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Generalized Generalized Spin Models Associated with Exactly Solvable Models

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Abstract.

We show a close connection between generalized generalized spin models (four-weight spin models) and exactly solvable models. The defining relations of four-weight spin models are discussed from the viewpoint of the star-triangle relations. For an illustration we derive symmetric spin models from the self-dual Z_N model, and then construct various four-weight spin models through gauge transformations. We also show that the gauge transformations do not change the values of the link invariants derived from the four-weight spin models.

§1. Introduction

In mathematical physics, there is a close connection between the theories of integrable models and the topological invariants of links [1, 22]. The key is the Yang-Baxter relation, which plays a central role in the 1-dim. many body system [24] and in solvable lattice models in statistical mechanics [5].

The formalism of spin model was introduced by Jones as a device for construction of link invariants and representations of the braid group [15]. We call topological invariant of knots and links link invariant. Various spin models and link invariants have been discussed [4, 10, 12, 13, 18]. Recently, the formalism of spin model due to Jones has been extended into that of generalized spin model [18] and further into that of generalized generalized spin model (four-weight spin model) [4]. Many four-weight spin models have been constructed explicitly [4, 16].

In this paper we discuss four-weight spin models from the viewpoint of the star-triangle relations of exactly solvable models in statistical mechanics. We show that certain limits of the Boltzmann weights of the

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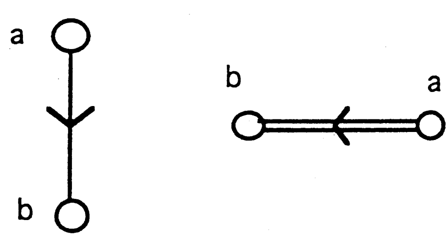
self-dual Z_N model give explicitly the symmetric spin models and the generalized generalized spin models. We introduce two gauge transformations on the Boltzmann weights of solvable models satisfying the star-triangle relations. Various four-weight spin models are constructed by applying the gauge transformations to the self-dual Z_N model. It is thus shown that these four-weight spin models can be generalized into the exactly solvable models, i.e., they can be "Yang-Baxterized". Furthermore, we show that the derived link invariants do not depend on the parameters related to the gauge transformations. We hope that the results of the paper might shed some light on connection of generalized generalized spin models to exactly solvable models in statistical mechanics.

After submission of the manuscript the author was informed that the gauge transformations for four-weight spin models have been independently introduced by F. Jaeger, who also proved that the derived link invariants are independent from the transformations [14].

§2. The star-triangle relations

2.1. The Z_N model

In statistical physics, a lattice model is called solvable if it has an infinite number of commuting transfer matrices [5]. If the Boltzmann weights satisfy the Yang-Baxter relations, then we can show that the transfer matrices commute, and that the system has an infinite number of "symmetries" or "conserved quantities". For Potts models and Z_N models the Yang-Baxter relations are called the star-triangle relations. Here Z_N denotes $\mathbf{Z}/N\mathbf{Z}$.



(1) (2)

Fig. 1. Two types of Boltzmann weights (1) $X(a, b; u)$ and (2) $Y(a, b; u)$. Arrows are often abbreviated.

Let us introduce a general Z_N model with nearest-neighbor interaction on an M-by-$2L$ rectangular lattice [9]. A lattice point is described by a vector $\vec{n} = (n_1, n_2)$, where $1 \leq n_1 \leq M$ and $1 \leq n_2 \leq 2L$. We assume the periodic boundary condition: $n_1 = n_1 + M(\mathrm{mod} M)$ and $n_2 = n_2 + 2L(\mathrm{mod} 2L)$.[1] To every lattice point $\vec{n}$ we associate a spin variable $\alpha(\vec{n})$, which takes Z_N values.

We introduce two types of Boltzmann weights $X(\alpha, \beta; u)$ and $Y(\alpha, \beta; u)$, where α and β are spin variables on two nearest-neighboring lattice points and u is a complex parameter called spectral parameter. Mathematically, the Boltzmann weights are functions on the spin variables and spectral parameter: $\mathbf{Z}_N \otimes \mathbf{Z}_N \otimes \mathbf{C} \to \mathbf{C}$. The physical meaning of the Boltzmann weight is that it denotes the probability of the two nearest-neighboring spin variables taking a particular value in $Z_N \otimes Z_N$, where the spectral parameter may play the role of the "temperature".

We now consider the solvability condition of the Z_N model. The star-triangle relations are given by the following.

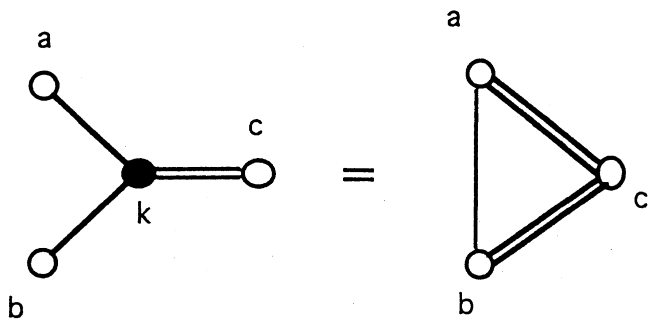


Fig. 2. Star-triangle relations:
$\sum X(a, k; u) X(k, b; v) Y(c, k; w)$
$= C(u, v) X(a, b; w) Y(c, b; u) Y(c, a; v)$

$$\sum_{k \in Z_N} X(\alpha, k; u) X(k, \beta; v) Y(\gamma, k; w) \tag{2.1}$$
$$= C(u, v, w) X(\alpha, \beta; w) Y(\gamma, \beta; u) Y(\gamma, \alpha; v),$$

$$\sum_{k \in Z_N} X(k, \alpha; v) X(\beta, k; u) Y(k, \gamma; w) \tag{2.2}$$
$$= C(u, v, w) X(\beta, \alpha; w) Y(\beta, \gamma; v) Y(\alpha, \gamma; u).$$

[1] Under the periodic boundary condition we may assume $\vec{n} \in \mathbf{Z}_M \otimes \mathbf{Z}_{2L}$.

