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Singularities of Solutions to System of Wave Equations with Different Speed

Keiichi Kato

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§1. Introduction and results

We consider the following system of wave equations

$$(1.1)\qquad \begin{cases} \Box_{c_1} u = f(u,v) \\ \Box_{c_2} v = g(u,v) \end{cases}$$

where $\Box_c = (1/c^2)\partial^2/\partial t^2 - \sum_{j=1}^n \partial^2/\partial x_j^2$ and c_1 and c_2 are positive constants. We assume that $f(\cdot,\cdot)$ and $g(\cdot,\cdot)$ are in C^∞. In what follows, we shall study the singularities of the solutions to (1.1) when the solutions are 'conormal distributions' to some hyperplanes. Before the statement of main theorems, we define conormal distributions.

Definition (Conormal distributions). *Let $\Omega \subset \mathbb{R}^n$ be a domain. Let L be a C^∞-manifold in Ω. We call that u is in $H^s(L,\infty)$ in Ω if*

$$M_1 \circ M_2 \circ \cdots \circ M_l u \in H^s_{loc}(\Omega) \quad \textit{for } l = 0,1,2,\dots,$$

where each M_j is a C^∞ vector field which is tangent to L.

We can define the space of conormal distributions not only for a C^∞-manifold but also for a union of two hypersurfaces which intersect each other transversally.

Now we shall state the main results. Let $\omega \in S^{n-1}$ and $L_{ij} = \{(t,x) \in \mathbb{R}^n; c_i t + (-1)^j \omega \cdot x = 0\}$ for i, $j = 1$, 2.

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Theorem 1. *Let Ω be a neighborhood of the origin of $\mathbb{R}^{n+1}$, $i = 1$ or 2 and $j = 1$ or 2. Suppose that u, v are in $H^s_{\mathrm{loc}}(\Omega)$ for $s > (n+1)/2$, u and v are solutions to (1.1) and*

$$u,\, v \in\; H^s(L_{ij}, \infty) \quad in\; \Omega \cap \{t < 0\},$$

then

$$u,\, v \in\; H^s(L_{ij}, \infty) \quad in\; K$$

where K is the domain of dependence with respect to $\Omega \cap \{t < 0\}$.

Theorem 2. *Let Ω be a neighborhood of the origin of $\mathbb{R}^{n+1}$ and i, i', j, $j' \in \mathbb{N}$ with $i + i' = 3$, $j + j' = 3$. Suppose that $0 < c_1 < c_2$, u, v are in $H^s_{\mathrm{loc}}(\Omega)$ for $s > (n+1)/2$, u and v are solutions to (1.1) and*

$$u,\, v \in\; H^s(L_{ij} \cup L_{i'j}, \infty) \quad in\; \Omega \cap \{t < 0\},$$

then

$$u,\, v \in\; H^s(L_{ij} \cup L_{i'j} \cup L_{ij'} \cup L_{i'j'}, \infty) \quad in\; K$$

where K is the domain of dependence with respect to $\Omega \cap \{t < 0\}$.

Theorem 3. *Let Ω be a neighborhood of the origin of $\mathbb{R}^{n+1}$ and i, i', j, $j' \in \mathbb{N}$ with $i + i' = 3$, $j + j' = 3$. Suppose that $0 < c_1 < c_2$, u, v are in $H^s_{\mathrm{loc}}(\Omega)$ for $s > (n+1)/2$, u and v are solutions to (1.1) and*

$$u,\, v \in\; H^s(L_{ij} \cup L_{ij'}, \infty) \quad in\; \Omega \cap \{t < 0\},$$

then

$$u,\, v \in\; H^s(L_{ij} \cup L_{i'j} \cup L_{ij'} \cup L_{i'j'}, \infty) \quad in\; K$$

where K is the domain of dependence with respect to $\Omega \cap \{t < 0\}$.

J.M. Bony has obtained the same result for scalar strictly hyperbolic equations in [3]. So our results are not full of originalities. But the author believes that our proofs are new and simple.

