group algebra" of the coset: $\mathbb{C}[L+\lambda] = \coprod_{\alpha\in L}\mathbb{C}e^{\alpha+\lambda}$. In particular, V_L is holomorphic (4.6) if, and only if, L is a self-dual lattice.

There is a canonical action of the torus $\mathbb{R}^n/L^0$ as automorphisms of V_L. (Here, n is the rank of L.) Namely if, in the notation of (3.1), we take $u \in S(H_1 \oplus H_2 \oplus \cdots)$ and $\alpha \in L$ then set

$$g(\nu)\colon u \otimes e^{\alpha} \mapsto e^{2\pi i(\nu,\alpha)} u \otimes e^{\alpha} \tag{5.1}$$

for $\nu \in \mathbb{R}^n = \mathbb{R} \otimes_{\mathbb{Z}} L$. Now $g(\nu)$ has finite order when ν lies in $Q^n = Q \otimes_{\mathbb{Z}} L$. Then it is shown in [DM1] that the simple $g(\nu)$-twisted V_L-modules arise precisely from the Fock species $V_{L+\lambda-\nu}$, where λ runs over coset representatives L^0/L. So for such automorphisms of V_L, (4.8) is indeed an equivalence of categories.

We give one more example along these lines: let $\theta\colon L \to L$ be the automorphism which acts as -1. Then θ naturally induces an automorphism of V_L, which we also denote by θ. There is then a θ-twisted V_L-module $V_{L,\theta}$ with Fock space defined via

$$V_{L,\theta} = S(H_{1/2} \oplus H_{3/2} \oplus \cdots) \otimes T. \tag{5.2}$$

Here, each $H_{i/2}$ is linearly isomorphic to $H = \mathbb{C} \otimes_{\mathbb{Z}} L$. The space T is finite of dimension $2^{n/2}$ (n is necessarily even since L is an even lattice). It is best regarded as an irreducible module for a so-called extra-special 2-group Q (see [Gor, Chapter 5]) of order 2^{n+1} which is constructed from L as follows. The map $x \mapsto (-1)^{(x,x)/2}$ defines a quadratic form $q\colon L \to \pm 1$ with associated bilinear form $\beta\colon L \times L \to \pm 1$. We may regard β as a 2-cocycle in the group $C^2(L, \pm 1)$ and as such it defines a 2-fold central extension $\hat{L}$ of L. Then T is a simple module for $\hat{L}$ where the kernel is isomorphic to $2L$ (which splits over the center of $\hat{L}$), and $Q = \hat{L}/2L$.

This example is discussed at length in [FLM]. Generalizations may be found in various places, e.g., [Do3], [Le], [DGM2]. Historically, this has been an important example, both for understanding the Moonshine module and for coming to grips with the "philosophy" of twisted sectors.

(v) Parallel to the spinor construction of V_L for L the D_ℓ root lattice, which was discussed in Example (v) of Section 3, existence and uniqueness results for twisted sectors corresponding to certain kinds of automorphisms are given in [DM1]. The Fock spaces can again be described as Clifford modules for "twisted" Clifford algebras of a certain kind. The only point we want to mention here is that, in some senses, the only difference between the twisted and *untwisted sectors* (this latter being the VOA itself) is in the grading of the Fock spaces.

(vi) We can now say something more about the Moonshine module $V^\natural$ (Example (vi), Section 3). Let Λ be the Leech lattice ([CS]) and let θ be as in the previous example. Then θ is an automorphism of V_Λ, whence the subspace V_Λ^θ of θ-invariants is itself a VOA. (More on this sort of thing later!) Similarly θ acts on $V_{\Lambda,\theta}$ (5.2) if we take θ to act trivially on T. If we let $V_{\Lambda,\theta}^{\pm}$ be the ± 1 eigenspace of θ on this space, then the Fock space of the Moonshine module is

$$V^{\natural} = V_{\Lambda}^{\theta} \oplus V_{\Lambda,\theta}^{-}. \tag{5.3}$$

Dong has established [Do2] the important fact that $V^{\natural}$ is a holomorphic VOA. Moreover every simple weak $V^{\natural}$-module is itself isomorphic to $V^{\natural}$: This follows from the existence of the VOA subalgebra $L(1/2, 0)^{\otimes 48}$ discussed in Example (viii) of Section 3 together with Example (ii) of the present section and the theory of tensor products.

Less is known about twisted modules for $V^{\natural}$. Of course $\mathrm{Aut}(V^{\natural}) \cong \mathbb{M}$ (the Monster), and as a special case of Problem 4 one has

Problem 7. *For each $g \in \mathbb{M}$, prove that there is a unique simple g-twisted $V^{\natural}$-module. Prove also that any simple weak g-twisted $V^{\natural}$-module necessarily has finite-dimensional homogeneous subspaces.*

An affirmative answer to Problem 7 is provided in [DLiM3] in case g is one of the elements $2A$, $2B$, $4A$. (This is standard notation for certain elements in $\mathbb{M}$ of orders, 2, 2, 4, respectively. See [ATLAS].) Again the theory of $L(1/2, 0)^{\otimes 48}$-modules is the crucial ingredient. Huang also solves Problem 7 for g of type $2B$ in [Hu], where he does much more, including a fresh approach to the VOA structure on the Fock space (5.3). The work of Tuite [Tu1], [Tu2] on generalized Moonshine implicitly assumes the uniqueness of twisted sectors for $V^{\natural}$.

