Advanced Studies in Pure Mathematics 23, 1994
Spectral and Scattering Theory and Applications
pp. 143–152

H^1-Blow up Solutions for Peker-Choquard Type Schrödinger Equations

Hitoshi Hirata

§1. Introduction and the main results

In this paper, we study the H^1-solution for the following nonlinear Schrödinger equation

$$(1\text{-}1) \qquad \begin{cases} i\partial_t u = -\Delta_x u - (r^{-\gamma} * |u|^2)u \\ u(0,x) = u_0(x) \in H^1(\mathbf{R}^N) \end{cases},$$

where $r = |x|$ and $2 \leq \gamma < 4$, $\gamma \leq N-1$, and show a sufficient condition of 'H^1-blowing up'. Here we say that u is an H^1-local solution of (1-1) when for some $T > 0$, $u \in C([0,T); H^1)$ and satisfies next integral equation

$$(1\text{-}2) \qquad u(t) = U(t)u_0 - i\int_0^t U(t-s)\{(r^{-\gamma} * |u^2|)u\}(s)ds,$$

where $U(t) = \exp(it\Delta_x)$ is the evolution operator for the free Schrödinger equation. Above type nonlinear Schrödinger equation is appeared in some approximations of many body problems, so-called Hartree approximation. As for detailed arguments of this approximation, see e.g. [5], [6] and [7].

Before stating the main results, we define several notations. For $p \in [1,\infty]$ and $k \in \overline{\mathbf{N}}$, we define Sobolev space

$$W^{k,p} \equiv \{f \in \mathcal{S}' : \|f\|_{W^{k,p}} \equiv \sum_{|\alpha|\leq k} \|\partial_x^\alpha f\|_p < \infty\},$$

where $\|\cdot\|_p$ is usual L^p-norm. $H^k \equiv W^{k,2}$ and $H^{-k} \equiv (H^k)^*$. For an interval I and a Banach space X, $C^k(I;X)$ is the space of X-valued C^k-functions on I, $k = 0,1,2...$ and $L^p(I;X)$ is the space of L^p-functions. We say $u \in L^p_{\mathrm{loc}}(I;X)$ if $u \in L^p(J;X)$ for any compact $J \subset I$.

Received January 5, 1993.

For the existence of H^1-local solution of (1-1) and (1-2), we have obtained following theorem. (e.g. [2],[3])

Theorem 0. *Let $2 \leq \gamma \leq 4$, $\gamma < N$ and $u_0 \in H^1$. Then, there exist $T^* > 0$ and $u \in C([0,T^*);H^1)$, which satisfies (1-2), and has following properties (1) $\sim$ (4).*

(1) u is unique solution of (1-2) in $L^\theta_{\mathrm{loc}}(0,T^*;W^{1,p})$, where $1/p = 1/2 - (\gamma-2)/4N$ and $\theta = 8/(\gamma-2)$.

(2) u satisfies following conservation laws.

$$\text{(1-3)} \quad \|u(t)\|_2 = \|u_0\|_2,$$

$$\text{(1-4)} \quad E(u(t)) \equiv \|\nabla_x u(t)\|_2^2 - 1/2(|u(t)|^2, r^{-\gamma} * |u(t)|^2) = E(u_0),$$

for $t \in [0,T^*)$. Here $(\cdot,\cdot)$ is L^2-dual coupling.

(3) If $2 \leq \gamma < 4$ and $T^* < \infty$, then $\|\nabla_x u(t)\|_2 \to \infty$ as $t \to T^*$.

(4) u satisfies (1-1) in H^{-1} sense.

Remark. (1) If u satisfies $\|u(t)\|_2 \to \infty$ as $t \to T^*$ for some $T^* < \infty$, we say u blows up at blow up time T^*.

(2) The assumption $2 \leq \gamma$ is not essential. Since the space in which u is unique becomes simple, we state this assumption. On the other hand, the assumption $4 \geq \gamma$ is essential for the existence of H^1-local solution.

