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On Spectral Theory for Schrödinger Operators with Magnetic Potentials

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Abstract.

In this survey, we want to analyze the effect of the presence of a magnetic potential on the spectrum of the Schrödinger operator with magnetic field. We consider three connected problems:
— study of the bottom of the spectrum
— study of the bottom of the essential spectrum
— study of the decay of the eigenfunctions.
We think this survey is complementary to other presentations of the subject in [12], [20] and [49].

§1. Qualitative Theory

Let $V \in C^\infty(\mathbb{R}^n)$ be an electrical potential s.t.

$$V \geq C \qquad \text{for some constant } C, \tag{1.1}$$

and let $A = (A_1, \ldots, A_n)$ be a magnetic potential in $C^\infty(\mathbb{R}^n, \mathbb{R}^n)$. We denote by

$$\omega_A = \sum_j A_j \, dx_j \tag{1.2}$$

the corresponding 1-form and by

$$\sigma_B = d\sigma_A = \sum_{j<k} b_{jk} \, dx_j \wedge dx_k \tag{1.3}$$

the corresponding magnetic 2-form.

The Schrödinger operator with magnetic field is usually defined by

$$P_{A,V}(h) = \sum_{1 \leq j \leq n} (hD_{x_j} - A_j)^2 + V \tag{1.4}$$

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and we shall denote by $P^{\Omega}_{A,V}$ the Dirichlet realization in a connected open set Ω with bounded regular boundary (cf. [57]). If the operator is with compact resolvent, for example (see also the results in Section 2) if

(1.5) $$V \text{ tends to } \infty, \text{ as } |x| \to \infty$$

we know by the Kato's inequality that (cf. [12])

(1.6) $$\lambda^{\Omega}_{0,A,V}(h) \geq \lambda^{\Omega}_{0,0,V}(h)$$

where $\lambda^{\Omega}_{0,A,V}(h)$ is the first eigenvalue of $P^{\Omega}_{A,V}$.

In the case when $P^{\Omega}_{A,V}$ is not with compact resolvent, one easily get

(1.7) $$\inf \operatorname{Sp} P^{\Omega}_{A,V} \geq \inf \operatorname{Sp} P^{\Omega}_{0,V}$$

observing that it is true (cf. (1.6)) when V is replaced by $V_\varepsilon = V + \varepsilon |x|^2$ and that

(1.8) $$\inf \operatorname{Sp} P^{\Omega}_{A,V_\varepsilon} \to \inf \operatorname{Sp} P^{\Omega}_{A,V} \qquad \text{as } \varepsilon \to 0 \ (\varepsilon > 0).$$

Finally let us observe that due to the characterization of the essential spectrum by Persson [54] (see also Agmon [1]) we have also for the essential spectrum

(1.9) $$\inf \operatorname{EssSp} P^{\Omega}_{A,V} \geq \inf \operatorname{EssSp} P^{\Omega}_{0,V}.$$

We are now interested to the cases where we have equality. Let us first recall the following result due essentially to Lavine-O'Caroll [41], (see also [21]).

Proposition 1.1. *Let $h > 0$ be fixed and Ω as above; let us assume that we have the assumptions* (1.1)–(1.5); *then the following properties are equivalent:*

(i) $\lambda^{\Omega}_{0,A,V}(h) = \lambda^{\Omega}_{0,0,V}(h)$

(ii) $P^{\Omega}_{A,V}$ *and* $P^{\Omega}_{0,V}$ *are unitary equivalent.*

(iii) (a) $\sigma_B = 0$ *in* Ω *and*

(b) *for all closed path in* Ω, $(2\pi h)^{-1} \int_\gamma \omega_A \in \mathbb{Z}$.

Sketch of the proof. If u_0 is the first eigenfunction of $P^{\Omega}_{0,V}(h)$ attached to the eigenvalue $\lambda^{\Omega}_0(h)$, (we know that u_0 does not vanish in Ω and we can then assume that $u_0 > 0$ in Ω and $\|u_0\| = 1$) we have the following identity

(1.10) $$\|(h\nabla - iA - h(\nabla u_0/u_0))\phi\|^2 = \langle (P^{\Omega}_A(h) - \lambda^{\Omega}_0)\phi \mid \phi\rangle \qquad \forall \phi \in C^\infty_0(\Omega)$$

The first consequence is of course that we get another proof of (1.6). Let us briefly sketch the proof of (i) $\Rightarrow$ (iii) (which is the non trivial part of the statements). From (1.10) we deduce, using a minimizing sequence tending in L^2 to a normalized eigenfunction of $P^{\Omega}_{A,V}(h)$ u_A corresponding to $\lambda_A = \lambda_0$

$$(1.11) \qquad (h\nabla - iA - h(\nabla u_0/u_0))u_A = 0 \qquad \text{in } \mathcal{D}'(\Omega).$$

We rewrite (1.11) on the form

$$(1.12) \qquad (h\nabla - iA)\varphi_A = 0 \qquad \text{in } \mathcal{D}'(\Omega), \text{ with } \varphi_A = u_A/u_0.$$

It is easy to prove that

$$(1.13) \qquad \varphi_A \neq 0 \qquad \text{in } \Omega.$$

By differentiation we get $\varphi_A d\omega_A = 0$ and finally $d\omega_A = 0$. In the case when Ω is simply connected we get the existence of θ such that $\omega_A = d\theta$ and we have immediately

$$\int_\gamma \omega_A = \int_\gamma d\theta = 0.$$

In the general case, we use (1.12) which can be written locally

$$(1.14) \qquad hd(\operatorname{Log}\varphi_A) = i\,\omega_A.$$

Hence $|\varphi_A|$ is locally constant (and then constant by connectedness) and because φ_A is univalued, we get (iii)$_{\mathrm{b}}$. (iii) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i) are much easier.

Remark 1.2. The same result can be obtained under the weaker assumption (replacing (1.5)).

(1.15)
The bottom of the spectrum of $P^{\Omega}_{0,V}$ is an isolated eigenvalue $\lambda^{\Omega}_{0,0,V}$,

with (i) replaced by the apparently different

$$(\text{i})' \qquad \inf \operatorname{Sp} P^{\Omega}_{A,V} = \lambda^{\Omega}_{0,0,V}.$$

We observe indeed that (1.15) implies

$$(1.16) \qquad \inf \operatorname{EssSp} P^{\Omega}_{0,V} > \lambda^{\Omega}_{0,0,V}.$$

Using (1.16) and (1.9), we get that if (i)$'$ is satisfied then there is at least one eigenvalue $\lambda^{\Omega}_{0,A,V}$ and the proof goes after in the same way.

§2. More on the essential spectrum

In this section, we present essentially the results of Helffer-Mohamed ([22], [23]) with more recent improvements due to Iwatsuka [34], Mohamed-Nourrigat [47], Meftah [44] .... We consider an electric potential of the form

$$(2.1) \quad V(x) = \sum_{j=1}^{p} V_j(x)^2 + V_0(x) \qquad \text{where } V_j \in C^\infty(\mathbb{R}^n),\ V_0(x) \geq 0$$

and a C^∞ magnetic potential $\omega_A = \sum_j A_j\, dx_j$. Because V is semibounded we know that $P_{A,V}$ admits a unique selfadjoint realization on $L^2(\mathbb{R}^n)$ (cf. Schechter [58], Avron-Herbst-Simon [3] or Reed-Simon [57]). Moreover $C_0^\infty(\mathbb{R}^n)$ is dense in $D(P_{A,V})$. In Avron-Herbst-Simon [3], Dufresnoy [13], Helffer-Mohamed [22], sufficient conditions were given which imply compact resolvent. These sufficient conditions are not far to be necessary (cf. Dufresnoy [13] and Iwatsuka [34], and also Remark 5 in Mohamed [45]). We shall give here two extensions of the basic result given in [22]. It is probably possible to establish a unique statement containing the two results. For the sufficient conditions we recall that it is sufficient to prove (cf. Avron-Herbst-Simon [3] or Iwatsuka [34]) the following inequality

$$(2.2) \qquad \forall u \in C_0^\infty(\mathbb{R}^n), \qquad \|\phi u\|^2 \leq C(\langle P_{A,V}u, u\rangle + \|u\|^2)$$

where ϕ is a continuous function tending to $+\infty$ as $|x|$ tends to ∞. For all $r \in \mathbb{Z}$, we introduce

$$(2.3) \qquad m_r(x) = 1 + |V_0(x)| + \sum_{j=1}^{p} \sum_{|\alpha|=0}^{r} |\partial_x^\alpha V_j(x)| + \sum_{i,j=1}^{n} \sum_{|\alpha|=0}^{r-1} |\partial_x^\alpha b_{ij}|.$$