The extended automorphism groups of the twisted sectors of $V^{\natural}(2A)$ and $V^{\natural}(2B)$ are known [DLiM3]. We have $\mathrm{Aut}^e(V^{\natural}(2A)) \cong \mathbb{C}^* \times 2Baby$. Here $2Baby$ is the non-split extension of the Baby Monster ([ATLAS]) by $\mathbb{Z}_2$, and is the centralizer of $2A$ in the Monster $\mathbb{M}$. The group $\mathrm{Aut}^e(V^{\natural}(2B))$ is a central product $\mathbb{C}^* \cdot \hat{C}$, where $\hat{C}$ is the non-split extension of the (non-simple) Conway group $\cdot O$ (= Automorphism group of the Leech lattice) by the extra-special group 2^{1+24}. The subgroup of $\mathbb{C}^*$ of order 2 is identified in $\mathbb{C}^* \cdot \hat{C}$ with a certain central subgroup of $\hat{C}$. Note in this case that the centralizer of $2B$ in $\mathbb{M}$ is *not* isomorphic to a subgroup of $\mathbb{C}^* \cdot \hat{C}$, i.e., the map α which intervenes in the definition of extended automorphism (4.11) is not an embedding in this case.

(vii) Given VOAs V_i and V_i-modules M_i, $1 \le i \le n$, the tensor product $M_1 \otimes \cdots \otimes M_n$ is a module for the VOA $V_1 \otimes \cdots \otimes V_n$ in a canonical way. See [FHL] for details.

(viii) *Contragredient Module.* Let $M = \coprod M_n$ be a g-twisted V-module, with corresponding Y-map Y_M. Let $M_n^* = \mathrm{Hom}_{\mathbb{C}}(M_n, \mathbb{C})$

be the dual of M_n and set $M' = \coprod M_n^*$, the *restricted dual* of M. Then there is a g^{-1}-twisted V-module $(M', Y_{M'})$ where $Y_{M'}\colon V \to (\mathrm{End}\, M')[[z^{1/N}, z^{-1/N}]]$ is defined by the condition

$$\langle Y'_M(v,z)m', m\rangle = \langle m', Y_M(e^{zL_1}(-z^{-2})^{L_0}v, z^{-1})m\rangle\,. \tag{5.4}$$

Here $\langle\ ,\ \rangle\colon M' \times M \to \mathbb{C}$ is the canonical pairing and m', m are arbitrary elements in M', M respectively. The case $g = 1$ is discussed in [FHL], following the original remark of Borcherds [Bo1].

Note that the assignment $(M, Y_M) \to (M', Y_{M'})$ defines a contravariant equivalence

$$g\text{-}V\text{-Mod} \xrightarrow{\cong} g^{-1}\text{-}V\text{-Mod}. \tag{5.5}$$

§6. Zhu algebras

One approach to establishing equivalences such as (4.8) is this: define associative algebras whose module categories are equivalent to categories of V-modules (of various kinds). Then prove that the associative algebras are themselves Morita equivalent. Zhu algebras are associative algebras which arise from one attempt in this direction and we have already referred to them on several occasions. The ideas leading to their construction were first explained in [Z], and recently generalized in [DLiM2], which we follow here.

Let (V, Y) be a VOA and $g \in \mathrm{Aut}\, V$ be of finite order. If $u \in V$ is homogeneous and satisfies $gu = e^{2\pi i r}u$, $0 < r \leq 1$, define a bilinear product $*_g$ on V to be the linear extension of the following:

$$u *_g v = \begin{cases} \mathrm{Res}_z\, Y(u,z)\dfrac{(z+1)^{wt(u)}}{z}v, & r = 1, \\ 0, & r < 1. \end{cases} \tag{6.1}$$

Explicitly, we have if $r = 1$,

$$u *_g v = \sum_{i \geq 0} \binom{wt(u)}{i} u_{i-1}(v). \tag{6.2}$$

Similarly, let

$$u \circ_g v = \begin{cases} \mathrm{Res}_z\, Y(u,z)\dfrac{(z+1)^{wt(u)}}{z^2}v, & r = 1, \\ \mathrm{Res}_z\, Y(u,z)\dfrac{(z+1)^{wt(u)+r-1}}{z}v, & r < 1. \end{cases} \tag{6.3}$$

Let $O_g(V)$ be the linear span of all $u \circ_g v$ over all such u, v, r. Note that $u \circ_g 1 = u$ if $gu = e^{2\pi i r}u$ and $0 < r < 1$. Thus we have $V = V^{\langle g \rangle} + O_g(v)$ where $V^{\langle g \rangle}$ is the space of g-invariants of V. Define

$$A_g(V) = V/O_g(V). \tag{6.4}$$

At first blush it appears as though $A_g(V)$ is merely a linear space, but in fact the product $*_g$ defined by (6.1) descends to an associative product on $A_g(V)$. More precisely, formula (2.5) continues to hold for the action of $Y_M(v, z) = \sum_n v_n z^{-n-1}$ on a weak g-twisted V-module (M, Y_M). In particular, if we set $o_g(v) = v_{wt(v)-1}$ then, as mentioned following (2.5), there is a linear map $V \to \mathrm{End}(M_c)$, $v \mapsto o_g(v)$ where M_c is the top level of M (which may be of infinite dimension). It turns out that $O_g(V)$ lies in the kernel of this map, and that the resulting map $A_g(V) \to \mathrm{End}(M_c)$ is a morphism of associative algebras.

What is crucial is the following result, due to Zhu [Z] if $g = 1$. If M is a *simple weak* g-twisted V-module then M_c affords a simple module for $A_g(V)$. Moreover, the map $M \mapsto M_c$ induces a bijection between equivalence classes of simple weak g-twisted V-modules and simple $A_g(V)$-modules. Finally if weak g-twisted V-modules are completely reducible then $A_g(V)$ is semi-simple. So in this latter case, there is an equivalence of categories

$$W\text{-}g\text{-}V\text{-Mod} \xrightarrow{\cong} A_g(V)\text{-Mod}. \tag{6.5}$$

It is these results that make weak twisted V-modules important, even though we may be primarily interested in (ordinary) twisted V-modules. It is hardly necessary to say that one tries to use the Zhu algebras to reduce questions about twisted V-modules to questions about representations of associative algebras, which are presumably more manageable. Actually calculating $A_g(V)$ is difficult; we refer to [FZ], [DMZ], [W] for examples.