On the blow up of H^1-solutions, $2 \leq \gamma$ is a necessary condition, i.e. when $0 \leq \gamma < 2$, the H^1-solution with any initial data $u_0 \in H^1$ is global. On the other hand, it is well-known that when $2 \leq \gamma$, $u_0 \in H^1 \cap L^2(\mathbf{R}^N; |x|^2 dx)$ and $E(u_0) < 0$, the H^1-solution of (1-1) blows up in finite time (e.g. [1]). K. Kurata and T. Ogawa ([4]) dealt with more complicated potential $-(r^{-\gamma_1} * |u|^2)u - (r^{-\gamma_2} * |u|^2)u$, and showed there exists a blow up solution under the assumption $\gamma_1 < 2 < \gamma_2 < 4$ and $\gamma_2 < N-1$. Recently, in the local nonlinear case, i.e. $-|u|^{p-1}u$ instead of $-(r^{-\gamma} * |u|^2)u$, T. Ogawa and Y. Tsutsumi ([8]) showed that for any radially symmetric H^1-initial data u_0, the H^1-solution of corresponding equation blows up in finite time. We shall prove that we can use their methods in the non-local nonlinear case in this paper. Our main result is following.

Theorem 1. *Let $2 \leq \gamma < 4$ and $\gamma + 1 \leq N$. Suppose that u_0 be radially symmetric in $H^1(\mathbf{R}^N)$ and $E(u_0) < 0$. Then the H^1-solution u blows up in finite time.*

Remark. (1) Since u_0 is unique in $L^\theta_{\mathrm{loc}}(0,T^*;W^{1,p})$ and the equation is symmetric by spatial rotation, u is also radially symmetric.

(2) Since $E(K\phi) = K^2\|\nabla_x\phi\|_2^2 - K^4/2 \cdot (|\phi|^2, r^{-\gamma} * |\phi|^2)$ for any $\phi \in H^1$ and $K > 0$, $E(u_0) < 0$ is attained by some $u_0 \in H^1$. This observation shows the assumption $E(u_0) < 0$ means 'u_0 is not small'.

§2. General lemmas

In this chapter, we state two well-known lemmas which hold in H^1. The first one is so-called Gagliardo-Nirenberg's inequality.

Lemma 2-1. *Let $u \in H^1(\mathbf{R}^N)$ and $N \geq 3$. Then, there exists a constant C such that*

$$\|u\|_p \leq C\|\nabla_x u\|_2^a \|u\|_2^{1-a}, \tag{2-1}$$

where $1/p = 1/2 - a/N$.

The second one holds on radially symmetric functions.

Lemma 2-2 (Strauss[9]). *Let u be a radially symmetric function in $H^1(\mathbf{R}^N)$. Then, there exists a constant C such that for any $R > 0$ and $p \in [2, \infty]$,*

$$\|u\|_{L^p(R<|x|)} \leq CR^{-(1/2-1/p)(N-1)}\|u\|_{L^2(R<|x|)}^{1/2+1/p}\|\nabla_x u\|_{L^2(R<|x|)}^{1/2-1/p}. \tag{2-2}$$

§3. Proof of Theorem 1

Choose $\phi \in W^{3,\infty}([0,\infty))$ such that

$$\phi(r) = \begin{cases} r & \text{for } 0 \leq r \leq 1, \\ r - (r-1)^3 & \text{for } 1 \leq r \leq 1 + \sqrt{3}/3, \\ \text{smooth and } \phi' \leq 0 & \text{for } 1 + \sqrt{3}/3 \leq r \leq 2, \\ 0 & \text{for } 2 \leq r, \end{cases} \tag{3-1}$$

and put

$$\phi_m(r) = m \cdot \phi(r/m),$$
$$\psi_m(x) = x/|x| \cdot \phi_m(|x|).$$

Remark that if we put $\Phi(r) = \int_0^r \phi_m(s)ds$, $\Phi \in L^\infty(\mathbf{R}^N)$ and $\nabla_x\Phi = \psi_m$. We also obtain next lemma.