The following theorem is due to Meftah [44] and is an improvement of [22] (see also Mohamed-Raikov [49] or Simon [61]):

Theorem 2.1. *Let us assume that* (2.1) *is satisfied and that there exists $r \in \mathbb{N}$, $0 \leq \delta < 1$ and $c_1 > 0$ such that*

$$(2.4) \quad |\operatorname{grad} V_0| + \sum_{j=1}^{p} \sum_{|\alpha|=r+1} |\partial_x^\alpha V_j(x)| + \sum_{i,j=1}^{n} \sum_{|\alpha|=r} |\partial_x^\alpha b_{ij}| \leq c_1 m_r(x)^{1+\delta},$$

then there exists a constant c_2 s.t.

$$(2.5) \qquad \|(m_r(x))^k u\|^2 \leq c_2(\langle P_{A,V}u \mid u\rangle + \|u\|^2) \qquad \forall u \in C_0^\infty(\mathbb{R}^n)$$

where $k = (1 - \delta(2^{r+1} - 3))/2^r$.

Corollary 2.2. *If we assume in addition that*

$$m_r(x) \to +\infty \qquad as\ |x| \to \infty \tag{2.6}$$

and

$$\delta < 1/(2^{r+1} - 3) \tag{2.7}$$

then $P_{A,V}$ *is with compact resolvent.*

Remark 2.3. The case $\delta = 0$ corresponds to the result given in [22]. As $r = 1$, $V_j = 0$, $n = 2$, Corollary 2.2 says that, if $|b_{12}(x)| \to \infty$ as $|x| \to \infty$ and if there exists $C > 0$ and $\delta < 1$ s.t. $|\nabla b_{12}| \leq C(|b_{12}|^{1+\delta}+1)$, then $P_{A,V}$ is with compact resolvent. The counterexamples given by Iwatsuka [34] and Dufresnoy [13] correspond to the case where $|\nabla b_{12}|$ is of the order of $|b_{12}|^2$.

The proof is an adaptation of the proof given in [22] (cf. also Helffer [20] or Mohamed-Raikov [49] for a presentation) and is based on ideas coming from a proof given by J.J. Kohn [37] for the hypoellipticity of Hörmander's operators.

Remark 2.4. As observed in Mohamed-Nourrigat [47], the choice of V of the form (2.1) is not necessary. We refer also to Guibourg [16] for other proofs in this direction or to the surveys of Mohamed-Raikov [49] and Nourrigat [51]. Other generalizations are given in Iftimie [32].

Remark 2.5. Necessity. Under the assumption (2.4), Corollary 2.2 gives in fact a necessary and sufficient condition for compactness of the resolvent. Indeed if there exists a sequence of points in $\mathbb{R}^n y_k$ such that $|y_k|$ tends to ∞ and s.t. $m_r(y_k)$ is bounded, then (taking possibly a subsequence) $m_r(x)$ remains bounded in a union of disjoints balls $B(y_k, C)$ and using the proof (see p.102–103 in Helffer-Mohamed [22]) characterizing the essential spectrum we get the existence of some essential spectrum. Let us also observe that an assumption like (2.4) permits the control of the variation of $m_r(x)$ in suitable balls and the comparison of the above statements with the statements of Iwatsuka [34].

In order to characterize the essential spectrum of $P_{A,V}$ in the case when $m_r(x)$ does not tend to ∞ we introduce stronger assumptions in place of (2.4). Let us first consider a slowly varying function ϕ on $\mathbb{R}^n$ that satisfies for some τ, $c > 0$ the conditions

$$\forall x \in \mathbb{R}^n, \qquad \phi(x) \geq 1 \tag{2.8a}$$

$$|x-y| < \tau\phi(x) \Longrightarrow c^{-1}\phi(y) \leq \phi(x) \leq c\,\phi(y) \tag{2.8}$$

and

$$\phi(x) \to +\infty \qquad \text{as } |x| \to \infty \tag{2.9}$$

and let us assume now that our potentials have a polynomial behavior in the following sense

$$|\operatorname{grad} V_0| + \sum_{j=1}^{p} \sum_{|\alpha|=r+1} |\partial_x^\alpha V_j(x)| + \sum_{i,j=1}^{n} \sum_{r\leq|\alpha|\leq(r+2)} \phi^{(|\alpha|-r)}|\partial_x^\alpha b_{ij}| \leq C\,\phi(x)^{-1}. \tag{2.10}$$

We then introduce the following "limit set" at ∞.

Definition 2.6. $\mathcal{B}_\infty$ is described as the set of the

$$z = (v_0, (v_j^\alpha)_{|\alpha|\leq r, j=1,\ldots,p}, (B_{ij}^\alpha)_{|\alpha|\leq(r-1), 1\leq i\leq j\leq n})$$

s.t. there exists a sequence y_ν ($\nu \in \mathbb{N}$) with the following properties:

$$\text{(a)} \qquad |y_\nu| \to \infty \qquad \text{as } |\nu| \to \infty \tag{2.11a}$$

$$\text{(b)} \qquad \partial_x^\alpha V_j(y_\nu) \to (v_j^\alpha) \qquad \text{as } |\nu| \to \infty \tag{2.11b}$$

$$\text{(c)} \qquad \partial_x^\alpha b_{ij}(y_\nu) \to (B_{ij}^\alpha) \qquad \text{as } |\nu| \to \infty \tag{2.11c}$$

We now associate to each $z \in \mathcal{B}_\infty$
— an electric potential:

$$V_z(x) = v_0 + \sum_{1\leq j\leq p} \Big(\sum_{|\alpha|\leq r} x^\alpha v_j^\alpha/\alpha!\Big)^2, \tag{2.12a}$$

— a magnetic potential:

$$(A_z(x))_i = \sum_{1\leq j\leq n} \Big(\sum_{|\alpha|\leq(r-1)} x^\alpha B_{ij}^\alpha x_j/(\alpha!\cdot(2+|\alpha|)\Big), \tag{2.12b}$$

and the corresponding Schrödinger operator P_{A_z,V_z}.

We then introduce the following subset of $\mathbb{R}$

$$S_\infty = \bigcup_{z\in\mathcal{B}_\infty} \operatorname{Sp}(P_{A_z,V_z}). \tag{2.13}$$

The theorem of [22] gives the link between the union of the spectra of these "limit Schrödinger operators" and the essential spectrum of $P_{A,V}$. This is quite natural if you remember the statement of Persson's Theorem (see [54] or Agmon [1])

$$\inf \operatorname{EssSp}(P_{A,V}) = \sup_{K\in\mathcal{K}} \inf \operatorname{Sp}(P_{A,V}^{\mathbb{R}^n\setminus K}) \tag{2.14}$$

where $\mathcal{K}$ is the family of the compacts in $\mathbb{R}^n$ or the second version $\inf \operatorname{EssSp}(P_{A,V}) = \lim_{R\to+\infty} \inf \operatorname{Sp}(P_{A,V}^{\mathbb{R}^n\setminus B(0,R)})$

Theorem 2.7 (cf. Helffer-Mohamed [22]). *Under assumption* (2.10), *we have*

$$\operatorname{EssSp}(P_{A,V}) = \overline{S_\infty}. \tag{2.15}$$

Actually we shall give in Section 6 a sketch of the unpublished result of Helffer-Mohamed [23] saying that

Theorem 2.8.