To give a different example, we know from [Do2] that $V^\natural$ is both rational and holomorphic (cf. Example (iv), Section 5). It follows that $A(V^\natural)$ is a matrix algebra, and since the top level of $V^\natural$ is spanned by the vacuum vector then in fact $A(V^\natural) \cong \mathbb{C}$. Similarly from [DLiM3] we find that if $g \in \mathbb{M} = \mathrm{Aut}(V^\natural)$ is of type $2A$ then $A_{2A}(V^\natural) \cong \mathbb{C}$; on the other hand $A_{2B}(V^\natural) \cong \mathrm{Mat}_{24}(\mathbb{C})$, the algebra of 24×24 matrices over $\mathbb{C}$. None of these facts seems to be readily deducible from the definitions (6.1)-(6.3) themselves.

One of the drawbacks (at present) with the algebras $A_g(V)$ is that there are problems in trying to make the connection between g-twisted

V-modules and $A_g(V)$ functorial. Note also that even though $A(V)$ has a unit (the image of the vacuum vector), it is not clear if $A_g(V)$ is non-zero in general. We suggest some problems along these lines.

Problem 8. *Prove that $A_g(V)$ is non-zero. That is, V possesses non-zero weak g-twisted V-modules.*

Problem 9. *Suppose that $A_g(V)$ is semi-simple. Is it true that W-g-V-Mod is semi-simple?*

§7. Group cohomology

We discuss the connections between group cohomology and VOAs.

In the following we fix a VOA V and finite group of automorphisms $G \leq \mathrm{Aut}(V)$. We assume given a family $\{(V(g), Y_g)\}$ of simple twisted modules, one for each $g \in G$, such that $V(1) = V$ and the following condition holds:

$$\varepsilon(h)(V(g)) \cong V(h^{-1}gh) \tag{7.1}$$

for $h \in G$ (cf. (4.7)). Thus the group G permutes the twisted modules $V(g)$ via the action (4.7). (7.1) is equivalent to choosing, for a fixed g in each conjugacy class, a $V(g)$ with the property that $\varepsilon(h)V(g) \cong V(g)$ for $h \in C_G(g)$. By way of example, if G is a group of *inner* automorphisms of V in the sense that G leaves invariant each (untwisted) V-module, then the truth of conjecture (4.8) would tell us that every $V(g)$ has the desired property. The case we are mainly interested in is when V is holomorphic (cf. (4.10)). Then according to (4.8) there is a unique simple g-twisted V-module, in which case we take the family $\{V(g)\}$ to consist of the simple g-twisted V-modules, $g \in G$.

By definition, there are linear maps $\phi_g(h)\colon V(g) \to V(hgh^{-1})$ satisfying $\phi_g(h)Y_g(v,z)\phi_g(h)^{-1} = Y_{hgh^{-1}}(hv,z)$. (Compare with (4.11) and the accompanying discussion.) Then there are $\alpha_g(h,k) \in \mathbb{C}^*$ (or S^1) satisfying $\phi_g(h,k) = \alpha_g(h,k)^{-1}\phi_{kgk^{-1}}(h)\phi_g(k)$.

Set

$$V^* = \bigoplus_{g\in G} V(g) \tag{7.2}$$

and introduce the group algebra $\mathbb{C}[G]$ of G and its dual $\mathbb{C}[G]^*$. The latter algebra is spanned by $e(g)$, $g \in G$ with relations $e(g)e(h) = e(g)\delta_{g,h}$. As in [Ba1], [Ba2], one may think of $e(g)$ in this context as projection of V^* onto $V(g)$. The group of units U of $\mathbb{C}[G]^*$ arises in this context because of the following: if $\alpha\colon G\times G\to U$ is given by $\alpha(h,k)=\sum_g \alpha_g(h,k)e(g)$ then we have $\alpha\in C^2(\mathbb{Z}G,U)$. That is, α is a 2-cocycle on G with coefficients in U, where U is a multiplicative $\mathbb{Z}G$ module with right G-action $e(g)\cdot h=e(h^{-1}gh)$.

Let $HH^*(\mathbb{Z}G)$ denote the Hochschild cohomology of $\mathbb{Z}G$ (coefficients in $\mathbb{Z}G^*$ by assumption). We have $H^*(\mathbb{Z}G,U)\cong HH^{*+1}(\mathbb{Z}G)$, so that the compatible family $\{V(g)\}$ provides us with an element $[\alpha]\in HH^3(\mathbb{Z}G)$. Now we have ([Lo], [Be]) $HH^3(\mathbb{Z}G)\cong\bigoplus_* H^3(C_G(g),\mathbb{Z})\cong\bigoplus_* H^2(C_G(g),\mathbb{C}^*)$ where here and below, $*$ denotes a sum over one element from each conjugacy class of G. From (4.10) and the accompanying discussion we see that the restriction of $[\alpha]$ to $H^3(C_G(g),\mathbb{Z})=H^2(C_G(g),\mathbb{C}^*)$ defines a central extension $\widehat{C_G(g)}$ of $C_G(g)$ on $V(g)$ which has the following property: a pre-image of g in $\widehat{C_G(g)}$ lies in the center of $\widehat{C_G(g)}$.

Let $HH_0^3(\mathbb{Z}G)$ be the group of Hochschild 3-cohomology classes satisfying this condition. Thus $\{V(g)\}$ defines an element $[\alpha]\in HH_0^3(\mathbb{Z}G)$.

Let $HC(\mathbb{Z}G)$ denote cyclic cohomology ([Lo], [Be], [Bu]). We have $HC^3(\mathbb{Z}G)\cong\bigoplus_* H^2(XC_G(g),\mathbb{C}^*)$. Here, $XC_G(g)$ is the so-called *extended centralizer* defined as the push-out

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{1\mapsto g} & C_G(g) \\ \downarrow & & \downarrow \\ \mathbb{R} & \longrightarrow & XC_G(g) \end{array}$$

If $H_0^2(XC_G(g),\mathbb{C}^*)$ is the subgroup of cohomology classes in $H^2(XC_G(g),\mathbb{C}^*)$ whose restriction to $\mathbb{R}$ (or rather its image in $XC_G(g)$) is trivial, and if $HC_0^3(\mathbb{Z}G)=\bigoplus_* H_0^2(XC_G(g),\mathbb{C}^*)$, then there is a natural isomorphism $HC_0^3(\mathbb{Z}G)\xrightarrow{\cong}HH_0^3(\mathbb{Z}G)$ given by restriction to $C_G(g)$. Conclusion: our family $\{V(g)\}$ determines an element $[\alpha]\in HC_0^3(\mathbb{Z}G)$. Note that the discussion in Example (vi), Section 5, shows that in the case of the Moonshine module $V^\natural$, the class $[\alpha]$ will be non-zero.