§2. Proof of Theorem 1

We set $M = t\partial_t + x \cdot \partial_x$ and $M_k = \omega_k \partial_t + c_i \partial_{x_k}$ for $k = 1, \ldots, n$. It is easy to prove the following proposition.

Proposition 1. *$M_1, \ldots, M_n$ are linearly independent on $\mathbb{R}^{n+1}$ and M, $M_1, \ldots, M_n$ are linearly independent on $\mathbb{R}^{n+1} \backslash L_{ij}$.*

Proof of Theorem 1.

$$\begin{aligned} \square(Mu) =&[\square, M]u + Mf(u,v) \\ =&2\square u + Mf(u,v) \\ =&2f(u,v) + Mf(u,v). \end{aligned} \tag{2.1}$$

Similarly we have

$$\square(Mu) = 2g(u,v) + Mf(u,v). \tag{2.2}$$

Since u and v are in $H^s_{\text{loc}}(\Omega)$, we have that $2f(u,v) + Mf(u,v)$ and $2g(u,v)+Mf(u,v)$ are in $H^{s-1}_{\text{loc}}(\Omega)$ and Mu, Mv are in $H^s_{\text{loc}}(\Omega \cap \{t < 0\})$. Using the energy estimate for $\square_{c_1}$ and $\square_{c_2}$, we consequently have that Mu, $Mv \in H^s_{\text{loc}}(K)$. Repeating this argument, we have

$$M^l u,\ M^l v \in H^s_{\text{loc}}(K). \tag{2.3}$$

It is easy to see that

$$M^l_k u,\ M^l_k v \in H^s_{\text{loc}}(K) \quad \text{for } \forall k,\ \forall l \in \mathbb{N}. \tag{2.4}$$

(2.3) and (2.4) yield Theorem 1.

§3. Proof of Theorem 2 and Theorem 3

Proof of Theorem 2. We put $M_a = t\partial_t + (x-a)\cdot\partial_x$ for $a \in \mathbb{R}^n$. Using the same argument as in the proof of Theorem 1, we have

$$M^l_a u,\ M^l_a v \in H^s_{\text{loc}}(K) \quad \text{for } \forall a \text{ with } a\cdot\omega = 0 \text{ and } \forall l \in \mathbb{N}. \tag{3.1}$$

We divide $K \backslash \bigcup_{i,j=1}^2 L_{ij}$ into the following three parts,

$$\begin{aligned} K_1 &= \{(t,x) \in K; c_1 t - \omega\cdot x > 0,\ c_1 t + \omega\cdot x > 0\} \\ K_2 &= \{(t,x) \in K; c_1 t - \omega\cdot x < 0,\ c_2 t - \omega\cdot x > 0\} \cup \\ &\qquad \{(t,x) \in K; c_2 t + \omega\cdot x > 0,\ c_1 t + \omega\cdot x < 0\} \\ K_3 &= \{(t,x) \in K; c_2 t - \omega\cdot x < 0 \text{ or } c_2 t + \omega\cdot x < 0\}. \end{aligned}$$

We prove first that u, $v \in C^\infty$ in K_1. Let (t_0, x_0) be any point in K_1. Let $(t_0, x_0, \tau_0, \xi_0)$ be any point in $T^*_{(t_0,x_0)} \backslash 0$. We use the same argument as in the proof of the main theorem of M. Beals [1]. If M_a is elliptic at $(t_0, x_0, \tau_0, \xi_0)$ for some $a \in \mathbb{R}^n$, then from (3.1) we have

$$u,\ v \in H^{s+1} \quad \text{at} \quad (t_0, x_0, \tau_0, \xi_0). \tag{3.2}$$

When M_a is not elliptic at $(t_0, x_0, \tau_0, \xi_0)$ for all $a \in \mathbb{R}^n$, $\Box_{c_1}$ and $\Box_{c_2}$ are elliptic at $(t_0, x_0, \tau_0, \xi_0)$. In fact, we can choose $a_0 \in \mathbb{R}^n$ with $a_0 \cdot \omega = 0$ such that $c_1^2 t_0^2 - |x_0 - a_0|^2 > 0$. Then we have

$$\begin{aligned} c_1 t_0 \left(\frac{1}{c_1} |\tau_0| - |\xi_0| \right) &< t_0|\tau_0| - |\xi_0||x_0 - a_0| \\ &= |\xi_0 \cdot (x_0 - a_0)| - |\xi_0||x_0 - a_0| \\ &\leq 0. \end{aligned}$$