$$S_\infty \textit{ is closed in } \mathbb{R}. \tag{2.16}$$

With this theorem we can effectively give a reasonable answer to the question of the equality

$$\inf \operatorname{EssSp}(P_{A,V}) = \inf \operatorname{EssSp}(P_{0,V}).$$

But first we can understand from a new point of view the inequality (1.9). For this, we compare $\mathcal{B}_\infty(A,V)$ and $\mathcal{B}_\infty(0,V)$. We observe first of all that

$$\mathcal{B}_\infty(A,V) \subset \mathcal{B}_\infty(0,V). \tag{2.17}$$

If we use what we know for the spectrum (cf. (1.7)), we get from (2.16) the existence of $z \in \mathcal{B}_\infty(A,V)$ s.t.

$$\inf S_\infty(A,V) = \inf \operatorname{Sp} P_{A_z,V_z} \geq \inf \operatorname{Sp} P_{0,V_z}$$

Then (2.17) implies

$$\inf S_\infty(0,V) \leq \inf \operatorname{Sp} P_{0,V_z} \leq \inf S_\infty(A,V). \tag{2.18}$$

because $z \in \mathcal{B}_\infty(A,V) \subset \mathcal{B}_\infty(0,V)$. In order to simplify we just discuss the case where $V = 0$ and we get

Proposition 2.9. *Under assumptions* (2.1), (2.10) *with* $V = 0$. *Then* $\inf \mathrm{EssSp}\, P_{A,0} = \inf \mathrm{EssSp}\, P_{0,0}$ *if and only if there exists a sequence* y_ν *s.t.* $|y_\nu|$ *tends to* ∞ *and* $|\partial_x^\alpha b_{ij}(y_\nu)| \to 0$ *as* $|\nu| \to \infty$ *for* $|\alpha| \leq (r-1)$ *and* $1 \leq i < j \leq n$.

§3. Semi-classical results

3.1. The Schrödinger case

In [21], we gave an estimate as h tends to 0 of $\lambda^\Omega_{0,A,V}(h) - \lambda^\Omega_{0,0,V}(h)$ when condition (iii) is not satisfied. Under suitable assumptions on V (V has a unique non degenerate minimum in Ω[1] at a point x_0, $V(x_0) = 0$ and V creates a sufficiently strong barrier around $\partial\,\Omega$), we prove that a magnetic potential (with 0 corresponding σ_B in Ω) creates a splitting of the type

$$\begin{aligned}(3.1)\quad & \lambda^\Omega_{0,A,V}(h) - \lambda^\Omega_{0,0,V}(h) \\ & = h^{1/2}\exp(-S_1/h)(a(h)(1-\cos(\int_\gamma \omega_A/h)) + O(\exp(-\varepsilon)/h)))\end{aligned}$$

where

- $a(h)$ is a symbol (independent of A) which is (under suitable generic assumptions) elliptic,
- ε is strictly positive,
- S_1 is the minimal length of a closed path starting of x_0 and not homotop to the trivial path in Ω.

Here the length is measured according to the Agmon metric $V \cdot dx^2$. The sentence "*creates a sufficiently strong barrier*" means mathematically that

$$S_1 < 2S_0$$

where S_0 is the Agmon distance of x_0 to $\mathbb{R}^2 \setminus \Omega$.

The proof is based on a comparison of $\lambda_A(h)$ with a problem (independent of A) on the covering of Ω. Another important fact in the proof is the decay of the eigenfunctions which is controlled by Agmon estimates (cf. Agmon [1], Helffer-Sjöstrand [28] and Section 5). As a consequence of these estimates we get also by perturbation

$$\begin{aligned}(3.2)\quad & \lambda_{0,A,V}(h) - \lambda_{0,0,V}(h) \\ & = h^{1/2}\exp(-S_1/h)a(h)((1-\cos(\int_\gamma \omega_A/h)) + O(\exp(-\varepsilon)/h)))\end{aligned}$$

[1]We have assumed to simplify that Ω was the complementary of a disc in $\mathbb{R}^2$

where $\lambda_{0,A,V}$ is now attached to the problem in $\mathbb{R}^n$.

3.2. The direct effect

When $2S_0 < S_1$, it is explained in Helffer [21] how to produce under suitable assumptions a direct effect of the magnetic field whose order is effectively $\exp(-2S_0/h)$.

3.3. The paramagnetic inequality

As a first application we obtain (following [21]) a new version of the counterexample (given by Avron-Simon [7]) to a conjecture on the existence of a paramagnetic inequality due to Hogreve-Schrader-Seiler [30] and we think that this gives also some interesting information in the discussion around the existence of the Bohm-Aharonov effect (cf. [2], [54], [8], and the references in this paper). We treat the case of dimension 2 but the arguments are more general in nature. Let us consider the Dirac operator in $\mathbb{R}^2$ with a magnetic field

$$D(A)(h) = \sum_{j=1}^{2} \sigma_j (hD_{x_j} - A_j), \tag{3.3}$$

where the σ_j are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

which is a selfadjoint operator on $L^2(\mathbb{R}^2) \otimes \mathbb{C}^2$.

Then the Pauli operator is classically defined as the square of the Dirac operator

$$P(A)(h) = (D(A)(h))^2 = \sum_{j=1}^{2} (hD_{x_j} - A_j)^2 \cdot \mathrm{Id} + h \begin{pmatrix} B & 0 \\ 0 & -B \end{pmatrix} \tag{3.4}$$

with $B(x) = (\partial_{x_2} A_1 - \partial_{x_1} A_2)$. If V satisfies (1.1), we are interested in the validity of the paramagnetic inequality

$$\inf \mathrm{Sp}((D(A)(h))^2 + V \cdot I) \leq \inf \mathrm{Sp}(-h^2 \Delta + V). \tag{3.5}$$

If $\lambda_{0,0,V}(h)$ denotes the first eigenvalue of $(-h^2\Delta + V)$ and if we denote by $\lambda^{\pm}_{A,V}$ the first eigenvalues of $((D(A)(h))^2 + V \cdot I)$, the question is to know if the following inequality is true:

$$\inf_{\pm} \lambda^{\pm}_{A,V}(h) \leq \lambda_{0,0,V}(h) \tag{3.6}$$

Let us recall that in Section 1 we have mentioned the opposite inequality:

$$\lambda_{0,A,V}(h) \geq \lambda_{0,0,V}(h) \tag{3.7}$$

It is then an easy corollary of (3.2) that, under the same assumptions, (3.6) is false for a convenient choice of A and h small enough. We observe indeed that according to the decay properties of the corresponding eigenfunctions, we have

$$\lambda^{\pm}_{A,V}(h) - \lambda^{0}_{A,V}(h) = O(\exp(-2(S_0 - \varepsilon)/h)), \qquad \forall \varepsilon > 0$$

which is a smaller effect that the effect due to the flux (this was the argument we use to go from (3.1) to (3.2)).

3.4. The Dirac operator in dimension 3

We consider the Dirac operator with magnetic potential A

$$(\sum_{j=1}^{3} \alpha_j (hD_{x_j} - A_j) + \beta + V) \tag{3.8}$$

in $L^2(\mathbb{R}^3; \mathbb{C}^4)$, where $(\alpha_j)_{j=1,2,3}$ and $\beta = \alpha_4$ are the Dirac matrices, $(A_j)_{j=1,2,3}$ is a magnetic vector potential and V a scalar potential. Let us assume that:

$$\limsup_{|x| \to \infty} V(x) < 1$$

which implies that the spectrum is discrete in the neighborhood of 0. We assume also that Ω is the complementary of an infinite cylinder $\mathcal{C}$ in the x_3 direction and that $B = 0$ in Ω. We assume that V creates a sufficiently strong barrier around $\mathcal{C}$ and that V has a unique non-degenerate extremum in Ω at some point say $x_0 = (0,1,0)$ and that $V(x_0) = 1$. Finally we assume generical assumptions on V (unique "non degenerate" minimal path around $\mathcal{C}$ starting from x_0). In the case where A is zero we know from X.P. Wang [64] that due to the Kramers theorem all the eigenspaces appear with even dimension (see also [53]). Near 0 the "first" eigenvalue $\lambda_0(h)$ is determined modulo $O(h^2)$ by some quadratic approximation and separated from the rest of the spectrum by (h/C) (cf. [64]). Moreover the multiplicity is exactly 2. The argument fails as the magnetic field is introduced and the purpose of the work of B. Parisse [53] was to study the effect of the magnetic field on the splitting which by perturbation arguments will in this context be "exponentially small". B. Parisse proves the following theorem:

Theorem 3.1. *If $\lambda_0(h)$ is the double eigenvalue of $D_0(h)$, then for h small enough, the operator $D_A(h)$ admits two eigenvalues $\lambda_A^{\pm}(h)$ satisfying to*

$$(3.9)\quad \lambda_A^{\pm}(h) = \lambda_0(h) + h^{1/2}\exp(-S_1/h)(\Re(c(h)\cdot[\exp(\pm i\phi/h) - 1]) + O(\exp(-\varepsilon)/h)))$$

where $\varepsilon > 0$, $c(h) = a(h) + i\, b(h)$ is a complex elliptic symbol, $\phi = \int_\gamma \omega_A$, $\varepsilon > 0$ and S_1 is the minimal length of a closed path starting of x_0 and not homotop to the trivial path in Ω. Here the length is measured according to the Agmon metric $(1 - V^2)_+ dx^2$.