Given an element of $HC^3(\mathbb{Z}G)$, one can follow the recipe of Kontsevich [Ko] (see also [PS]) to derive certain topological invariants of moduli spaces. This is an example of how one can expect to derive topological invariants from VOAs.

Elliptic cohomology is a newer subject (see [La], [Hi]). In the work of Baker and Thomas ([Bak], [Th1], [Th2]) and others one can see how it may well be reasonable to expect that families $\{V(g)\}$ of the type we are considering will lead to elements in $\mathrm{Ell}^*(BG)$, the elliptic cohomology of the classifying space BG of G. The true geometric meaning of this theory remains a closed book, so we will simply recast our preceding discussion in bundle-theoretic language which might be part of the prescription for $\mathrm{Ell}^*(BG)$.

First, if $V = V(1)$ is graded as in (2.1), we may form the Borel space $EG \times_G V_n$ for each n (here, $EG \to BG$ is the universal G-bundle), so that we may consider $EG \times_G V \to BG$ as a $\mathbb{Z}$-graded vector bundle over BG with fiber V. Next, if LBG is the free loop space of BG then LBG is (homotopic to) a space with components $BC_G(g)$, one for each conjugacy class. Thus $H^*(LBG,\mathbb{Z}) \cong \bigoplus_* H^*(BC_G(g),\mathbb{Z}) = \bigoplus_* H^*(C_G(g),\mathbb{Z}) = HH^*(\mathbb{Z}G)$. Moreover we may form the Borel construction $EC_G(g) \times_{C_G(g)} \mathbb{P}V(g)$ where $\mathbb{P}V(g)$ is the projective space of $V(g)$ (recall that $C(g)$ only acts projectively on $V(g)$). We could also form $EC_G(g) \times_{\widehat{C_G(g)}} V(g)$ in the obvious way. In this way we get a vector bundle $\xi_V \colon E \to LBG$ which is naturally associated with $\{V(g)\}$.

In this picture, $HC^*(\mathbb{Z}G)$ arises as the integral cohomology of the Borel construction $ES^1 \times_{S^1} LBG$, where S^1 acts on LBG by rotation of loops. If $V(g) = V = \coprod_{n=0}^{\infty} V(g)_{c+\frac{n}{N}}$ is graded as in (4.9) then the action (4.10), essentially that of g on $V(g)$, is part of an action of $\mathbb{R}$ given by $t|V(g)_{c+\frac{n}{N}} = e^{-2\pi int/N}$, $t \in \mathbb{R}$. So $1 \in \mathbb{R}$ acts as g does, in other words we have an action of $XC_G(g)$ on $V(g)$, at least projectively. Since $ES^1 \times_{S^1} LBG$ is homotopic to $\bigcup_* BXC_G(g)$ we can repeat the above construction to get a bundle $\pi_V \colon M \to ES^1 \times_{S^1} LBG$ whose fiber over a point in $BXC_G(g)$ is $\mathbb{P}V(g)$. Note that in this situation, the grading on $V(g)$ is now indexed by a character of S^1. This is related to modular-invariance properties, which we discuss further in Section 10. This aspect of the theory is crucial to the very existence of elliptic cohomology, though our own manipulations have been purely formal. The reader might compare our discussion with Segal's notion of "elliptic object" [Seg2].

Problem 10. *Clarify the relationship between elliptic cohomology and VOA theory.*

To complete this section we consider another aspect of group cohomology in VOA theory, which we call the *Dijkgraaf–Witten cocycle.* To be on safe(r) ground, let us assume that V is holomorphic and that $\{V(g)\}$ constitutes all the simple twisted V-modules.

In [DW] it is argued on topological grounds that there is a cohomology class $[\delta] \in H^4(G,\mathbb{Z})$ which determines $[\alpha]$, and thereby puts conditions on the nature of $[\alpha]$ which go beyond the containment $[\alpha] \in HC_0^3(\mathbb{Z}G)$ which we have already established. We call δ the Dijkgraaf–

Witten cocycle. To be precise, if we think of δ as an element of $C^3(G,\mathbb{C}^*)$ then one expects that the following holds (up to coboundaries):

$$\alpha_g(h,k) = \frac{\delta(hkgk^{-1}h^{-1},h,k)\delta(h,k,g)}{\delta(h,kgk^{-1},k)}. \tag{7.3}$$

See* [DPR] for more information on this point. We note here only that δ is related to the issue of tensor products of V-modules, which we address in Section 9. Note that one see easily that the r.h.s. of (7.3) indeed defines an element of $HC_0^3(\mathbb{Z}G)$, so that it defines a map

$$DW\colon H^4(G,\mathbb{Z}) \to HC_0^3(\mathbb{Z}G). \tag{7.4}$$

In this notation we have

Problem 11. *Prove that $[\alpha]$ lies in the image of the map DW. Can one describe $[\delta]$ in terms of G and its action on V?*

An affirmative answer to the second part of Problem 11 would thus provide a canonical element of $H^4(G,\mathbb{Z})$ associated with the pair (V,G). A further topological significance of this is that one knows from the work of a number of authors ([DW], [FQ], [F], [Y] for example) that there is a canonical topological quantum field theory (TQFT) associated with a finite group G and an element of $H^4(G,\mathbb{Z})$. So one would get a map

$$\text{VOA} \to \text{TQFT}. \tag{7.5}$$

This is in accord with G. Segal's definition of conformal field theory (CFT) in [Seg1], which is categorical in nature. From this point-of-view (7.5) is a forgetful functor from CFT to TQFT in which the conformal structure is forgotten, leaving only a topological theory.