Here the spectral parameter w is to be determined from the parameters u and v under a certain rule, which may be different depending on models.

In order to show the merit of the star-triangle relations we consider the transfer matrices of the Z_N model. Let $\vec{\alpha}$ and $\vec{\beta}$ be the sequences of the spin variables on the k-th and $(k+1)$-th rows of the M-by-$2L$ lattice, respectively, i.e., $\alpha_j = \alpha(k,j)$ and $\beta_j = \alpha(k+1,j)$. We define the transfer matrices $V_M(u)$ and $W_M(u)$ as follows

$$
\begin{aligned}
V_M(u)_{\vec{\alpha},\vec{\beta}} &= \prod_{j=1}^{M} \left(X(\alpha_j, \beta_j; u) Y(\alpha_{j+1}, \beta_j; u) \right), \\
W_M(u)_{\vec{\alpha},\vec{\beta}} &= \prod_{j=1}^{M} \left(X(\alpha_j, \beta_j; u) Y(\beta_{j+1}, \alpha_j; u) \right).
\end{aligned}
\tag{2.3}
$$

Here we recall the periodic boundary condition that $\alpha_{M+1} = \alpha_1$ and $\beta_{M+1} = \beta_1$.

The transfer matrices are important in calculation of the partition function of the model. In terms of the transfer matrices the partition function Z of the lattice model is given by $Z = \mathrm{Tr}\left((V_M(u) W_M(u))^L \right)$.

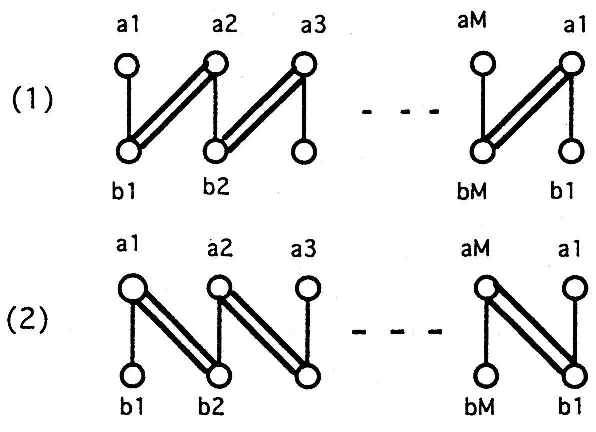


Fig. 3. (1) Transfer matrix
$V(u) = \prod X(\alpha_j, \beta_j; u) Y(\alpha_{j+1}, \beta_j; u)$
(2) Transfer matrix
$W(u) = \prod X(\alpha_j, \beta_j; u) Y(\beta_{j+1}, \alpha_j; u)$
Periodic boundary condition is assumed:
$\alpha_{M+1} = \alpha_1, \beta_{M+1} = \beta_1$.

Proposition 2.1. *If the star-triangle relations* (2.1) *and* (2.2) *hold, then we can show for any M that the transfer matrices commute:*

$$\begin{aligned} V_M(u)W_M(v) &= W_M(v)V_M(u), \\ V_M(u)V_M(v) &= V_M(v)V_M(u), \\ W_M(u)W_M(v) &= W_M(v)W_M(u). \end{aligned} \tag{2.4}$$

(For proof, see for instance, §7.2 of [5].)

2.2. Gauge transformations on solvable models

We consider two transformations on the Boltzmann weights of the models satisfying the star-triangle relations. [2]

For an arbitrary function $f(\alpha)$ on $\alpha \in Z_N$ we introduce the following transformation of weights from X and Y into $\tilde{X}$ and $\tilde{Y}$

$$\tilde{X}(\alpha,\beta;u) = X(\alpha,\beta;u) \times f(\alpha)/f(\beta), \quad \tilde{Y}(\alpha,\beta;u) = Y(\alpha,\beta;u), \tag{2.5}$$

for any $\alpha, \beta \in Z_N$.

Proposition 2.2. *Let us assume that the Boltzmann weights $X(\alpha,\beta;u)$ and $Y(\alpha,\beta;u)$ satisfy the star-triangle relations* (2.1) *and* (2.2)*, and that $\tilde{X}$ and $\tilde{Y}$ are obtained from X and Y by the gauge transformation* 1*. Then the weights $\tilde{X}$ and $\tilde{Y}$ also satisfy the star-triangle relations* (2.1) *and* (2.2)*.*

We now introduce another transformation by the following.

$$\tilde{Y}(\alpha,\beta;u) = Y(\alpha,\beta+\gamma;u), \quad \tilde{X}(\alpha,\beta;u) = X(\alpha,\beta;u), \tag{2.6}$$

for any α and $\beta \in Z_N$. We call the transformation gauge transformation 2.