Modulo some technicalities due to the fact that we now deal with systems, the scheme of the proof is the same as for Schrödinger. It is more delicate to prove that $c(h) \neq 0$ and this a consequence of the WKB constructions.

Let us remark that as a consequence of (3.9) we get the following formula for the splitting

$$(3.10)\quad \lambda_A^{+}(h) - \lambda_A^{-}(h) = -2h^{1/2}\exp(-S_1/h)\cdot(b(h)\sin(\phi/h) + O(\exp(-\varepsilon)/h)))$$

It would be very interesting to prove that generically $b(h)$ is elliptic or that under additional symmetries $b(h)$ is exponentially small. As suggested by B. Parisse it would also be interesting to look to the non relativistic limit where we will find a problem similar to the case treated in Subsection 3.3.

§4. The case of systems

(after Hebbar, Kuwabara, Manabe, Shigekawa ...)

4.1. Introduction

The idea to look at systems is very natural and physically motivated (see for example T.T. Wu and C.N. Yang [65]). But O. Hebbar found more recently that R. Kuwabara treats the case with $V = 0$ in 1982 [40]. As we shall see, the case $V \neq 0$ is not essentially more difficult. Anyway the result of Hebbar [18] is a little more general that the result of [40] *also* in the case $V = 0$.

4.2. The results of Kuwabara revisited

Let (M, g) be a compact n-dimensional C^∞ manifold without boundary and E a be a complex vector bundle over M with rank r. We assume that E has a C^∞ Hermitian structure $\langle\cdot\rangle$. Let us denote by $A^0(M,E) = C^\infty(E)$ the set of the C^∞ sections of E. More generally we denote by $A^p(M)$ the set of the C^∞ p-forms on M and by $A^p(M,E)$ the set of E-valued C^∞ p-forms on M. Let $\tilde{d}$: $A^0(M,E) \to A^1(M,E)$ be a linear connection on E compatible with the Hermitian structure. There is also a natural extension of $\tilde{d} = \tilde{d}_0$ on the p-forms given by

$$\tilde{d}_p(s \otimes t) = (\tilde{d}_0 s) \otimes t + s \otimes dt \tag{4.1}$$

for all $s \in A^0(M,E)$ and $t \in A^p(M)$. There is a natural inner product on $A^p(M,E)$ and we can then define the L^2 p-forms with a natural Hilbertian structure. The Laplace operator on the p-forms is then defined by:

$$L^{(p)} = \tilde{d}_p^* \tilde{d}_p + \tilde{d}_{p-1}.\tilde{d}_{p-1}^* \tag{4.2}$$

We shall concentrate on: $L = L^0$ and will write sometimes $L(E,\tilde{d})$ to mention the dependence with respect to the fiber bundle and the connection. Of course L is an elliptic operator (of order 2) with compact resolvent and admits as spectrum an increasing sequence of eigenvalues $\lambda_j(E,\tilde{d})$ tending to $+\infty$ and because the Laplacian is positive, we have of course $\lambda_0(E,\tilde{d}) \geq 0$. If $E = M \times \mathbb{C}$, and if we take the trivial connection d, we get the usual spectrum of the Laplace-Beltrami operator $\lambda_j(M)$ with $\lambda_0(M) = 0$. The problem we want to address is now: *Under which conditions on E and $\tilde{d}$ do we have $\lambda_0(M) = \lambda_0(E,\tilde{d})$, or more generally $\lambda_0(M) = \lambda_j(E,\tilde{d})$ for $j = 0,\ldots,k-1$ for some k.* Let us remark that if a section s satisfies $Ls = 0$ (we shall say that s is harmonic) then it satisfies $\tilde{d}s = 0$ (that is s is a parallel section). Kuwabara proves the following proposition (Proposition 3.1 in [40]):

Proposition 4.1. (i) *If L has a zero eigenvalue with multiplicity k $(k \leq r)$ then*

$$E = E' \oplus T_k \qquad (\textit{Whitney sum}), \tag{4.3}$$

where T_k is a trivial bundle of rank k.
(ii) *If L has zero eigenvalue with multiplicity r, then E is a trivial bundle and the curvature Ω of the connection vanishes.*

The proof is a direct consequence of the fact that an orthonormal system of k independent eigenfunctions u_k gives actually a system of k independent sections giving a natural orthogonal basis for a trivial subbundle of E. The second point is as in the study of the scalar Bohm-Aharonov effect.

The second result given in [40] is the following:

Proposition 4.2. *If L has zero eigenvalue, then* $\mathrm{Sp}(M,g) \subset \mathrm{Sp}(M,g,E,\tilde{d})$. *Moreover, if L has zero eigenvalue with multiplicity r, then* $\mathrm{Sp}(M,g,E,\tilde{d}) = r \cdot \mathrm{Sp}(M,g)$ *where* $r \cdot \mathrm{Sp}(M,g) = \mathrm{Sp}(M,g) \cup \cdots \cup \mathrm{Sp}(M,g)$ *(r times).*

Proof. Since $0 \in \mathrm{Sp}(M,g,E,\tilde{d})$, there is a non zero f in $C^\infty(E)$ s.t.

$$\tilde{d}f = 0. \tag{4.4}$$

We have already seen that it does not vanish anywhere. Suppose $\lambda \in \mathrm{Sp}(M,g)$ and let ϕ be a non zero eigenvector

$$-\Delta\phi = \lambda\phi. \tag{4.5}$$

Then, using elementary computations, (4.4) and (4.5), we get that $s = \phi f$ is an eigenvector for L. The other part is also easy.

Actually, O. Hebbar will deduce these results from the following:

Lemma 4.3 (see [18]). *If L has a zero eigenvalue with multiplicity k ($k \leq r$) then the connection split according to the decomposition:*

$$\begin{aligned} E &= T_k^\perp \oplus T_k \qquad (\textit{orthogonal decomposition}) \\ \tilde{d} &= \tilde{d}_1 \oplus \tilde{d}_2 \end{aligned}$$

As a consequence we have a direct decomposition of the Laplacian

$$L(M,g,E,\tilde{d}) = L(M,g,T_k^\perp,\tilde{d}_1) \oplus L(M,g,T_k,\tilde{d}_2)$$

with

$$\mathrm{Sp}(M,g,E,\tilde{d}) = \mathrm{Sp}(M,g,E,\tilde{d}_1) \cup \mathrm{Sp}(M,g,E,\tilde{d}_2)$$

and moreover $L(M,g,T_k,\tilde{d}_2)$ has zero eigenvalue with multiplicity k. Then we get the following improvement of Proposition:

Proposition 4.4. *If L has a zero eigenvalue with multiplicity k ($k \leq r$) then*

(i) $E = T_k^\perp \oplus T_k$ *(Whitney sum),* $\tilde{d} = \tilde{d}_1 \oplus \tilde{d}_2$
(ii) $L(M,g,T_k,\tilde{d}_2)$ *has zero eigenvalue with multiplicity k*
(iii) T_k *is a trivial bundle and the curvature of $\tilde{d}_2$ vanishes*
(iv) $\mathrm{Sp}(L(M,g,E,\tilde{d}) \supset \mathrm{Sp}\, L(M,g,T_k,\tilde{d}_2) = k\,\mathrm{Sp}(M,g)$.