The same argument works for $\Box_{c_2}$. Hence

$$u,\ v \in H^{s+1} \quad \text{at} \quad (t_0, x_0, \tau_0, \xi_0). \tag{3.3}$$

From (3.2) and (3.3), we have

$$u, v \in H^{s+1} \quad \text{at } (t_0, x_0).$$

Repeating this argument, we have

$$u, v \in C^\infty \quad \text{at } (t_0, x_0). \tag{3.5}$$

Next we prove that u, v is in C^∞ on K_2. Let (t_0, x_0) be any point in K_2. Let $(t_0, x_0, \tau_0, \xi_0)$ be any point in $T^*_{(t_0,x_0)}\backslash 0$. When M_a is elliptic at $(t_0, x_0, \tau_0, \xi_0)$ for some $a \in \mathbb{R}^n$, then from (3.1) we have

$$u,\ v \in H^{s+1} \quad \text{at} \quad (t_0, x_0, \tau_0, \xi_0). \tag{3.5}$$

When M_a is not elliptic at $(t_0, x_0, \tau_0, \xi_0)$ for all $a \in \mathbb{R}^n$, the same method as in the first step proves that $\Box_{c_2}$ is elliptic at $(t_0, x_0, \tau_0, \xi_0)$. So it suffices to show that $\Box_{c_1}$ is elliptic at $(t_0, x_0, \tau_0, \xi_0)$. Since $\tau_0 t_0 + (x_0 - a) \cdot \xi_0 = 0$ for all $a \in \mathbb{R}^n$ with $a \cdot \omega = 0$, $\tau_0 t_0 + x_0 \cdot \xi_0 = a \cdot \xi_0 = 0$. Then $a \cdot \xi_0 = 0$ for all $a \in \mathbb{R}^n$ with $a \cdot \omega = 0$. Hence ξ is parallel to ω. We decompose $x_0 = x_0^{(1)} + x_0^{(2)}$ such that $x_0^{(1)}$ is parallel to ω and $x_0^{(2)}$ is perpendicular to ω. We put $a_0 = x_0^{(2)}$. Hence $x_0 - a_0 = x_0^{(1)}$ is parallel to ω. Since $c_1^2|t_0|^2 < |x_0 - a|^2$ for all $a \in \mathbb{R}^n$, we have

$$\begin{aligned} c_1 t_0 \left(\frac{1}{c_1} |\tau_0| - |\xi_0| \right) &> t_0|\tau_0| - |\xi_0||x_0 - a_0| \\ &= t_0|\tau_0| - |\xi_0 \cdot (x_0 - a_0)| \\ &= 0 \quad (\text{since } t_0\tau_0 - \xi_0 \cdot (x_0 - a_0) = 0). \end{aligned}$$

Consequently we have

$$u,\ v \in H^{s+2} \quad \text{at} \quad (t_0, x_0, \tau_0, \xi_0). \tag{3.6}$$

From (3.5) and (3.6), we have

$$u, v \in H^{s+1} \quad \text{at } (t_0, x_0).$$

Repeating this argument, we have

$$u, v \in C^{\infty} \quad \text{at } (t_0, x_0). \tag{3.7}$$

The same argument for u in the second step yields that

$$u, v \in C^{\infty} \quad \text{in } K_3. \tag{3.8}$$

(3.1), (3.4), (3.7) and (3,8) imply Theorem 2.

We can prove Theorem 3 by the same argument as in the proof of Theorem 2.

References

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Department of Mathematics
Osaka University
Toyonaka 560
Japan