Finally, we mention

Problem 12. *Let $[\alpha]$ lie in the image of DW. Is there a holomorphic VOA V with $G \le \operatorname{Aut} V$ such that $[\alpha]$ arises as described above?*

§8. Fixed-point theory

We return to the theme of Section 6. It is a basic philosophy of the subject that the category of V-modules should be equivalent to the module category of some quasi-quantum group. In the case that V is

*Some of our formulae differ in appearance from those of [DPR]. See the footnote on page 123 in this regard.

a holomorphic VOA and G is a finite subgroup of $\mathrm{Aut}(V)$ one may consider the space V^G of G-invariants, which is evidently itself a VOA. In this case there is a precise conjecture, essentially described in [DPR], for what the quasi-quantum group is. We describe what is known about this situation, following [DM2]. But first we mention

Problem 13. *Let V be a holomorphic VOA with a finite group $G \leq \mathrm{Aut}(V)$. Prove that V^G is rational.*

We will always assume that V is holomorphic and that there is a unique simple g-twisted module $V(g)$ for each $g \in G$. Thus the results of Section 7 show that there is an element $[\alpha] \in HC_0^3(\mathbb{Z}G)$ canonically associated to this situation.

Let $D_\alpha(G)$ denote the smash product of $\mathbb{C}[G]$ with $\mathbb{C}[G]^*$ modified by α, where G acts on $\mathbb{C}[G]^*$ by right conjugation. Thus $D_\alpha(G) = \mathbb{C}[G] \otimes \mathbb{C}[G]^*$ as vector space with product*

$$a \otimes e(x) \cdot b \otimes e(y) = \alpha_y(a,b)ab \otimes e(b^{-1}xb)e(y). \tag{8.1}$$

So $D_\alpha(G)$ is essentially a twisted version of the quantum double of G (cf. [Dr]).

The space V^* ((7.2)) becomes a left $D_\alpha(G)$-module via the action $a \otimes e(x) \cdot m = \delta_{x,g}\phi_x(a)m$ for $m \in V(g)$ and $\phi_x(a)\colon V(x) \to V(axa^{-1})$ as in Section 7. Now from definition (4.3) we see that the restriction of the action of V on each $V(g)$ to V^G yields an ordinary V^G-module, so that V^* is in fact a V^G-module. Moreover the actions of V^G and $D_\alpha(G)$ commute, so that what obtains is an action of $D_\alpha(G) \otimes V^G$ (in the obvious sense) on V^*.

The simple modules for $D_\alpha(G)$ are easily classified ([DPR], [Ma2]), and it is known ([DM2],[DM3]) that *every* such module occurs in V^*. Thus we may write

$$V^* = \bigoplus_{\chi} M_\chi \otimes V_\chi \tag{8.2}$$

where χ ranges over the simple characters of $D_\alpha(G)$ and M_χ is a $D_\alpha(G)$-module affording χ. We may take V_χ to be a V^G-module, and if it has a grading of the form $V_\chi = \coprod_{n\in\mathbb{C}} V_{\chi,n}$ then we may think of $\sum_n(\dim V_{\chi,n})\, q^n$ as the *graded multiplicity* of M_χ in V^*. (For now, q is merely an indeterminate.)

The two main problems are the following:

*The map $a \otimes e(x) \mapsto a \otimes e(axa^{-1})$ will transform (8.1) into the definition used in [DPR].

Problem 14. *Prove that each V_χ in (8.2) is a simple V^G-module.*

Problem 15. *Prove that every simple V^G-module is contained in V^*.*

Problem 14 is proved in [DM2] for the case that G is a nilpotent group, and a similar proof will almost certainly deal with G solvable. But the general case is more difficult.

Now $D_\alpha(G)$ is semi-simple. So if we know also that V^G-modules are completely reducible, then affirmative solutions of Problems 14 and 15 provide us with a categorical equivalence

$$D_\alpha(G)\text{-Mod} \xrightarrow{\cong} V^G\text{-Mod} \tag{8.3}$$

$$M_\chi \mapsto V_\chi.$$

Suppose that α is trivial. Then as explained in [DPR] or [Ma2], the simple $D_\alpha(G) = D(G)$-modules are naturally indexed by pairs (g, χ_g) where g ranges over one element in each conjugacy class of G and χ_g ranges over the simple characters of $C_G(g)$. Hence, granted the equivalence (8.3), there are bijections between the following sets:

$$\begin{aligned}
&\{\text{simple } D(G)\text{-modules}\},\\
&\{\text{simple } V^G\text{-modules}\},\\
&\mathrm{Hom}(\mathbb{Z}^2, G)/\text{conjugation by } G,\\
&\{\text{principle } G\text{-bundles over } S^1 \times S^1\}/G,\\
&\{\text{homotopy classes of maps } S^1 \times S^1 \to BG\}/G.
\end{aligned} \tag{8.4}$$

Thus again we observe the convergence of ideas from topology and VOA theory.

§9. Tensor products

We discuss the quasi-quantum structure of $D_\alpha(G)$ and its conjectured meaning for V. We retain the notation of the preceding section.

Suppose to begin with that the cocycle α of the last section is trivial. Then $D_\alpha(G) = D(G)$ becomes a Hopf algebra with comultiplication $\Delta\colon D(G) \to D(G) \otimes D(G)$ defined by

$$\Delta\colon a \otimes e(x) \mapsto \sum_{\substack{h,k\in G\\ hk=x}} a \otimes e(h) \otimes a \otimes e(k). \tag{9.1}$$

(There is also a counit and antipode which, however, will not concern us here. See [DPR].) With this structure we may define a (strictly associative) tensor product of $D(G)$-modules, so that $D(G)$-Mod becomes a *strict monoidal category.* (For this and other categorical ideas in this section, we refer the reader to [SJ] or [SS].)