To go further, we have to analyze more precisely and introduce the notion of gauge transformations. Recall that a gauge transformation on a vector bundle E with the Hermitian structure is a diffeomorphism $\Phi\colon E \to E$ which maps each fiber E_x isometrically and linearly onto itself. For a linear connection $\tilde{d}$ on E, we get a new connection $\Phi^*\tilde{d} = \Phi^{-1}\tilde{d}\,\Phi$. Two connections $\tilde{d}$ and $\tilde{d}'$ on E are called gauge equivalent to each other (and we write $\tilde{d} \sim \tilde{d}'$) if there exists a gauge transformation such that: $\tilde{d}' = \Phi^*\tilde{d}$. Of course, we have in this case

$$\mathrm{Sp}(L(M,g,E,\tilde{d}) = \mathrm{Sp}(L(M,g,E,\tilde{d}').$$

The problem we are looking at is to give now a good characterization of two gauge equivalent connections. Kuwabara [40] gives the following criterion:

Proposition 4.5. *Let E be a line-bundle on M then $\tilde{d} \sim \tilde{d}'$ if and only if the corresponding connection 1-forms ω and ω' satisfy $(\omega - \omega')/2\pi$ [2] is an integral 1-form.*

This was already observed in Section 1. For a general fiber bundle, there is a similar criterion using the notion of matrix of holonomy attached to a connection and a closed path γ. Using the theorem that a connection with 0 curvature is locally gauge-equivalent to 0, it is natural to attach to each curve γ a class of equivalence of unitary matrices in $U(\mathbb{C}^r)$: $U_\gamma = I$. We have then the following criterion (cf. for example [18] but it is probably well known in Topology):

Proposition 4.6. *Let E be a trivial hermitian fiber bundle on M and let d_0 be the connection associated to the 1-form 0; then $\tilde{d} \sim d_0$ if and only if the corresponding connection 1-form ω*

(a) *ω has 0-curvature*

and

(b) *$U_\gamma = I$ for any closed path γ.*

As a conclusion of this subsection, we get following Hebbar [18] the following extension of the results in [40]:

---

[2] which is a global 1-form on M

Theorem 4.7. *Let E be a Hermitian bundle over (M, g), and $\tilde{d}$ a linear connection on E which is compatible with the Hermitian structure. Then the following properties are equivalent:*

(i) *L has a zero eigenvalue with at least multiplicity k $(k \leq r)$*

(ii) *$E = T_k^{\perp} \oplus T_k$ (Whitney sum), $\tilde{d} = \tilde{d}_1 \oplus \tilde{d}_2$ with*[3] *$\tilde{d}_2 \sim \tilde{d}_0$ where d_0 denotes the canonical connection on the trivial bundle T_k whose 1-form is 0.*

(iii) $\mathrm{Sp}(L(M, g, E, \tilde{d}) \supset k\,\mathrm{Sp}(M, g)$

4.3. Extension to the Bochner-Laplace-Schrödinger equation

Here we explain the results of [18]. More precisely we shall explain how to deduce the results with non zero V from the corresponding results with $V = 0$. But note that it is possible because we are on a compact manifold. For other cases (boundary problems) we must of course take the problem directly (as Hebbar did). The theorem obtained by Hebbar [18], generalizing results of ([21], [40], [59], [43]), is the following (we limit ourselves to the case when M is compact):

Theorem 4.8. *Let E be a Hermitian bundle over (M, g), and $\tilde{d}$ a linear connection on E which is compatible with the Hermitian structure. Let V be a C^∞ potential on M. Let $\lambda_0(M, g, V)$ be the first eigenvalue of the Laplace-Beltrami-Schrödinger operator on M: $-\Delta + V$. Then the following properties are equivalent:*

(i) *$L + V$ has $\lambda_0(M, g, V)$ with at least multiplicity k $(k \leq r)$.*

(ii) *$E = T_k^{\perp} \oplus T_k$ (Whitney sum), $\tilde{d} = \tilde{d}_1 \oplus \tilde{d}_2$ with (cf. preceding Footnote) $\tilde{d}_2 \sim d_0$ where d_0 denotes the canonical connection on the trivial bundle T_k whose 1-form is 0.*

(iii) $\mathrm{Sp}(L(M, g, V, E, \tilde{d})) \supset k\,\mathrm{Sp}(M, V, g)$

Remark 4.9. In particular, if $k = r$, we get the equivalent of the theorem given in Section 1.

Corollary 4.10. *Let E be a Hermitian bundle over (M, g) with rank r; then the following properties are equivalent:*

(i) *$L + V$ has $\lambda_0(M, g, V)$ as an eigenvalue with multiplicity r.*

(ii) *E is a trivial bundle and $\tilde{d} \sim d_0$ where d_0 denotes the canonical connection on E whose 1-form is 0.*

---

[3] and Proposition 4.6 gives a good criterion to verify the property

(iii) *$L(M, g, V, E, \tilde{d})$ is gauge equivalent to $(-\Delta + V)$. Id defined on the trivial fiber bundle $M \times \mathbb{C}^r$.*

The equivalence of (i) and (iii) was proved in [43].

Sketch of the proof (following partially [18]*).* We extend the Lavine-O'Caroll formula to this case. For $s \in C^\infty(E)$, we have the identity

$$\|\tilde{d}s - (du_0/u_0) \otimes s\|^2 = \langle L(M, g, V, E, \tilde{d})s \mid s\rangle - \lambda_0(M, g, V)\|s\|^2 \tag{4.6}$$

where we take the L^2-canonical scalar products. From this, we get that an eigenfunction s_j of $L(M, g, V, E, \tilde{d})$ with eigenvalue $\lambda_0(M, g, V)$ has the property that (s_j/u_0) is parallel for $\tilde{d}$. This was the essential point to get all the statements in Subsection 4.2.

Remark 4.11. It is possible to quantify this result by semi-classical methods in the spirit of the results of Section 3. The problem is studied by Hebbar in [18].

§5. Some decay results for the eigenfunctions

5.1. Decay at ∞

We want to present in this subsection some results on the decay at ∞ (or locally as the Planck constant tends to 0) of the eigenfunctions of $P_{A,V}$. For the first result, we consider the simpler case where A and V are polynomials with

$$V \geq 0. \tag{5.1}$$

As in Helffer-Nourrigat [24] and also Feffermann [12] we introduce

$$M(x) = \sum_\alpha |\partial^\alpha V(x)|^{1/(|\alpha|+2)} + \sum_{\alpha,j,k} |\partial^\alpha b_{j,k}(x)|^{1/(|\alpha|+2)}. \tag{5.2}$$

In this simpler case, the compactness criterion given in Corollary 2.2 was obtained in [24], where it is also proved that, if $M(x)$ tends to ∞, every solution in $\mathcal{S}'(\mathbb{R}^n)$ of $H\psi = \lambda\psi$, $\lambda > 0$ is actually in $\mathcal{S}(\mathbb{R}^n)$. In the case when $V(x)$ itself tends to ∞, the decay of the eigenfunction ψ is associated with the Agmon metric $(V - \lambda)_+ dx^2$. Of course it is not necessary to assume that we have compact resolvent and it is for example sufficient to assume that λ satisfies

$$\lambda < \inf \operatorname{EssSp} P_{A,V} \tag{5.3}$$

in order to get some decay like $\exp((\lambda - \inf \operatorname{EssSp} P_{A,V})|x|)$. We refer to [1] and references therein for a discussion. But let us come back to the case when $M(x)$ tends to ∞. The heuristic idea is that the role played by V is replaced by $M(x)^2$. We shall loose a little in precision because one has to remember that $M(x)$ is only defined up to some multiplicative constant. For all λ we introduce the "well"

$$U(\lambda) = \{x \in \mathbb{R}^n, M(x)^2 \leq \lambda\} \tag{5.4}$$

and denote by $d_\lambda(x) = d(x, U(\lambda))$ the distance of x to $U(\lambda)$ for the modified Agmon's metric $ds^2 = M(x)^2 dx^2$. The principal result obtained in [25] is the following:

Theorem 5.1. *There exist constants $C > 0$ and $\varepsilon > 0$, depending only on the dimension n of the space and on the largest degree r of the polynoms A_j and $V \geq 0$, s.t. for any solution $\psi \in \mathcal{S}(\mathbb{R}^n)$ of $P_{A,V}\psi = \lambda\psi, \lambda > 0$, the following inequality is satisfied*

$$|\psi(x)| \leq C\lambda^{n/4} \exp(-\varepsilon d_{C\lambda}(x))\|\psi\|_{L^2}, \qquad \textit{for all } x \in \mathbb{R}^n. \tag{5.5}$$

Remark 5.2. As we have implicitly seen in Section 3 (and as it appears clearly in [26] or in [60], [62]), the Agmon's type estimates have a natural transcription in the semi-classsical context and play a basic role in the estimate of the tunneling effect. The estimates are then local but asymptotic for h tending to 0. A semi-classical version of this theorem was obtained by Brummelhuis [10] (see also [25] Section 6).