Proposition 2.3. *Let us assume that the Boltzmann weights $X(\alpha,\beta;u)$ and $Y(\alpha,\beta;u)$ satisfy the star-triangle relations* (2.1) *and* (2.2)*, and that $\tilde{X}$ and $\tilde{Y}$ are obtained from X and Y by the gauge transformation* 2*. If there is a function $g(m;u)$ on $Z_N \otimes \mathbf{C}$ such that $Y(\alpha,\beta;u) = g(\alpha-\beta;u)$, then the transformed Boltzmann weight $\tilde{Y}$ and the weight X satisfy the star- triangle relations* (2.1) *and* (2.2)*.*

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[2] For four-weight spin models essentially the same transformations were discussed independently by F. Jaeger [14].

§3. The self-dual Z_N model

3.1. The Boltzmann weights

We now introduce the Boltzmann weights of the self-dual Z_N model. Let r denote an integer coprime to N, $(N,r)=1$. For a positive integer m we introduce weight $x_m(u)$ by the following. [3]

(3.1)
$$x_m(u)=\prod_{k=0}^{m-1}\sin\left(\frac{r-N}{2N}u+\frac{\pi rk}{N}\right)\Big/\sin\left(\frac{r-N}{2N}u-\frac{\pi r(k+1)}{N}\right).$$

For $m=0$ we assume $x_0(u)=1$. We may define x_{-m} for $m>0$ by $x_{-m}=x_m(u)$, since from (3.1) we have $x_{N-m}=x_m(u)$ for $0<m<N$.

We define $y_{\alpha-\beta}(u)$ by $y_{\alpha-\beta}(u)=x_{\alpha-\beta}(\pi-u)$. The Boltzmann weights $\{x_m(u)\}$ satisfy the star-triangle relation [9]:

(3.2)
$$\sum_{k=0}^{N-1}x_{\alpha-k}(u)x_{k-\beta}(v)y_{\gamma-k}(u+v)=C(u,v)x_{\alpha-\beta}(u+v)y_{\gamma-\beta}(u)y_{\gamma-\alpha}(v),$$

where $C(u,v)=F_0(u)F_0(v)/F_0(u+v)$. $F_j(u)$ denotes the Fourier transform of $x_m(u)$

$$F_j(u)=\sum_{k=0}^{N-1}x_k(u)\omega^{jk}, \tag{3.3}$$

where ω is given by $\omega=\exp(2\pi\sqrt{-1}/N)$. The star-triangle relations (3.2) correspond to (2.1) and (2.2) by

$$X(\alpha,\beta;u)=x_{\alpha-\beta}(u),\qquad Y(\alpha,\beta;u)=y_{\alpha-\beta}(u)=x_{\alpha-\beta}(\pi-u). \tag{3.4}$$

We make a remark on the proof of the star-triangle relations (3.2) of the model. The self-dual Z_N model corresponds to the cyclotomic solution in[17], whose star-triangle relations are explicitly proved in Ref. [17].

One of the most characteristic properties of the self-dual Z_N model is the following relation called self-duality [9]:

$$x_m(\pi-u)=\tilde{x}_m(u)=F_m(u)/F_0(u). \tag{3.5}$$

The Boltzmann weights $x_m(u)$ also satisfy the following basic properties.

[3] The parametrization in Ref. [9] corresponds to the case $r=N+1$.

1. The initial conditions:

$$x_m(0) = \delta_{m0}, \qquad x_m(\pi) = 1. \tag{3.6}$$

2. The inversion relations:

$$x_m(\pi + u)x_m(\pi - u) = 1, \tag{3.7}$$

$$\sum_k x_{\alpha-k}(u)x_{k-\beta}(-u) = C(u,\pi)\delta_{\alpha,\beta}. \tag{3.8}$$

It is straightforward to see the initial conditions (3.6) and the inversion relation (3.7) from the parametrization (3.1). We can show (3.8) by putting $v = \pi$ in the star-triangle relation (3.2) and using (3.6) and (3.7).

We give a comment on solvable models associated with the self-dual Z_N model. Two families of generalizations are known for the model. The chiral Potts model [2] and the broken Z_N symmetric model [17]. The latter model has been reconstructed from the representation of the Sklyanin algebra [11], and has also been generalized into $Z_N^{\otimes n-1}$-symmetric model [21]. We may say that the chiral Potts model and the broken Z_N symmetric model are associated with the generalized generalized spin models which will be discussed in §4. For the chiral Potts model a method for constructing multicomponent models has been given [3].

3.2. Modification of the self-dual Z_N model

We now apply the gauge transformations (2.5) and (2.6) to the Boltzmann weight $x_m(u)$ (3.1) of the self-dual Z_N model so that we obtain a modified solvable model. Let us introduce two types of Boltzmann weights $y(\alpha,\beta;u)$ and $x(\alpha,\beta;u)$ by $y(\alpha,\beta;u) = y_{\alpha-\beta+s}(u) = x_{\alpha-\beta+s}(\pi-u)$ and $x(\alpha,\beta;u) = x_{\alpha-\beta}(u) \times f(\alpha)/f(\beta)$, respectively. Here $\alpha,\beta \in Z_N$. Then the Boltzmann weights satisfy the star-triangle relations (2.1) and (2.2) in §2 and the following basic relations.