Problem 16. *Assume that α is trivial. Show that there is a notion of tensor product of V^G-modules such that the equivalence* (8.3) *is one of strict monoidal categories.*

In fact, the problem of defining tensor products of (twisted) modules for general VOAs is fundamental, and impinges on a number of the problems that we discuss in this paper. Given two modules X, Y for a VOA V, the tensor product $X \boxtimes Y$ should again be a V-module, but $X \boxtimes Y$ will not resemble the ordinary tensor product. For example, if V is holomorphic, then one expects that $V \boxtimes V \cong V$.

Huang and Lepowsky ([HL1], [HL2]) have considered this problem at great length. The paper [Li2] should also be consulted, along with the work [KL] which announces a significant advance concerning tensor products of Kac–Moody modules.

There are complications if the cocycle α is non-trivial, and we consider this case next. The main point is that $D_\alpha(G)$ should now be a *quasi-Hopf* algebra ([Dr], [SS]). This entails a modification of the coproduct (9.1) of the form

$$\Delta\colon a \otimes e(x) \mapsto \sum_{\substack{h,k \in G \\ hk=x}} \gamma_a(h,k)a \otimes e(h) \otimes a \otimes e(k) \tag{9.2}$$

where $\gamma_a(h,k) \in \mathbb{C}^*$ can be expressed in terms of the Dijkgraaf–Witten cocycle δ in a way that is entirely analogous to (7.3). One still requires Δ to be an algebra morphism from $D_\alpha(G)$ to $D_\alpha(G) \otimes D_\alpha(G)$ but it will no longer be associative. Instead one requires only that there is $\varphi \in D_\alpha(G)^{\otimes 3}$, φ invertible, such that for $d \in D_\alpha(G)$ we have

$$(\Delta \otimes \mathrm{id}) \circ \Delta(d) = \varphi((\mathrm{id} \otimes \Delta) \circ \Delta(d))\varphi^{-1}. \tag{9.3}$$

When this holds, the tensor product of $D_\alpha(G)$-modules is *quasi-associative* in the sense that $(X \otimes Y) \otimes Z$ and $X \otimes (Y \otimes Z)$ are isomorphic by an isomorphism which depends on X, Y, Z. One takes φ to be a linear combination of element of the form $1 \otimes e(g) \otimes 1 \otimes e(h) \otimes 1 \otimes e(k)$ as (g,h,k) range over G^3, with coefficients given by the Dijkgraaf–Witten cocycle (thought of as an element of $C^3(G,\mathbb{C}^*)$).

There is even more structure available: $D_\alpha(G)$ has a so-called *R-matrix*, which in this case is the element

$$R = \sum_{g,h \in G} 1 \otimes e(g) \otimes 1 \otimes e(h) \tag{9.4}$$

in $D_\alpha(G) \otimes D_\alpha(G)$. The first property of R is that it satisfies for $d \in D_\alpha(G)$:

$$\tau \circ \Delta(d) = R\Delta(d)R^{-1} \tag{9.5}$$

where τ is the twist operator on $D_\alpha(G) \otimes D_\alpha(G)$. Equation (9.5) ensures that $X \otimes Y \cong Y \otimes X$ for $D_\alpha(G)$-modules via an isomorphism that depends once again on X and Y. Two further properties of R allow us to deduce the so-called *quasi-Yang–Baxter equations*, and together these axioms give $D_\alpha(G)$ the structure of a *quasi-triangular quasi-Hopf algebra.*

Needless to say, all of this structure on $D_\alpha(G)$ and its module category is expected to reflect analogous properties of the category V^G-Mod. We will merely state the main problem, the extension of Problem 17 to the general case.

Problem 17. *Show that there is a notion of tensor product of V^G-modules such that the equivalence* (8.3) *is one of braided monoidal categories.*

§10. Modular-invariance

We discuss the conjectured modular-invariance properties of the characters of modules of rational VOAs, concentrating on the case of holomorphic orbifolds, i.e., VOAs of the form V^G where V is holomorphic and $G \leq \operatorname{Aut} V$ is finite (cf. Section 8). We assume some familiarity with elliptic modular functions.

Assume to begin with that V is a VOA and M a simple V-module. Then according to (4.5), M is graded with shape $M = \coprod_{n=0}^{\infty} M_{c+n}$ with $M_c \neq 0$. The (graded) character of M is defined to be

$$ch\, M = q^c \sum_{n=0}^{\infty} (\dim M_{c+n}) q^n. \tag{10.1}$$

According to convenience, we take q to be either an indeterminate or of the form $q = e^{2\pi i \tau}$ with $\tau \in \mathbb{C}$ satisfying $\mathrm{im}\tau > 0$. Thus τ lies in the upper-half plane $\mathfrak{H}$.

Problem 18. *Is it true that* $ch\,M$ *is a holomorphic function on* $\mathfrak{H}$*?*

Let us now take V to be a rational VOA (cf. (4.5)) with simple V-modules $M_1, \dots, M_k$. Let c' denote the central charge of V, and set

$$ch'\,M_i = q^{-c'/24}\,ch\,M_i. \tag{10.2}$$

Inclusion of the factor $q^{-c'/24}$ may seem rather strange, but it considerably enhances the invariance properties of the character. If $\Gamma = SL(2,\mathbb{Z})$ is the modular group then Γ acts on $\mathfrak{H}$ via Möbius transformations in the usual way, and thence on holomorphic functions on $\mathfrak{H}$.

Problem 19. *Let E be the $\mathbb{C}$-space spanned by the functions $ch'\,M_i$, $1 \le i \le k$. Prove that E is invariant under the action of Γ. Prove, moreover, that the kernel of the action is a congruence subgroup of Γ.*

The work of Kac–Peterson [KP], [K] on the analogue of Problem 19 for Kac–Moody Lie algebras is very important. And the paper of Zhu [Z] establishes the truth of Problem 19 for VOAs satisfying certain additional finiteness conditions which are, so far as we know, satisfied in all of the known rational VOAs.