Example 5.3. $n = 2$; $A_1(x_1, x_2) = x_1^2 x_2$, $A_2(x_1, x_2) = -x_2^2 x_1$; $V = 0$. We have in this case: $b_{1,2}(x_1, x_2) = x_1^2 + x_2^2$ and $M(x_1, x_2)^2 \approx (1 + x_1^2 + x_2^2)$.

Remark 5.4. The polynomial character is only assumed for simplification. One can certainly extend the results under assumptions of the type given in (2.11) (see Guibourg [16] for results in this direction).

Some words on the proof. The L^2 estimates in (5.4) follows closely the Agmon's proof replacing V by M^2. In order to get the L^∞ estimates, a global Sobolev's theorem is used in [25] whose proof is based on the proof of maximal estimates in adapted Sobolev spaces appearing in [24]. The proof is then a consequence of the nilpotent Lie groups techniques which will be presented very shortly in Section 6 (See the book [24] or the surveys of Helffer [19] or Nourrigat [51]).

5.2. Semiclassical aspects for the decay

As it was already mentioned in the context of the study of the decay of the eigenfunctions at ∞, we can also study the decay in the semiclassical context and the first result proved in Helffer-Sjöstrand [28] is that if

$$P_{A,V}(h)u_h = \lambda(h)u_h \tag{5.5}$$

with $\lambda(h) \to E$ and $||u_h||_{L^2} = 1$ then we have on every compact K and for every $\varepsilon > 0$

$$|u_h(x)| \leq C_{\varepsilon,K} \exp(\varepsilon/h) \exp(-d_{(V-E)_+}(x, U_E)/h) \tag{5.6}$$

where U_E is the well: $V \leq E$ and $d_{(V-E)_+}(x,y)$ is the Agmon distance attached to the potential $(V-E)_+$. As we observed in Subsection 5.1 and as one can easily compute for examples of the type

$$-\sum_j (h\partial_{x_j} - i\sum_k b_{jk}x_k)^2 + |x|^2,$$

this estimate is not at all optimal. It can be useful (at least to understand heuristically the problem) to look for WKB constructions in the case where V has a unique non-degenerate minimum at 0 and is analytic in a neighborhood of 0. We assume here that

$$\inf V = 0.$$

It is proved in [28] that for t small enough it is possible to construct a WKB solution for $P_{tA,V}(h)$ of the form

$$h^{-n/4}a(t,x,h)\exp(-\phi(t,x,h)/h) \tag{5.7}$$

where $\phi(t,x,h)$ is a solution in a neighborhood of 0 of the eikonal equation

$$(\nabla_x\phi - itA)^2 = V \tag{5.8}$$

Admitting that this WKB approximation gives effectively an approximation of one eigenfunction (and this is proved for t small enough in [28]), then $\Re\,\phi$ gives the control of the decay with respect to t. We admit the existence of $\phi(t,x)$ (also proved in [28]) and taking the real part and the imaginary part of (5.8) we get

$$|\nabla(\Re\,\phi_t)|^2 = V + |\nabla(\Im\,\phi_t) - tA|^2 \tag{5.9}$$

and

$$\nabla(\Re\,\phi_t)\cdot(\nabla(\Im\,\phi_t)-tA)=0 \tag{5.10}$$

and we can take

$$\Re\,\phi_t \geq 0. \tag{5.11}$$

Equations (5.9) and (5.11) permit to say that in a neighborhood of 0, $\Re\,\phi_t$ is the Agmon distance to 0 for the potential: $(V+|\nabla\Im\,\phi_t - tA|^2)$. This gives the general inequality

$$\Re\,\phi_t \geq \Re\,\phi_0 \qquad \text{in a neighborhood of } 0. \tag{5.12}$$

Then we observe that

$$\nabla(\Re\,\phi_t+\phi_0)\nabla(\Re\,\phi_t-\phi_0)=|\nabla\Im\,\phi_t - tA|^2 \tag{5.13}$$

and

$$(\Re\,\phi_t-\phi_0)(0)=0. \tag{5.14}$$

Then we get in a suitable (but independent of t with $|t|\leq t_0$) neighborhood of 0 that $(\Re\,\phi_t-\phi_0)(x)=0$ implies that: $\nabla(\Im\,\phi_t)-tA=0$ along the integral curve of the vector field $\nabla(\Re\,\phi_t+\phi_0)$ joining x and 0. In particular if $(\Re\,\phi_t-\phi_0)(x_j)=0$ in an open set on some sphere around 0 then we get by analyticity that $\nabla(\Im\,\phi_t)-tA=0$ in a neighborhood of 0 which gives that A is locally exact.

§6. Nilpotent Lie group techniques

In this section we shall give the proof of Theorem 2.8. We assume that the reader is somewhat familiar with the theory of nilpotent Lie groups (see [15]) and we emphasize that all these techniques were developped first for the study of hypoellipticity. For $n,p,s\in\mathbb{N}$, let us introduce the "maximal" universal Lie Algebra $\mathcal{G}^{(n,p,s)}$ with the following properties

$$\mathcal{G}^{(n,p,s)} \text{ is graded of rank of nilpotency } s, \tag{6.1}$$

i.e.

$$\mathcal{G}^{(n,p,s)}=\mathcal{G}_1\oplus\mathcal{G}_2\oplus\ldots\oplus\mathcal{G}_s$$

and

$$[\mathcal{G}_i, \mathcal{G}_j] \subset \mathcal{G}_{i+j}, \qquad \text{if } (i+j) \leq s,$$
$$[\mathcal{G}_i, \mathcal{G}_j] = 0, \qquad \text{if } (i+j) \geq (s+1)$$

(6.2) $$\mathcal{G}_1 = \mathcal{G}_1' \oplus \mathcal{G}_1'', \qquad \text{with } \dim \mathcal{G}_1' = n,\ \dim \mathcal{G}_1'' = p$$

(6.3) $$\mathcal{G}_1 \text{ generates } \mathcal{G}.$$

(6.4) $$[\mathcal{G}_1'' \oplus \mathcal{G}^2, \mathcal{G}_1'' \oplus \mathcal{G}^2] = 0$$

where

(6.5) $$\mathcal{G}^2 = \mathcal{G}_2 \oplus \cdots \oplus \mathcal{G}_s$$

and

$$\mathcal{G}^{(n,p,s)} \text{ is of maximal dimension with the above properties.}$$

The algebra $\mathcal{G}^{(n,p,s)}$ has the following universal property: Let $(Y_j')_j$ be a basis of $\mathcal{G}_1'$, $(Y_k'')_k$ a basis of $\mathcal{G}_1''$; then there exists a partial homomorphism of rank s, λ, s.t.:

(6.6) $$\lambda(Y_j') = X_j' \qquad \lambda(Y_k'') = X_k'',$$

where

$$X_j' = \partial_{x_j} - iA_j(x) \qquad \text{for } j = 1, \dots, n;\ X_k'' = iV_k(x)$$

for $k = 1, \dots, p$ (with $s = r + 1$). We refer to R. Goodman [15] for this property or to [24] where this type of Lie Algebras is studied in Chapter XI. We observe (cf. Chapter XI of [24])