$$\begin{gathered} x(\alpha,\beta;0) = \delta_{\alpha,\beta}, \qquad x(\alpha,\beta;\pi) = 1, \\ x(\alpha,\beta;\pi+u)x(\beta,\alpha;\pi-u) = 1, \\ \sum_k x(\alpha,k;u)x(k,\beta;-u) = C(u,\pi)\delta_{\alpha,\beta}, \end{gathered} \tag{3.9}$$

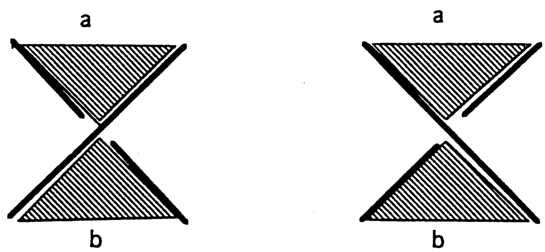
$$\begin{gathered} y(\alpha,\beta;0) = \delta_{\alpha,\beta}, \qquad y(\alpha,\beta;\pi) = 1, \\ y(\alpha,\beta;\pi+u)y(\alpha,\beta;\pi-u) = 1, \\ \sum_k y(\alpha,k;u)y(\beta,k;-u) = C(u,\pi)\delta_{\alpha,\beta}, \end{gathered} \tag{3.10}$$

for any $\alpha,\beta \in Z_N$.

§4. Spin models

4.1. Symmetric spin models

Let us introduce the symmetric spin models [15].



(1) (2)

Fig. 4. Double points (1) of type + and (2) of type −. Spin variables a and b are defined on the shaded regions.

Let S denote a finite set $S = \{1, \ldots, n\}$. We consider two types of weights w_j $(j = \pm)$, which are complex functions on $S \otimes S$. For α and $\beta \in S$, the weight w_j gives a complex number $w_j(\alpha, \beta)$ for $j = \pm$.

Definition 4.1 ([15]). A set of weights $\{w_j; j = \pm\}$ is called a symmetric spin model, if the weights satisfy the following relations.

$$w_+(\alpha,\beta) = w_+(\beta,\alpha), \qquad w_-(\alpha,\beta) = w_-(\beta,\alpha), \tag{4.1}$$

$$w_+(\alpha,\beta)w_-(\alpha,\beta) = 1, \tag{4.2}$$

$$\sum_{k\in S} w_+(\alpha,k)w_-(k,\beta) = n\delta_{\alpha\beta}, \tag{4.3}$$

$$\sum_{k\in S} w_+(\alpha,k)w_+(k,\beta)w_-(\gamma,k) = \sqrt{n}w_+(\alpha,\beta)w_-(\gamma,\alpha)w_-(\gamma,\beta). \tag{4.4}$$

Let us derive symmetric spin models from the self-dual Z_N model by taking advantage of the star-triangle relation (3.2). The set S is given by $S = Z_N$, i.e., n is given by N. Let $G(N,r)$ denote the Gaussian sum for integers N and r with $(N,r) = 1$: $G(N,r) = \sum_{k=0}^{N-1} \exp(\pi\sqrt{-1}rk^2/N)$. We now define weights $w_\pm(\alpha,\beta)$ by the limit of sending the spectral parameter to infinity: [4]

$$w_{\pm}(\alpha,\beta) = \lim_{u\to\infty} x_{\alpha-\beta}(\pm u/\sqrt{-1}) \left(\frac{\sqrt{N}}{G(N,r)}\right)^{\pm 1/2} \tag{4.5}$$

Then we have

$$w_{\pm}(\alpha,\beta) = \exp(\pm\pi\sqrt{-1}r(\alpha-\beta)^2/N) \left(\frac{\sqrt{N}}{G(N,r)}\right)^{\pm 1/2} \tag{4.6}$$

Here we have used the fact that

$$\lim_{u\to\infty} x_m(\pm u/\sqrt{-1}) = \exp(\pm\pi\sqrt{-1}rn^2/N)$$

for $r > N$.

Let us show that the weights (4.6) defined by the limit (4.5) satisfy the defining relations of symmetric spin models. We first note the following

$$\lim_{u\to\infty} F_0(u/\sqrt{-1}) = \lim_{u\to\infty} \sum_k x_k(u/\sqrt{-1}) = G(N,r), \tag{4.7}$$

$$\lim_{u\to\infty} C(u/\sqrt{-1},\pi) = \lim_{u\to\infty} \frac{F_0(u/\sqrt{-1})F_0(\pi)}{F_0(u/\sqrt{-1}+\pi)} = N. \tag{4.8}$$

Here $F_0(\pi) = N$ since $x_k(\pi) = 1$ (see (3.6)). We next note that the weights $w_{\pm}(\alpha,\beta)$ defined by (4.5) are symmetric, since $x_m(u) = x_{-m}(u)$. The inversion relation (3.7) leads to the relation (4.2). The relation (4.3) is derived from the inversion relation (3.8). Finally, from the star-triangle relation (3.2) we have the relation (4.4), since we have $\lim_{u,v\to\infty} C(u,v) = G(N,r)$ from (4.7) .

Thus we have shown that the weights defined by (4.5) give a symmetric spin model. From the derivation we see that the relations (4.3) and (4.4) are consequences of the star-triangle relation (3.2).