Lemma 3-1. *Let u be the H^1-solution of* (1-1). *Then,*

(3-2)
$$\Im\int u_0\,\psi_m\cdot\nabla_x\overline{u_0}\,dx-\Im\int u(t)\psi_m\cdot\nabla_x\overline{u(t)}dx$$
$$=\int_0^t[2\Re\sum_{j,k}\int\partial_j(\psi_m)_k\,\partial_j u(\tau)\partial_k\overline{u(\tau)}dx$$
$$-1/2\int\Delta_x(\nabla_x\cdot\psi_m)\cdot|u(\tau)|^2,dx+\gamma E(u_0)-\gamma\|\nabla_x u(\tau)\|_2^2$$
$$+\gamma/2\int\int_{|x|\vee|y|\geq m}a(x,y)|x-y|^{-\gamma-2}|u(\tau,x)|^2|u(\tau,y)|^2dxdy]d\tau$$
for all $t\in[0,T^*)$,

where $\Im$ and $\Re$ mean imaginary and real parts respectively, $(\psi_m)_k$ is k^{th} component of ψ_m and

(3-3)
$$a(x,y)=|x-y|^2-(\psi_m(x)-\psi_m(y))\cdot(x-y).$$

Now, remarking that u is radially symmetric, we have

(3-4)
$$2\Re\sum_{j,k}\int\partial_j(\psi_m)_k\partial_j u\,\partial_k\overline{u}\,dx$$
$$=2\int_{|x|\leq m}|\nabla_x u|^2dx+2\int_{m\leq|x|\leq 2m}\phi_m'|\nabla_x u|^2dx.$$

And, simple calculation shows that there exists a constant C such that

(3-5)
$$|\Delta_x(\nabla_x\cdot\psi_m(x))|\quad\begin{cases}\leq Cm^{-2} & \text{for}\quad m\leq|x|\leq 2m,\\ =0 & \text{for otherwise.}\end{cases}$$

The next lemma is the key estimate to obtain our result.

Lemma 3-2. *Let $0<\alpha<1$ and $m\gg 1$. For $|x|\vee|y|\geq m$ and $|x-y|\leq m^\alpha$, there exists a constant C, which is independent of x,y and m, such that*

(3-6)
$$a(x,y)\leq C(b(|x|)+b(|y|))|x-y|^2.$$

Here

(3-7)
$$b(r)=\begin{cases}0 & \textit{for}\quad r\leq m,\\ 1-\phi_m'(r) & \textit{for}\quad m\leq r\leq 2m,\\ 1 & \textit{for}\quad 2m\leq r.\end{cases}$$

Using this lemma, we obtain

(3-8)

$$\int\int_{|x|\vee|y|\geq m,|x-y|\leq m^\alpha} a(x,y)|x-y|^{-\gamma-2}|u(x)|^2|u(y)|^2dxdy$$

$$\leq C\int\int_{|x|\vee|y|\geq m,|x-y|\leq m^\alpha} (b(|x|)+b(|y|))|x-y|^{-\gamma}|u(x)|^2|u(y)|^2dxdy$$

$$\leq 2C\int_{|x|\geq m} b(r)|u(x)|^2(\{\chi(\{r\leq m^\alpha\})\cdot r^{-\gamma}\}*|u|^2)(x)dx$$

$$\leq 2C\|b^{1/2}(r)u(x)\|^2_{L^\infty(|x|\geq m)}\|\chi(\{r\leq m^\alpha\})\cdot r^{-\gamma}\|_1\cdot\|u_0\|_2^2$$

(by Hölder's and Young's inequalities)

$$\leq Cm^{-(N-1)}\|\nabla_x\{b^{1/2}(r)u(x)\}\|^2_{L^2(|x|\geq m)}\cdot m^{\alpha(N-\gamma)}\|u_0\|_2^2$$

(by Lemma 2-2)

$$\leq Cm^{\alpha(N-\gamma)-(N-1)}\|u_0\|_2^2\int b(r)|\nabla_x u(x)|^2dx + Cm^{\alpha(N-\gamma)-(N-1)}\|u_0\|_2^4.$$

Here we used L^2-conservation law (1-3) and defined

$$\chi(A)(x)=\begin{cases}1 & x\in A,\\ 0 & x\notin A.\end{cases}$$

On the other hand, since $|(\psi_m(x)-\psi_m(y))\cdot(x-y)|\leq\|\psi_m'\|_\infty|x-y|^2$, we get

$$\int\int_{|x|\vee|y|\geq m,|x-y|\geq m^\alpha} a(x,y)|x-y|^{-\gamma-2}|u(x)|^2|u(y)|^2dxdy$$