(6.7) $$P_{A_z, V_z} = \Pi_{\ell_z, \mathcal{H}}(-\Delta),$$

where

(6.8) $$\mathcal{H} = \mathcal{G}_1'' \oplus \mathcal{G}^2,$$

ℓ_z is the element of $\mathcal{G}^*$ (dual of $\mathcal{G} = \mathcal{G}^{(n,p,s)}$) associated to $z \in \mathcal{B}_\infty$ by the relations

$$\ell_z/\mathcal{G}_1' = 0, \tag{6.9}$$

$$\ell_z((adY')^\alpha Y_k'') = v_k^\alpha, \qquad \text{for } |\alpha| \leq s-1, \tag{6.10}$$

$$\ell_z((adY')^\alpha [Y_i', Y_j']) = B_{ij}^\alpha, \qquad \text{for } |\alpha| \leq (s-2), \tag{6.11}$$

$$\Delta := \sum_j Y_j'^2 + \sum_k Y_k''^2 \tag{6.12}$$

and $\Pi_{\ell,\mathcal{H}}$ is the induced representation associated to ℓ and to a subalgebra $\mathcal{H}$ satisfying

$$\ell([\mathcal{H},\mathcal{H}]) = 0.$$

Let us introduce

$$\Lambda(\ell,\mathcal{H}) = G\cdot(\ell + \mathcal{H}^\perp) \qquad \text{in } \mathcal{G}^*.$$

In a first step we use the techniques of [24] in order to prove:

Proposition 6.1.

$$\sigma(P_{A_z,V_z}) = \bigcup_{\rho\in\overline{\Lambda}} \sigma(\Pi_\rho(-\Delta)) \tag{6.13}$$

The map $\rho \to \Pi_\rho$ is the classical Kirillov's map from $\mathcal{G}^*$ onto $\hat{G}$ (the set of equivalence classes of irreducible representations of the simply connected Lie group associated to $\mathcal{G}$, $G := \exp \mathcal{G}$) and G acts on $\mathcal{G}^*$ by the coadjoint map.

Proof of Proposition 6.1. Let us first observe that the different operators appearing in formula (6.13) P_{A_z,V_z} and $\Pi_\rho(-\Delta)$ are essentially selfadjoint starting from respectively $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}_{\Pi_\rho}$, the space of C^∞-vectors of the representation. Proposition 2.21 of Chapter II in [24] gives immediately the following equivalences for $\lambda \in \mathbb{R}$ and $C > 0$

$$\|(P_{A_z,V_z} - \lambda)u\| \geq C^{-1}\|u\|, \qquad \forall u \in C_0^\infty(\mathbb{R}^n) \tag{6.14}$$

$$\|(\Pi_\rho(-\Delta - \lambda)f\|_{H^0_{\Pi_\rho}} \geq C^{-1}\|f\|_{H^0_{\Pi_\rho}}, \qquad \forall f \in \mathcal{S}_{(\Pi_\rho)} \text{ and } \forall \rho \in \overline{\Lambda(\ell_z,\mathcal{H})} \tag{6.15}$$

where $H^0_{\Pi_\rho}$ is the space of the representation Π_ρ.

We shall write $(6.14)_{\lambda,C}$ (resp. $(6.15)_{\lambda,C}$) in order to say that the inequality (6.14) (resp. (6.15)) is satisfied for specific constants (λ, C).

This equivalence between (6.14) and (6.15) implies immediately the property

$$\sigma(P_{A_z,V_z}) = \overline{\mathcal{C}} \tag{6.16}$$

with

$$\mathcal{C} = \left(\bigcup_{\rho \in \overline{(\Lambda(\ell_z,\mathcal{H})}} \sigma(\Pi_\rho(-\Delta)) \right),$$

and the way to go from (6.16) to the stronger (6.13) is of the same type as the object of Theorem 2.8.

Proof of (6.13). Let us assume that for some $\lambda \in \mathbb{R}$, we have the following property

$$\forall \rho \in \Gamma_z = \overline{\Lambda(\ell_z,\mathcal{H})},\ \exists C_\rho > 0$$

s.t. $(6.15)_{\lambda,c}$ is satisfied with $C = C_\rho$.

We wish to show $(6.15)_{\lambda,c}$ with C independent of $\rho \in \Gamma_z$. This problem is quite analogous to the problems solved in [24]. The only new point is that Γ_z is closed and invariant by G but not stable by dilation. We refer to [19] which is more adapted to our problem. A first important remark coming from the hypoellipticity of Δ in G is the existence of a constant $D > 0$ s.t.

$$\|u\|^2_{H^2_{\pi_{\ell_z,\mathcal{H}}}} \leq D(\|P_{A_z,V_z}u\|^2 + \|u\|^2), \qquad \forall u \in \mathcal{S}(\mathbb{R}^n), \tag{6.17}$$

and (cf. Proposition 2.2.1, Chapter II in [24]),

$$\|f\|^2_{H^2_{\pi_\rho}} \leq D(\|\Pi_\rho(-\Delta)f\|^2_{H^0_{\pi_\rho}} + \|f\|^2_{H^0_{\pi_\rho}}), \qquad \forall f \in \mathcal{S}_{\Pi_\rho}. \tag{6.18}$$

where H^m_π (for $m \in \mathbb{N}$ and π a representation) is the space of the $u \in H^0_\pi$ s.t. $\pi(Y)^\alpha u \in H^0_\pi$, for $|\alpha| \leq m$, with the natural Hilbertian norm. (6.18) shows that the problem to prove $(6.15)_{\lambda,C}$ with C independent of ρ is equivalent to the apparently stronger result (but more stable):

Property P₁. Let us assume that, for all $\rho \in \Gamma_z$, there exists $C_\rho > 0$ s.t.

(6.19)

$$(\|(\Pi_\rho)(-\Delta) - \lambda)f\|^2_{H^0_{\pi_\rho}} \leq C^{-1}\|f\|^2_{H^0_{\pi_\rho}}, \qquad \forall f \in \mathcal{S}_{\Pi_\rho} \text{ with } C = C_\rho,$$

then there exists $C > 0$ s.t. $(6.19)_C$ is satisfied for all $\rho \in \Gamma_z$.

On the same way, Proposition 6.1 results of the following stronger property:

Property $\mathbf{P_2}$. Let us assume that, for all $z \in \mathcal{B}_\infty$ and for all $\rho \in \Gamma_z$ there exists $C_{\rho,z} > 0$ s.t. $(6.19)_C$ is satisfied with $C = C_{\rho,z}$, then there exists $C > 0$ s.t. $(6.19)_C$ is satisfied for all $z \in \mathcal{B}_\infty$ and $\rho \in \Gamma_z$.

Here we introduce as a new subset of $\mathcal{G}^*$

$$\Gamma = \bigcup_{z \in \mathcal{B}_\infty} \Gamma_z \tag{6.20}$$

whose properties are given in the following:

Proposition 6.2. *$\ell \in \Gamma$ if and only if there exists $z \in \mathcal{B}_\infty$ s.t.: $\ell/(\mathcal{G}^2 \oplus \mathcal{G}_1'') = \ell_z/(\mathcal{G}^2 \oplus \mathcal{G}_1'')$ where ℓ_z is defined in* (6.9–6.11). *Moreover Γ is closed in $\mathcal{G}^*$ and stable by the action of G.*

Proof of Proposition 6.2. We can define Γ on the following way which is quite similar to Definition 2.4 in chapter I of [24]

$$\ell \in \Gamma \iff \exists((x_q, \xi_q)_{q\in\mathbb{N}}$$
$$\text{s.t. } |x_q| + |\xi_q| \to \infty \text{ as } q \to \infty \text{ and } \ell = \lim_{q\to\infty} \lambda^*_{x_q}\xi_q$$

where λ is the partial homomorphism of rank s introduced in (6.6): $(\lambda^*_{x,\xi})(Z) := i^{-1}\sigma(\lambda(Z))(x,\xi)$,

$$\forall Z \in \mathcal{G}$$

(If X is a vector field, $\sigma(X)$ is by definition the symbol of the corresponding differential operator). The proof that Γ is closed is the same as in [24] (Corollary 2.4, Section 2, Chapter IV). We observe that if

$$\ell = \lim_{q\to\infty} \lambda^*_{x_q}\xi_q$$

then, for $(y,\eta) \in \mathbb{R}^{2n}$,

$$\ell_{y,\eta} = \lim_{q\to\infty} \lambda^*_{x_q}(\xi_q + \eta)$$

is well defined in Γ and that we have

$$\ell_{(y,\eta)}/(\mathcal{G}_1'' \oplus \mathcal{G}^2) = (\exp(y \cdot Y')) \cdot \ell/(\mathcal{G}_1'' \oplus \mathcal{G}^2).$$

As (y, η) varies in $\mathbb{R}^{2n}$, we verify that $\ell_{(y,\eta)}$ describes the orbit of ℓ, which proves the stability of Γ by the action of G.