A version of symmetric spin model has been constructed in terms of the IRF model (the cyclotomic solution) and the link invariants derived from it have been extensively investigated [19]. It is interesting to note that the symmetric spin model is related to the invariants of 3 dimensional manifolds [20]. A certain class of the spin models associated with $sl(n)$ generalization of the Chiral Potts model [7] has been also shown to be derived from the invariants of 3 dimensional manifolds [20].

[4]Hereafter we assume that $r > N$ without loss of generality.

Recently, from the Boltzmann weights of the chiral Potts model an extended version of spin model has been constructed [23]. We can regard it as a four-weight spin model.

4.2. Four-weight spin models

Let us introduce four-weight spin model (generalized generalized spin model) [4]. Recall that S denotes a finite set $S = \{1, \ldots, n\}$. We consider four types of weights w_j $(j = 1, \ldots, 4)$, which are complex functions on $S \otimes S$. For α and $\beta \in S$, the weight w_j gives a complex number $w_j(\alpha, \beta)$.

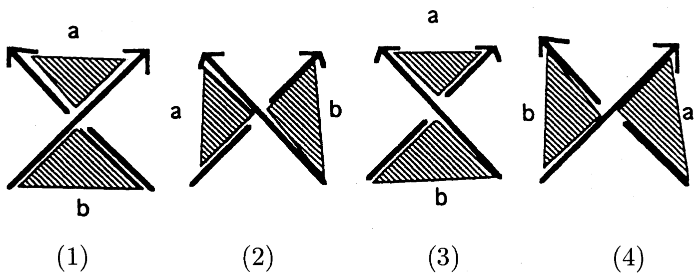


Fig. 5. Double point with index j is defined by figure (j) for $j = 1, 2, 3, 4$. Weight $w_j(a, b)$ is associated with figure (j).

Definition 4.2 ([4]). A set of weights $\{w_j; j = 1, \ldots, 4\}$ is called a four-weight spin model, if the weights satisfy the following relations.

$$w_1(\alpha, \beta) w_3(\beta, \alpha) = 1, \qquad w_2(\alpha, \beta) w_4(\beta, \alpha) = 1, \tag{4.9}$$

$$\begin{aligned} \sum_k w_1(\alpha, k) w_3(k, \beta) &= n \delta_{\alpha, \beta}, \\ \sum_k w_2(\alpha, k) w_4(k, \beta) &= n \delta_{\alpha, \beta}, \end{aligned} \tag{4.10}$$

$$\begin{aligned} \sum_k w_1(\alpha, k) w_1(k, \beta) w_4(\gamma, k) &= \sqrt{n} w_1(\alpha, \beta) w_4(\gamma, \alpha) w_4(\gamma, \beta), \\ \sum_k w_1(k, \alpha) w_1(\beta, k) w_4(k, \gamma) &= \sqrt{n} w_1(\beta, \alpha) w_4(\alpha, \gamma) w_4(\beta, \gamma), \end{aligned} \tag{4.11}$$

for any α, β and γ in S.

Let us introduce graph of an oriented link [6].

Definition 4.3 (graph of an oriented link). Let a regular link diagram be chess-board colored with colors black (or shade) and white. Assign to every double point A^k of the diagram an index 1, 2, 3, or 4 with respect to the colorings as defined by Fig. 5. Denote by S_j the black-colored (or shaded) regions of a link diagram. Define a graph Γ whose vertices P_i correspond to the S_i, and whose edges a_{ij}^k correspond to the double points A^k that is the intersection between regions S_i and S_j, where a_{ij}^k connects P_i and P_j and has the index of A^k.

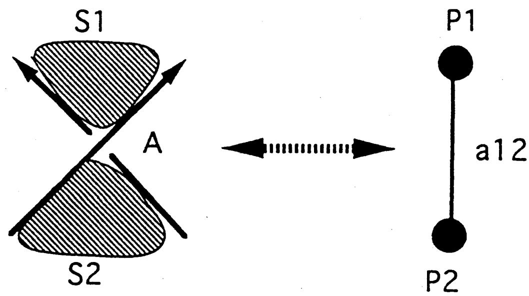


Fig. 6. Double point A with regions $S1$ and $S2$ corresponds to edge $a12$ with vertices $P1$ and $P2$. The index of A is 1.

We now define configuration of spin variables. To each region S_j we associate a spin variable which takes its values in Z_N. A configuration of the spin variables on the graph is given by assigning an element in Z_N to each spin variable on the graph. There are N^m configurations if the graph has m vertices.

We shall define the partition function of a four-weight spin model. Let us consider a link diagram, and then by chess-board coloring of the diagram we make the graph of the link with spin variables assigned on the shaded regions. We assign the weight w_j to each edge in the graph with index j. We make the product of the weights w_j for all the edges in the graph under a given configuration of the spin variables. Then we take the sum of the product over all the configurations of the spin variables. By the sum we define the partition function of the four-weight spin model for the graph of the link diagram.