(3-9)
$$\leq C\int\int_{|x|\vee|y|\geq m,|x-y|\geq m^\alpha} |x-y|^{-\gamma}|u(x)|^2|u(y)|^2dxdy$$

$$\leq Cm^{-\gamma\alpha}\|u_0\|_2^4.$$

After all, by (3-2),(3-4),(3-8) and (3-9), we have

$$\begin{aligned}
&\Im\int u_0\psi_m\cdot\nabla_x\overline{u_0}dx-\Im\int u(t)\psi_m\cdot\nabla_x\overline{u(t)}dx\\
&\le\int_0^t[\gamma E(u_0)-(\gamma-2)\|\nabla_x u(\tau)\|_2^2\\
&\qquad-2\int b(r)|\nabla_x u(\tau)|^2dx-Cm^{-2}\|u_0\|_2^2\\
&\qquad+C(m^{-\gamma\alpha}+m^{\alpha(N-\gamma)-(N-1)})\|u_0\|_2^4\\
&\qquad+Cm^{\alpha(N-\gamma)-(N-1)}\|u_0\|_2^2\int b(r)|\nabla_x u(\tau)|^2dx]d\tau.
\end{aligned}\tag{3-10}$$

Thus, if we take sufficiently large m such that

$$\gamma E(u_0)+C(m^{-\gamma\alpha}+m^{\alpha(N-\gamma)-(N-1)})\|u_0\|_2^4\equiv-\eta<0,$$

and

$$Cm^{\alpha(N-\gamma)-(N-1)}\|u_0\|_2^2-2\le 0,$$

we obtain

$$\Im\int u_0\,\psi_m\cdot\nabla_x\overline{u_0}\,dx-\Im\int u(t)\,\psi_m\cdot\nabla_x\overline{u(t)}dx\ge\eta t.\tag{3-11}$$

Since

$$d/dt(\int\Psi|u(t)|^2dx)=-2\Im\int u(t)\,\psi_m\cdot\nabla_x\overline{u(t)}dx,$$

integrating the both hands of (3-12), we deduce that

$$\begin{aligned}
\int\Psi|u(t)|^2dx&\le-\eta t^2-2t\,\Im\int u_0\,\psi_m\cdot\nabla_x\overline{u_0}\,dx\\
&\qquad+\int\Psi|u_0|^2dx\quad\text{for all } t\in[0,T^*).
\end{aligned}\tag{3-12}$$

Now, we assume u is a global solution. Then, (3-12) is satisfied for any $t<\infty$ and the r.h.s. of (3-12) is negative for sufficiently large t. This is contradiction since the l.h.s. of (3-12) is non-negative. Thus, u is not global solution and $T<\infty$. Using Theorem 0.(3), we obtain $\|\nabla_x u(t)\|_2\to\infty$ as $t\to T^*$. This means our desired result.

§4. The proofs of lemmas

Proof of Lemma 3-1. We first assume $u_0\in H^2$. Under this assumption, the solution u belongs to $C([0,T^*);H^2)\cap C^1([0,T^*);L^2)$ and satisfies (1-1) in L^2-sense (see e.g. [2]). Note that the maximum existence time T^* is the same as that of the H^1-solution. We take the real part of L^2-inner product between (1-1) and $\psi_m \cdot \nabla_x u$. Here, using equality (1-1) and integrating by parts, we have

$$
\begin{aligned}
&2\Re(i\partial_t u\,,\,\psi_m\cdot\nabla_x u)\\
&=i\int \partial_t u\,\psi_m\cdot\nabla_x\overline{u}\,dx - i\int \psi_m\cdot\nabla_x u\,\partial_t\overline{u}\,dx\\
&=i\,d/dt\int u\,\psi_m\cdot\nabla_x\overline{u}\,dx + \int \nabla_x\cdot\psi_m|u|^2(r^{-\gamma}*|u|^2)dx\\
&\quad-\int \nabla_x\cdot\psi_m|\nabla_x u|^2dx + 1/2\int \Delta_x(\nabla_x\cdot\psi_m)|u|^2dx,
\end{aligned}
\tag{4-1}
$$

$$
\begin{aligned}
&2\Re(-\Delta_x u\,,\,\psi_m\cdot\nabla_x u)\\
&\quad= 2\Re\sum_{j,k}\int \partial_j(\psi_m)_k\,\partial_j u\,\partial_k\overline{u}\,dx - \int \nabla_x\cdot\psi_m|\nabla_x u|^2dx,
\end{aligned}
\tag{4-2}
$$

and

$$
\begin{aligned}
&2\Re(u(r^{-\gamma}*|u|^2)\,,\,\psi_m\cdot\nabla_x\overline{u})\\
&=\int(\nabla_x\cdot\psi_m)|u|^2(r^{-\gamma}*|u|^2)dx + \int |u|^2\psi_m\cdot\nabla_x(r^{-\gamma}*|u|^2)dx.
\end{aligned}
\tag{4-3}
$$