Proof of P_2 (the proof of P_1 is similar). Let us assume that for each $\rho \in \Gamma$, we have $(6.19)_{C_\rho}$ with $C_\rho > 0$. In order to come back to a more homogeneous situation, we introduce a new Lie algebra $\hat{\mathcal{G}}$

$$\hat{\mathcal{G}} = \mathcal{G} \oplus \mathbb{R} \cdot Z, \tag{6.21}$$

where the law (and the graduation) for $\hat{\mathcal{G}}$ is deduced from $\mathcal{G}$'s law by imposing

$$\hat{\mathcal{G}}_1 = \mathcal{G}_1 \oplus \mathbb{R} \cdot Z \tag{6.22}$$

and

$$[\mathcal{G}, Z] = \{0\}. \tag{6.23}$$

Let us now introduce $\mathcal{P}_\lambda \in \mathcal{U}_2(\hat{\mathcal{G}})$ ($\mathcal{U}(\hat{\mathcal{G}})$ is the enveloping algebra of $\hat{\mathcal{G}}$ and $\mathcal{U}_2(\hat{\mathcal{G}})$ is the subspace of the 2-homogeneous elements for the natural dilation)

$$\mathcal{P}_\lambda = -\Delta + \lambda \cdot Z^2. \tag{6.24}$$

We associate to Γ the set $\hat{\Gamma}$ defined by

$$\hat{\Gamma} = \{\hat{\rho} \in \hat{\mathcal{G}}^*;\ \hat{\rho} = (\rho, \zeta),\ \rho \in \Gamma \text{ and } \zeta = 1\}. \tag{6.25}$$

It is clear that $\hat{\Gamma}$ is closed in $\hat{\mathcal{G}}^*$, $\hat{G}$-stable and that there is a natural identification

$$\Pi_\rho(-\Delta) - \lambda = \Pi_{\hat{\rho}}(\mathcal{P}_\lambda). \tag{6.26}$$

Consequently, we have

$$\forall \hat{\rho} \in \hat{\Gamma},\ \exists C_{\hat{\rho}} = C_\rho > 0$$

s.t.

$$\|\Pi_{\hat{\rho}}(\mathcal{P}_\lambda) f\|^2_{H^0_{\pi_{\hat{\rho}}}} \geq C^{-1} \|f\|^2_{H^0_{\pi_{\hat{\rho}}}}, \qquad \forall f \in \mathcal{S}_{\Pi_{\hat{\rho}}} \tag{6.27}$$

with

$$C = C_{\hat{\rho}}.$$

Unfortunately, we can not directly apply the statements of [24] but the proof of Theorem 4.7 as sketched in [19] can be adapted in our context by modifying the assumptions on the following way:

Theorem 6.3. *Let $\mathcal{G}$ be a graded Lie algebra of rank s and Γ a closed G-stable subset in $\mathcal{G}^*$. Let us assume moreover that: $[\mathcal{G}^2, \mathcal{G}^2] = 0$ and that $\mathcal{G}_1$ generates $\mathcal{G}$. Let $\mathcal{P} \in \mathcal{U}_m(\mathcal{G})$ and let us assume that*

(H1) $\quad \forall \rho \in \Gamma,\ \exists C_\rho > 0 \ s.t.\ \|f\|^2_{H^m_{\pi_\rho}} \leq C_\rho \|\Pi_\rho(\mathcal{P})f\|^2_{H^0_{\pi_\rho}},\ \forall f \in \mathcal{S}_{\Pi_\rho}$

(H2)
$\exists \hat{C} > 0 s.t\ \forall \rho \in \Gamma,\ \|f\|^2_{H^m_{\pi_\rho}} \leq \hat{C}[\|\Pi_\rho(\mathcal{P})f\|^2_{H^0_{\pi_\rho}} + \|f\|^2_{H^0_{\pi_\rho}}],\ \forall f \in \mathcal{S}_{\Pi_\rho}$

(H3)
$$\inf_{(g,\rho)\in G\times\Gamma} |g\cdot\rho| \geq (1/2).$$

Then there exists $C > 0$ s.t. for all $\rho \in \Gamma$ we have:

$$\|f\|^2_{H^m_{\pi_\rho}} \leq C\|\Pi_\rho(\mathcal{P})f\|^2_{H^0_{\pi_\rho}} \qquad \forall f \in \mathcal{S}_{\Pi_\rho} \tag{6.28}$$

It is easy to see, using (6.18) and (6.19), Proposition 6.2 and the property $|g\cdot\hat{\rho}| \geq |\zeta| = 1$, that all the assumptions of Theorem 6.3 are satisfied with $\mathcal{G} = \hat{\mathcal{G}}$, $\Gamma = \hat{\Gamma}$ and $\mathcal{P} = \mathcal{P}_\lambda \in \mathcal{U}_2(\hat{\mathcal{G}})$. (6.28) will give Property (P_2).

Indications on the proof of Theorem 6.3. We follow closely the sketch given in [19] p.228 (proof of Theorem 4.7). Let us mention that J. Nourrigat [50] has improved this theorem, but it is sufficient to use the above theorem in our context. If we compare with Theorem 4.7 in [19], we do not make a proof by induction nor a restriction to $|\ell_s| = 1$. Assumption (H2) replaces (4.21) and (H3) replaces (4.22) in [19]. Modulo these modifications the proof is the same (in this article $s = r$). We introduce for $j = 1, \ldots, s$ and $(\ell_1, \ldots, \ell_s)$ the set

$$\Gamma^j(\ell_1, \ldots, \ell_s) = \{\hat{\ell} \in \Gamma,\ \exists g \in G \text{ with } g\cdot\hat{\ell}/\mathcal{G}^j = (\ell_j, \ldots, \ell_s)\}$$

where $\mathcal{G}^j = \mathcal{G}_j \oplus \cdots \oplus \mathcal{G}_s$. Note that $(\Gamma^{s+1} = \Gamma)$ and that $\Gamma^j(\ell_j, \ldots, \ell_s)$ is just the orbit of $\ell \in \Gamma$ if $\ell \in \Gamma$ and $\emptyset$ if $\ell \notin \Gamma$.

Lemma 6.4. *Let us assume* (H2), (H3) *and the following property: For all $(\ell_j, \ldots, \ell_s) \in \mathcal{G}^*_j \times \cdots \times \mathcal{G}^*_s$, $\exists C(\ell_j, \ldots, \ell_s)$ s.t. $\forall \tilde{\ell} \in \Gamma^j(\ell_j, \ldots, \ell_s)$,*

$$\|f\|^2_{H^m_{\pi_{\tilde{\ell}}}} \leq C(\ell_j, \ldots, \ell_s)\|\Pi_{\tilde{\ell}}(\mathcal{P})f\|^2_{H^0_{\pi_{\tilde{\ell}}}} \qquad \forall f \in \mathcal{S}_{\Pi_{\tilde{\ell}}}, \tag{Q_j}$$

then we have Property $(Q_{(j+1)})$.

Note now that $(Q_1) = (H1)$ and that $(Q_{(s+1)})$ is the conclusion of the theorem. According to the remarks before the lemma, the proof of Lemma 6.4 is almost identical to the proof of Lemma 4.10 in [19] by observing that the assumptions of Theorem 4.9 in [19] are satisfied ($|\ell_s| = 1$ is no more true but (H3) replaces this assumption). This ends the proof of Theorem 6.3.

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