Proposition 4.4 ([4]). *The partition function of a four-weight model leads to a link invariant.*

We now construct a generalized generalized spin model from the modified solvable model given in §3.2. We define the weights $w_j(\alpha,\beta)$ for $j = 1,2,3,4$ by the following limit:

$$
\begin{aligned}
w_1(\alpha,\beta) &= \lim_{u\to\infty} x(\alpha,\beta;u/\sqrt{-1}) \left(\frac{\sqrt{N}}{G(N,r)}\right)^{1/2}, \\
w_2(\alpha,\beta) &= \lim_{u\to\infty} y(\beta,\alpha;-u/\sqrt{-1}) \left(\frac{\sqrt{N}}{G(N,r)}\right)^{1/2}, \\
w_3(\alpha,\beta) &= \lim_{u\to\infty} x(\alpha,\beta;-u/\sqrt{-1}) \left(\frac{\sqrt{N}}{G(N,r)}\right)^{-1/2}, \\
w_4(\alpha,\beta) &= \lim_{u\to\infty} y(\alpha,\beta;u/\sqrt{-1}) \left(\frac{\sqrt{N}}{G(N,r)}\right)^{-1/2},
\end{aligned}
\tag{4.12}
$$

where $\alpha,\beta \in Z_N$. Then we have

$$
\begin{aligned}
w_1(\alpha,\beta) &= \frac{f(\alpha)}{f(\beta)} \exp(\frac{\pi\sqrt{-1}}{N} r(\alpha-\beta)^2) \left(\frac{\sqrt{N}}{G(N,r)}\right)^{1/2}, \\
w_2(\alpha,\beta) &= \exp(\frac{\pi\sqrt{-1}}{N} r(\alpha-\beta-s)^2) \left(\frac{\sqrt{N}}{G(N,r)}\right)^{1/2}, \\
w_3(\alpha,\beta) &= \frac{f(\alpha)}{f(\beta)} \exp(-\frac{\pi\sqrt{-1}}{N} r(\alpha-\beta)^2) \left(\frac{\sqrt{N}}{G(N,r)}\right)^{-1/2}, \\
w_4(\alpha,\beta) &= \exp(-\frac{\pi\sqrt{-1}}{N} r(\alpha-\beta+s)^2) \left(\frac{\sqrt{N}}{G(N,r)}\right)^{-1/2},
\end{aligned}
\tag{4.13}
$$

for $\alpha,\beta \in Z_N$.

It is a consequence of the star-triangle relations that the weights defined by (4.12) satisfy the relations (4.11). In the same way as the symmetric spin model, we can derive the other defining relations from the basic relations (3.9) and (3.10). Thus all the defining relations are shown by using the star-triangle relations and the basic relations of the solvable model.

§5. Discussion

5.1. Gauge transformations on spin models

In terms of spin models we discuss the gauge transformations in §2 defined on the exactly solvable models.

We first consider the gauge transformation 1. For an arbitrary function $f(\alpha)$ we consider the following transformation of the weight w_j into $\tilde{w}_j$:

$$\begin{aligned} \tilde{w}_1(\alpha,\beta) &= w_1(\alpha,\beta) \times f(\alpha)/f(\beta), \quad \tilde{w}_2(\alpha,\beta) = w_2(\alpha,\beta), \\ \tilde{w}_3(\alpha,\beta) &= w_3(\alpha,\beta) \times f(\alpha)/f(\beta), \quad \tilde{w}_4(\alpha,\beta) = w_4(\alpha,\beta), \end{aligned} \tag{5.1}$$

for any α and $\beta \in S$. Then we have the following.

Proposition 5.1. *Assume that $\{w_j; j = 1, \dots, 4\}$ is a four-weight spin model, and also that $\tilde{w}_j$ is obtained from w_j by the gauge transformation 1. Then $\{\tilde{w}_j; j = 1, \dots, 4\}$ gives a four-weight spin model.*

We now consider the gauge transformation 2 given by the following.

$$\begin{aligned} \tilde{w}_1(\alpha,\beta) &= w_1(\alpha,\beta), \quad \tilde{w}_2(\alpha,\beta) = w_2(\alpha+s,\beta), \\ \tilde{w}_3(\alpha,\beta) &= w_3(\alpha,\beta), \quad \tilde{w}_4(\alpha,\beta) = w_4(\alpha,\beta+s), \end{aligned} \tag{5.2}$$

for any α and $\beta \in S$. It is easy to see the following.

Proposition 5.2. *Let us assume that $\{w_j; j = 1, \dots, 4\}$ is a four-weight spin model, and that $\tilde{w}_j$ is obtained from w_j by the gauge transformation 2. If there are functions z_1 and z_2 such that $w_2(\alpha,\beta) = z_1(\alpha-\beta)$ and $w_4(\alpha,\beta) = z_2(\alpha-\beta)$, then $\{\tilde{w}_j; j = 1, \dots, 4\}$ gives a four-weight spin model.*

We recall that the transformations 1 and 2 were independently introduced by F. Jaeger [14].

5.2. Link invariants being independent of the gauge transformations

Let us discuss the link invariants derived from those four-weight spin models that are obtained from the symmetric spin model by applying the gauge transformations 1 or 2. We show that these link invariants are independent of any parameter related to the gauge transformations.

In order to show the independence explicitly, we introduce Seifert cycle and smoothing operation [6]. Let $p(L)$ be a regular projection of an oriented link L. By altering $p(L)$ in the neighborhood of double points as shown in Fig. 7, $p(L)$ dissolves into a number of disjoint oriented

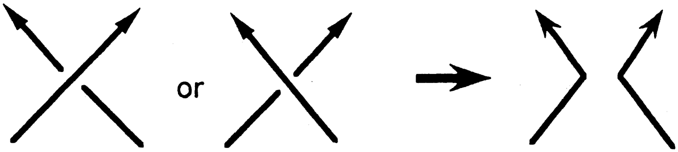


Fig. 7. Smoothing operation.

simple closed curves which are called Seifert cycles. We call this altering procedure smoothing operation.