As illustration, suppose that V is holomorphic so that $k=1$ and $M_1 = V$. Problem 19 suggests that $ch'\,V$ is then a (semi-)invariant of Γ, i.e., if we set $Z(\tau) = ch'(V)$ then $Z(\gamma\tau) = \varepsilon(\gamma)Z(\tau)$ for some character ε of Γ. This expectation is borne out in practice: for example, if L is a self-dual lattice and V_L the corresponding VOA (Section 3, Example (iv)) then V_L is holomorphic (Section 5, Example (iv)) and $ch'\,V_L = \theta_L(\tau)/\eta(\tau)^{\dim L}$. Here $\theta_L(\tau)$ is the theta-function of L and $\eta(\tau)$ the Dedekind eta-function. It is well-known [Se] that $\theta_L(\tau)$ is a form level one and that $\eta(\tau)^{\dim L}$ transforms under Γ with a character of order 1 or 3 (since 8 divides $\dim L$). If $V^\natural$ is the Moonshine module (Sections 3 and 5, Example (vi)) then $ch'\,V^\natural = J(q)$ is the modular function $q^{-1} + 0 + 196884q + \cdots$ of level one.

Now let V again be holomorphic of central charge c' with finite $G \le \operatorname{Aut} V$. In this case there is a beautiful and sharp reformulation of Problem 19 for the VOA V^G which goes to the heart of "Moonshine." To describe this we need to assume that the simple V^G-modules are as described in Problems 13–15. Thus after choosing a g in each conjugacy class of G, we assume existence and uniqueness of a simple g-twisted V-module, say $V(g)$. It is convenient to assume that the cocycle α is trivial—the general case involves only cosmetic changes.

Now $V(g)$ will be graded as in (4.9), that is $V(g)=\coprod_{n=0}^{\infty} V(g)_{c(g)+\frac{n}{N}}$ with $N = o(g)$ and $c(g)$ a scalar (presumably rational). As $V(g)$ admits $C(g)$ we may define for $h \in C(g)$,

$$Z(g,h,\tau) = q^{c(g)} \sum_{n=0}^{\infty} (\operatorname{tr} h|V(g)_{c(g)+\frac{n}{N}})q^{\frac{n}{N}}. \tag{10.3}$$

According to (8.2) the simple V^G-modules M contained in $V(g)$ are parameterized bijectively by the simple modules (or characters) of $C(g)$, and the character $ch\,M$ of M is the graded multiplicity of the corresponding character of $C(g)$. By the orthogonality relations for the characters of $C(g)$, the space spanned by $ch\,M$ as M ranges over the simple V^G-modules in $V(g)$ is that spanned by the functions $Z(g,h,\tau)$ as h ranges over one element in each conjugacy class in $C(g)$. Thus one sees that the space E of characters of V^G (Problem 19) is also spanned by the functions

$$Z'(g,h,\tau) = q^{-c'/24} Z(g,h,\tau) \tag{10.4}$$

as (g,h) ranges over the set $P = P(G)$. Here P is a set of representatives of $\{(x,y) \in G \times G | xy = yx\}$ modulo simultaneous conjugation by G. (Hence we have $|P(G)| = \#$simple V^G-modules—cf. (8.4).)

Now Γ acts on the right of P via $(x,y)\cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (x^a y^c, x^b y^d)$. An element of P may be thought of as an element of $\mathrm{Hom}(\pi_1(S^1 \times S^1), G)$, where the prescribed action of Γ is essentially that on a homology basis of the torus. Now we have

Problem 20. *Let E be the space spanned by the functions $ch'\,M$ as M ranges over the simple V^G-modules. Then E is also spanned by the trace functions $Z'(g,h,\tau)$, $(g,h) \in P$. Moreover E is a Γ-module and the action of $\gamma \in \Gamma$ satisfies*

$$Z'(g,h,\gamma\tau) = \varepsilon(g,h,\gamma) Z'((g,h)\gamma, \tau) \tag{10.5}$$

for some root of unity $\varepsilon(g,h,\gamma)$.

The condition (10.5) was introduced by Norton in an appendix to the paper [Ma1] on an empirical basis, and was subsequently extended to more general contexts in [Ma3], [Ma4]. Its origins in CFT can be traced through [DHW], [DGH]. In [DM1], (10.5) is actually established for certain fermionic orbifolds (Section 3, Example (v)). Little else is *known* to be true save for some calculations in [DLiM3] concerning the Moonshine module $V^\natural$ and its involutorial automorphisms.

Included in Problem 20 is the conjecture that (taking $g = 1$), the graded characters of $h \in G$ on V, more precisely the functions $Z'(1, h, \tau)$, are modular functions. Moreover one expects them to be semi-invariants of $\Gamma_0(M)$ where $M = o(h)$. Of course this is precisely how Monstrous Moonshine got started: the paper of Conway and Norton [CN] made very precise conjectures about the functions $Z'(1, h, \tau)$ in the case $V = V^\natural$ and $G = \mathbb{M}$ (Monster). These conjectures were subsequently established by Borcherds [Bo2], although his methods went outside the theory of VOAs as described here and utilized his theory of generalized Kac–Moody Algebras. Perhaps this is to be expected; after all, the Conway–Norton conjectures specify that in the case of $V^\natural$ the trace functions $Z'(1, h, \tau)$, are so-called *hauptmoduls*. These are very special kinds of modular functions, and certainly for other VOAs the corresponding trace functions are *not* hauptmoduls.

Problem 21. *Can Borcherds' theorem (Conway–Norton conjectures) be established via the representation theory of VOAs?*

Michael Tuite has studied this problem from the viewpoint of conformal field theory [Tu1], [Tu2]. His very interesting ideas suggest a specific path towards a solution to Problem 21 based on so-called orbifolding.

Problem 22. *Let $V = V^\natural$ be the Moonshine Module. Prove that each $Z'(g, h, \tau)$ is either a hauptmodul or is constant.*

This is Norton's Generalized Moonshine Conjecture. Actually, we have already pointed out in a previous section that for $V^\natural$ the cocycle α is not trivial. Thus one has to suitably interpret $Z'(g, h, \tau)$ in order to make the problem meaningful. Borcherds' methods do not appear to extend to this situation, whereas in principle those of Tuite do.