Here, since

$$
\begin{aligned}
&1/2\int |u(x)|^2\psi_m(x)\cdot\nabla_x(\int |x-y|^{-\gamma}|u(y)|^2dy)dx\\
=&1/2\int |u(x)|^2\{\nabla_x(\int \psi_m(x)|x-y|^{-\gamma}|u(y)|^2dy)\\
&-(\nabla_x\psi_m)(x)\cdot\int |x-y|^{-\gamma}|u(y)|^2dy\}dx\\
=&1/2\int |u(x)|^2\nabla_x\cdot[\int\{(\psi_m(x)-\psi_m(y))|x-y|^{-\gamma}|u(y)|^2\\
&+|x-y|^{-\gamma}\psi_m(y)|u(y)|^2\}dy]dx\\
&-1/2\int(\nabla_x\cdot\psi_m)(x)|u(x)|^2(\int |x-y|^{-\gamma}|u(y)|^2dy)dx\\
=&-1/2\int \nabla_x|u(x)|^2\cdot(\int \psi_m(y)|x-y|^{-\gamma}|u(y)|^2dy)dx
\end{aligned}
$$

$$+\,1/2\int|u(x)|^2[\int\nabla_x\cdot\{(\psi_m(x)-\psi_m(y))|x-y|^{-\gamma}\}|u(y)|^2dy]dx$$
$$-\,1/2\int(\nabla_x\cdot\psi_m)(x)|u(x)|^2(\int|x-y|^{-\gamma}|u(y)|^2dy)dx$$
$$=-\,1/2\int|u(y)|^2\psi_m(y)\cdot(\int\nabla_x|u(x)|^2|x-y|^{-\gamma}dx)dy$$
$$+\,1/2\int|u(x)|^2\{\int(\psi_m(x)-\psi_m(y))\cdot(\nabla r^{-\gamma})(x-y)|u(y)|^2dy\}dx$$
$$=-\,1/2\int|u(x)|^2\psi_m(x)\cdot\nabla_x(r^{-\gamma}*|u|^2)(x)dx$$
$$-\,\gamma/2\int|u(x)|^2\{\int(\psi_m(x)-\psi_m(y))\cdot(x-y)\,|x-y|^{-\gamma-2}|u(y)^2dy\}dx,$$

the second term of r.h.s. of (4-3) is equal to

$$-\gamma/2\int\int|u(x)|^2(a(x,y)|x-y|^{-\gamma-2}-|x-y|^{-\gamma})|u(y)|^2\,dy\,dx.$$

Thus, by (4-1)$\sim$(4-3), we get

$$i\,d/dt\int u\,\psi_m\cdot\nabla_x\overline{u}\,dx+1/2\int\Delta_x(\nabla_x\cdot\psi_m)|u|^2dx$$
$$=2\Re\sum_{j,k}\int\partial_j(\psi_m)_k\,\partial_j u\,\partial_k\overline{u}\,dx$$
$$+\,\gamma/2\int\int|u(x)|^2a(x,y)|x-y|^{-\gamma-2}|u(y)|^2dy\,dx$$
$$-\,\gamma/2\int|u|^2(r^{-\gamma}*|u|^2)dx.$$

Taking real part of b.h.s. and using the definition of energy (1-4), we obtain

$$-\,d/dt\,\Im\int u\,\psi_m\cdot\nabla_x\overline{u}\,dx$$
$$=2\Re\sum_{j,k}\int\partial_j(\psi_m)_k\,\partial_j u\,\partial_k\overline{u}\,dx-1/2\int\Delta_x(\nabla_x\cdot\psi_m)|u|^2dx \tag{4-4}$$
$$+\,\gamma E(u_0)-\gamma\|\nabla_x u\|_2^2$$
$$+\,\gamma/2\int\int|u(x)|^2a(x,y)|x-y|^{-\gamma-2}|u(y)|^2dy\,dx.$$

Thus, integrating (4-4) over $[0,T^*)$ by t, we obtain (3-3).