We first consider the gauge transformation 1. The weight $\tilde{w}_j$ may depend on complex-valued parameters. For instance, we can set $f(\alpha) = a^{\alpha}$, where a is a complex variable. However, we can show the following.

Proposition 5.3. *Let us assume that weights* $\{\tilde{w}_j; j = 1,2,3,4\}$ *are obtained from a four-weight spin model* $\{w_j; j = 1,2,3,4\}$ *through the gauge transformation 1 with the function* $f(\alpha)$. *Then the link polynomial derived from the weights* $\{\tilde{w}_j; j = 1,2,3,4\}$ *coincides with that of the spin model* $\{w_j; j = 1,2,3,4\}$.

Proof. To a given link diagram we apply the smoothing operation. Suppose that the diagram (or the graph of the link diagram) has at least one double point with index 1 or 3. Then the two black-colored regions separated by the double point are connected by the smoothing operation. Thus we see that all the edges with indices 1 or 3 make cycles in the graph. If we take the product of the weights $\{\tilde{w}_j; j = 1,2,3,4\}$ for all the edges, then the factors $f(\alpha)$ cancel each other out for each such cycles

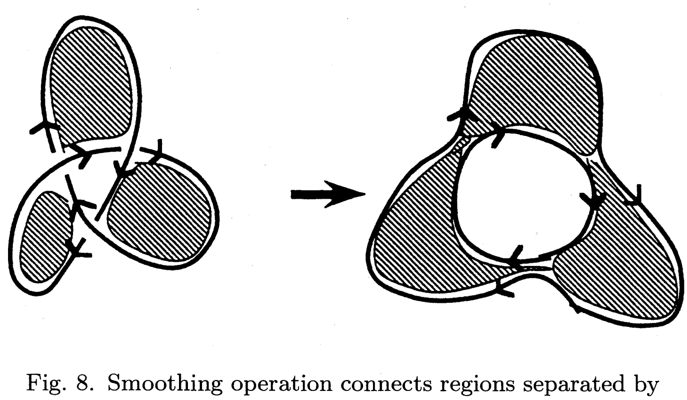


Fig. 8. Smoothing operation connects regions separated by the double points with index 1 or 3.

under any given configuration of spin variables. Therefore the partition function is independent of the function $f(\alpha)$. If the link diagram does not have any double point with index 1 or 3, then by definition the partition function does not depend on the gauge transformation 1.

We next consider the gauge transformation 2. Let us introduce special projection [6].

Definition 5.4. Let $P(L)$ be a regular projection of a link L on a plane. Choose a chess-board coloring on $P(L)$ such that the infinite region is white-colored. $P(L)$ is called a special projection if the union of black-colored regions is the image of a Seifert surface of L under the projection.

Using the same techniques in the proof of Prop. 13.15 of [6], we have the following.

Proposition 5.5. *Every link has a special projection.*

If we apply the smoothing operation to a special projection, then the image of the Seifert surface is divided into sub-regions encircled by the Seifert cycles. These sub-regions are disjoint and do not overlap each other, and they correspond to the black-colored regions by the definition of special projection. Thus we see that for a special projection every black-colored region is encircled by a Seifert cycle.

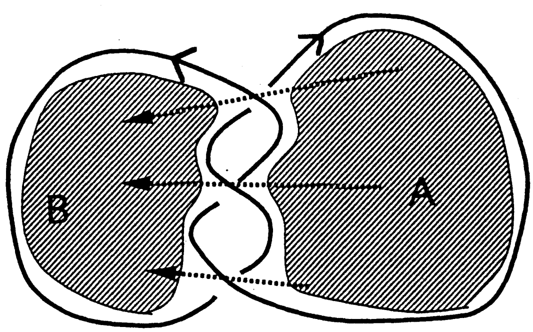


Fig. 9. A special projection of trefoil knot. Arrows denote transverse directions at double points. Positive region A and negative region B.

Note that a Seifert cycle has either a clockwise or counterclockwise orientation. Let us call a black-colored region positive (negative), if it is encircled by a clockwise (counterclockwise) Seifert cycle. We note the following.

Lemma 5.6. *If two Seifert cycles in a link diagram share a double point, then they have different orientations; if one has counterclockwise (clockwise) orientation then the other has clockwise (counterclockwise) orientation.*

We now consider the graph Γ of an oriented link L for its special projection $P(L)$. Any vertex of the graph Γ corresponds to either a positive or negative region of $P(L)$. We call a vertex positive (negative) if it corresponds to a positive (negative) region of $P(L)$. From Lemma 5.6 and the discussion in the previous paragraphs we have the following.

Lemma 5.7. *The graph of a special projection is a bipartite graph such that any of the vertices is either positive or negative and that every vertex connected to a positive (negative) vertex is negative (positive).*

For a special projection all the double points are of index 2 or 4. If there exists a double point of index 1 or 3, then the boundaries of the black-regions around the double point are not Seifert cycles, which is not consistent with the definition of special projection. Thus, every edge of the graph of a special projection is of index 2 or 4.

Let us introduce two directions with respect to a double point as shown in Fig. 10. For a double point we define smoothing direction by the direction of the tangential vector of the Seifert cycle at the double point. Transverse direction is defined by the direction obtained by rotating the smoothing direction 90 degree counterclockwise around the double point.