Problem 23. *What is the significance of the hauptmodul property of the trace functions on the Moonshine Module?*

One can say that the hauptmodul property is a deep-seated consequence of properties of the Leech lattice, but this doesn't seem to help much. We also note that in [DM4] it is shown that a certain fermionic orbifold, with group G being the large Mathieu group M_{24}, enjoys the hauptmodul property. In fact it satisfies the conclusions of Problem 22. The Leech lattice plays no rôle in this latter work though M_{24} is of course intimately related to it.

§11. Quantum Galois theory

It is well-known that VOAs behave in some ways like commutative, associative algebras. So a *simple* VOA V (i.e., one with no proper V-submodule) may be likened to a commutative field. Quantum Galois Theory (QGT) is a manifestation of this analogy.

We fix a simple VOA (V, Y) with finite group $G \leq \operatorname{Aut} V$. A sub VOA of V is a subspace $W \subseteq V$ preserved by all component operators w_n of $Y(w, z)$ for $w \in W$, and with the same vacuum vector and conformal vector as V.

Problem 24. *Prove that there is a Galois correspondence (i.e., a containment-reversing bijection) between subgroups of G and sub VOAs of V containing V^G given by*

$$H \mapsto V^H. \tag{11.1}$$

This was conjectured in [DM3], and the case in which G is *nilpotent* or *dihedral* is settled affirmatively in [DM2], [DM3]. It is also known (*loc. cit.*) that (11.1) is an injection, and that if G is *solvable* then (11.1) gives a bijection between normal subgroups of G and G-invariant sub VOAs of V which contain V^G. An interesting point is that we know (*loc. cit.*) that if V is a simple VOA and $G \leq \operatorname{Aut} V$ is finite then V^G is also simple. This enables one to proceed inductively and, if G is solvable at least, to reduce certain questions to the case in which G is abelian. But if G is not solvable we have no clue at present; what is lacking is any perspective which would at least explain why there *should* be a Galois correspondence.

Another point is this: if in (8.2) we restrict our attention to $V(1) = V$, then we obtain

$$V = \bigoplus_{\chi} M_\chi \otimes V_\chi \tag{11.2}$$

where χ ranges over the simple characters of G, M_χ a simple G-module affording χ, and V_χ a certain V^G-module. Now (8.2) was a decomposition for holomorphic V, but (11.2) holds for a simple VOA V.

Problem 25. *Prove that in* (11.2) *with V simple, each V_χ is a non-zero simple V^G-module.*

This is known for solvable groups (*loc. cit.*). We may regard (11.2) as the QGT analogue of the normal basis theorem, which says that if

E/K is a normal extension of fields with (finite) Galois group G, then E is a regular KG-module.

Armed with a character ψ of G and normal extension E/K with Galois group G, one may construct the Artin L-series $L_{E/K}(\psi, s)$ and ask after its analytic properties. The QGT analogue might then be the analytic properties of the characters of the V_χ in (11.2): of course this is an *additive* theory, whereas $L_{E/K}(\psi, s)$ is, by definition, an Euler product.

Let us take V holomorphic to be on safe ground (a churlish phase at this stage of the proceedings!). Then the character $ch' V$ (cf. Section 10) is the analogue of the zeta-function $\zeta_{E/K}(s)$, and one expects $ch' V$ to be a modular function of level 1 (possibly with character). The characters $ch' V_\chi$ are the analogues of the L-series $L_{E/K}(\chi, s)$. The decomposition of $\zeta_{E/K}(s)$ into factors $\zeta_{E/K}(s) = \prod_\chi L_{E/K}(\chi, s)^{\deg \chi}$ is just a formal consequence of the basis theorem, and its QGT analogue comes from (11.2), namely $ch' V = \sum_\chi (\deg \chi)\, ch' V_\chi$.

Problem 26. *Is there a QGT-analogue of the Euler products which define the L-series?*

Put another way, one might ask if there is a local theory of VOAs. Recent remarkable work of Borcherds and Ryba [R], [BR], [Bo3] suggests that the answer is "yes"! Not only that, but there seem to be connections between local VOAs (whatever they are) and twisted sectors.

Added in Proof (November 3rd. 1995): Since this paper was written there has been some progress towards solving several of the 26 problems. We briefly discuss these newer results here.

Following Problem 2 we introduced the notion of a weak g-twisted V-module. There is a useful variant of this which is intermediate between weak module and ordinary module, namely 'admissible' module. An admissible (g-twisted) V-module is a weak module such that the component operators of $Y(v, z)$ for homogeneous v preserve the homogeneous spaces of the module in the usual way. This idea is important in Zhu's work [Z]. It is now established in [DLiM2] that if all admissible g-twisted V-modules are completely reducible (a strengthening of (4.5)(b)), then (4.5)(a) holds and also (4.5)(c) holds for admissible modules. Compare these results with Problem 3. Further results in this direction are contained in another preprint (Regularity of rational vertex operator algebras) of the authors and H. Li. Let us now call a VOA g-rational if indeed all of its admissible g-twisted modules are completely reducible.

We referred to certain finiteness conditions in [Z] following Problem 19. Essentially, these are (g-)rationality in the sense of the last paragraph together with a certain "C_2 Condition"(loc.cit.). In forthcoming work of the authors and Li (Elliptic functions and orbifold theory) the following are established: affirmative solution of Problem 4 in case V is both rational, g-rational, and satisfies C_2; affirmative solution of Problem 5 under the same conditions; affirmative solution of Problems 6 and 8 if V is rational and satisfies C_2; affirmative solution of Problem 7. Problem 20 is also solved under similar conditions on V, though with the proviso that there is no information available with regard to the intervening constant — which should be a root of unity.

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Department of Mathematics
University of California
Santa Cruz, CA 95064-1099
U.S.A.