For the case of $u_0 \in H^1$, we take $\{u_{0,l}\} \subset H^2$ such that $u_{0,l} \to u_0$ in H^1 as $l \to \infty$. For each $u_{0,l}$, we can construct strong solutions $u_l(t)$ of (1-1) in a certain common time interval $[0, T]$, and $\{u_l(t)\}$ converges to the H^1-solution $u(t)$ in H^1 uniformly. (See [2].) Thus, we obtain (3-2) on $[0, T]$. Since T is depend only on $\|u_0\|_{H^1}$, we can repeat this procedure, and we obtain (3-2) as long as $u(t)$ exists. Q.E.D.

Proof of Lemma 3-2. It suffices to consider on x, y 2-dimensional plain, then let $x = (r\cos\theta, r\sin\theta)$ and $y = (\rho, 0)$. By taking m sufficiently large and using renormalization, we can assume $m = 1$ and $\theta \ll 1$. For the case of $1 \le r, \rho \le 1 + \sqrt{3}/3$, we calculate

$$
\begin{aligned}
&|x-y|^2 - (\phi(x) - \phi(y)) \cdot (x-y) \\
=&(r-\rho)\{(r-\phi(r)) - (\rho - \phi(\rho))\} \\
&\quad + (1-\cos\theta)\{r(\rho - \phi(\rho)) + \rho(r - \phi(r))\} \\
=&(r-\rho)\{(r-1)^3 - (\rho-1)^3\} + (1-\cos\theta)\{r(\rho-1)^3 + \rho(r-1)^3\} \\
=&(r-\rho)^2\{(r-1)^2 + (r-1)(\rho-1) + (\rho-1)^2\} \\
&\quad + (1-\cos\theta)\{r(\rho-1)^3 + \rho(r-1)^3\}.
\end{aligned}
$$

Since $b(r) = 3(r-1)^2$ on $1 \le r \le 1+\sqrt{3}/3$, it suffices to show that there exists a constant C, independent of r and ρ, such that

$$
\begin{aligned}
&(r-\rho)^2\{(r-1)^2 + (r-1)(\rho-1) + (\rho-1)^2\} \\
&\quad + (1-\cos\theta)\{r(\rho-1)^2 + \rho(r-1)^2\} \\
\le& C[(r-\rho)^2\{(r-1)^2 + (\rho-1)^2\} \\
&\quad + 2(1-\cos\theta)r\rho\{(r-1)^2 + (\rho-1)^2\}].
\end{aligned}
$$

This is possible obviously since $1 \le r, \rho$. For the case of $r \wedge \rho < 1$, the similar calculation shows the statement, and we omit the details.

Q.E.D.

References

[1] R.T. Glassey, On the blowing up of solutions to Cauchy problem for nonlinear Schrödinger equations, J. Math. Phys., **18** (1977), 1794–1797.

[2] H. Hirata, The Cauchy problem for Hartree type Schrödinger equations in weighted Sobolev space, J. Fac. Sci. Univ. Tokyo Sect. IA, Math., **38-3** (1991), 567–588.

[3] H. Hirata, The Cauchy problem for Hartree type Schrödinger equations in H^s, preprint.
[4] K. Kurata and T. Ogawa, Remarks on blowing-up of solutions for some nonlinear Schrödinger equations, Tokyo J. Math., **13-2** (1990), 399–419.
[5] E.H. Lieb and B. Simon, The Hartree-Fock theory for Coulomb systems, Comm. Math. Phys., **53** (1977), 185–194.
[6] P.L. Lions, Solutions of Hartree-Fock equation for Coulomb systems, Comm. Math. Phys., **109** (1987), 33–97.
[7] P.L. Lions, The Choquard equation and related questions, Nonlinear Anal., **4** (1980), 1063–1073.
[8] T. Ogawa and Y. Tsutsumi, Blow-up of H^1-solution for the nonlinear Schrödinger equation, J. Diff. Eq., **92** (1991), 317–330.
[9] W.A. Strauss, Existence of solitary waves in higher dimensions, Comm. Math. Phys., **55** (1977), 149–162.

Department of Mathematical Science
University of Tokyo
Komaba, Meguro-ku, Tokyo 153, Japan