For a special projection the transverse direction at a double point is consistent with the direction from the positive to negative regions around the double point. Denote $S(\alpha_i)$ ($S(\beta_j)$) the positive (negative) regions of a special projection. We also denote by α_i (β_j) the spin variables defined on the positive (negative) regions. If a double point A^k is in the intersection of the boundaries of positive region $S(\alpha_i)$ and negative region $S(\beta_j)$ in a special projection, then the transverse direction at A^k is consistent with the direction from $S(\alpha_i)$ to $S(\beta_j)$. (See Fig. 9.)

Let us introduce weights y_2 and y_4 given by the following

(5.3)
$$y_2(\gamma_1, \gamma_2) = w_2(\gamma_2, \gamma_1), \quad y_4(\gamma_1, \gamma_2) = w_4(\gamma_1, \gamma_2), \qquad \gamma_1, \gamma_2 \in Z_N.$$

Then at the double point A^k the weight $y_k(\alpha_i, \beta_j)$ with k=2 or 4 is assigned.

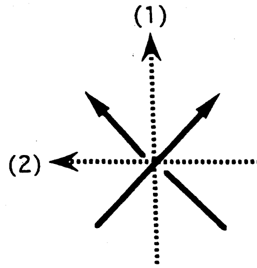


Fig. 10. (1) Smoothing direction. (2) Transverse direction.

Under the gauge transformation 2 the weights y_2 and y_4 are transformed into $\tilde{y}_2$ and $\tilde{y}_4$ defined by the following

$$\tilde{y}_k(\alpha,\beta) = y_k(\alpha,\beta+s), \quad \alpha,\beta \in Z_N, \tag{5.4}$$

for k=2 and 4.

For the gauge transformation 2 we can show the next proposition.

Proposition 5.8. *Let $\{w_j; j = 1,\dots,4\}$ be a such four-weight spin model that has functions z_1 and z_2 such that $w_2(\alpha,\beta) = z_1(\alpha-\beta)$ and $w_4(\alpha,\beta) = z_2(\alpha-\beta)$. Assume also that $\tilde{w}_j$ is obtained from w_j by the gauge transformation 2. Then the link polynomial derived from the transformed weights $\{\tilde{w}_j; j = 1,2,3,4\}$ coincides with that of the spin model $\{w_j; j = 1,2,3,4\}$.*

Proof. Let us consider calculation of the partition function of the four-weight spin model $\{\tilde{w}_j; j = 1,2,3,4\}$ for a special projection $P(L)$ of a link L. We first recall Lemma 5.7. We denote by $S(\alpha_i)$ $(S(\beta_j))$ the positive (negative) regions of $P(L)$ and by α_i (β_j) the spin variables assigned on $S(\alpha_i)$ $(S(\beta_j))$. Note that all the weights in the partition function are given in the form $\tilde{y}_k(\alpha_i,\beta_j)$ for k=2 or 4. Then we see that the partition function of the four-weight spin model $\{\tilde{w}_j; j = 1,2,3,4\}$ is reduced to that of the spin model $\{w_j; j = 1,2,3,4\}$ by replacing all the spin variables β_j on $S(\beta_j)$ with $\beta_j - s$. Here we have used the eq. (5.4) and the fact that the summation of the partition function is invariant under such a replacement that $\gamma \to \gamma - s \pmod N$ for any spin variable γ. Thus we have the proposition.

Let us consider the four-weight spin models constructed by Kac and Wakimoto in Ref. [16] from the viewpoint of the gauge transformations. Let $\mathbf{L} = \mathbf{Z}^m$ and $<,>$ denote a symmetric bilinear form on $\mathbf{L}$. We set $< e_i, e_j >= a_{ij}$, where e_i is the standard basis of $\mathbf{L}$, $a_{ij} \in \mathbf{Q}$ and

$a_{ii} = p/q$, $\gcd(p,q) = 1$ with pq is even. Let $\mathbf{M} = \{\alpha \in \mathbf{L}; <\alpha, \beta> \in \mathbf{Z}$, for all $\alpha \in \mathbf{M}\}$. We define the set S by $S = \mathbf{L}/\mathbf{M}$. We have $n = |S|$. Let us define t_α and D by

$$D = \sum_{\beta \in S} t_\beta, \quad t_\alpha = \exp(\pi i < \alpha, \alpha >), \quad \alpha \in S. \tag{5.5}$$

Then the weights are given by [16]

$$\begin{aligned} &w_1(a,b) = At_{\xi+a-b}, w_2(a,b) = Bt_{\eta+a-b}, \\ &w_3(a,b) = A^{-1}t^{-1}_{\xi+a-b}, w_4(a,b) = B^{-1}t^{-1}_{\eta+a-b}, \end{aligned} \tag{5.6}$$

where $AB = \sqrt{n}/(Dt_\xi)$.

By applying the Propositions 5.3 and 5.8 we can show that the variables ξ and η correspond to the parameters of the gauge transformations 1 and 2, respectively, and that the link invariant derived from the four-weight spin model (5.6) does not depend on ξ or η, essentially. The invariant may depend on the normalization factors of the weights. However, such dependence can be determined by the writhe or the numbers of the positive and negative crossing points of the link diagram. Thus as far as evaluation of link invariants is concerned, we may only consider the case $\xi = \eta = 0$.

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