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The Relativistic Boltzmann Equation Near Equilibrium

Robert T. Glassey and Walter A. Strauss

§1. A remark on spectral theory

The Boltzmann equation, linearized around the equilibrium, has the form

$$\frac{\partial f}{\partial t} + A f + K f = 0.$$

We want to deduce the exponential decay of $f(t)$ as $t \to \infty$. The operator $A + K$ is neither symmetric nor skew-symmetric. Nor is K compact. However, it enjoys the following properties:

(i) Spec $(A + K) \subset \{\text{Re } \lambda \geq 0\}$

(ii) $A + K$ has no point spectrum on Re $\lambda = 0$.

(iii) Spec $(A) \subset \{\text{Re } \lambda \geq \alpha_0\}$ for some $\alpha_0 > 0$.

(iv) K is A -smoothing.

Property (iv) means, roughly, that the operator

$$e^{-t_1 A} K e^{-t_2 A} K \dots e^{-t_\ell A} K$$

is compact for all $t_1 > 0, \dots, t_\ell > 0$.

Theorem [Vidav, Shizuta]. *The spectrum of $A + K$ in the strip $\{0 \leq \text{Re } \lambda < \alpha_0\}$ is discrete, and*

$$\| e^{-t(A+K)} \| \leq e^{-\alpha_1 t}$$

for some $\alpha_1 > 0$.

This is a generalization of Weyl's classical theorem on the perturbation of spectra. We will see at the end of the lecture how this theorem proves the stability of the equilibrium of the relativistic Boltzmann equation.

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§2. The relativistic Boltzmann equation

Consider a gas with particle density $F(t, x, v)$ where $t =$ time, $x =$ position, and $v =$ momentum. The particles interact only through collision. Thus

$$v_0 \, \partial_t F + v \cdot \nabla_x F = \text{scattering term}.$$

If the particles are treated relativistically, then the momentum v is any vector in $\mathbf{R}^3$ and the velocity $\hat{v}$ satisfies $|\hat{v}| < c$. They are related by

$$v_0^2 - |v|^2 = m^2 c^2, \quad \hat{v} = c \frac{v}{v_0}.$$

The mass of a particle is m and the energy is cv_0. Henceforth we set $c = m = 1$ and rewrite the equation as

$$\text{(RB)} \qquad \partial_t F + \hat{v} \cdot \nabla_x F = Q(F)$$

with the scattering term

$$Q(F)(v) = \int_{\mathbf{R}^3} \int_{S^2} V_M \sigma [F(u')F(v') - F(u)F(v)] \, d\Omega \, du.$$

Here u and v are interpreted as the momenta of a pair of incoming particles, and u' and v' as the scattered ones. Thus the term $F(u')F(v')$ represents the gain and $F(u)F(v)$ the loss. Conservation of momentum and energy is expressed by

$$u + v = u' + v', \quad u_0 + v_0 = u_0' + v_0',$$

where $v_0 = \sqrt{1 + |v|^2}$, $u_0' = \sqrt{1 + |u_0'|^2}$, etc. (This is in contrast to the classical non-relativistic case where $v_0 = \text{const} \cdot |v|^2$). The scattering kernel is the product of two quantities. The Møller velocity V_M is given by

$$V_M^2 = |\hat{v} - \hat{u}|^2 - |\hat{v} \times \hat{u}|^2.$$

The scattering cross-section $\sigma = \sigma(g, \Theta)$ is a function of the generalized momentum difference g and the generalized scattering angle Θ. Notice that, for a given incoming momentum v, the three vectors u, u' and v' are constrained by the four scalar conservation laws given above. The integration in the scattering term runs over the five remaining variables.

A solution of (RB) has the conserved quantities

$$\int \int F \, dv \, dx, \int \int v \, F \, dv \, dx, \int \int v_0 \, F \, dv \, dx,$$

the mass, momentum and energy, respectively. Furthermore, the entropy increases:

$$\frac{d}{dt} \int \int F \log F \, dv \, dx \leq 0.$$

(The last integral is the negative entropy.) The equilibrium of greatest entropy comes from minimizing the negative entropy subject to fixed mass, momentum and energy. It is

$$\mu(v) = e^{a+b\cdot v - c\sqrt{1+|v|^2}},$$

the maxwellian distribution. Our goal is to prove the asymptotic stability of $\mu(v)$.

In the classical case, $\mu(v)$ is a gaussian. After the introduction of the equation by Boltzmann in 1872, it was not until Carleman in 1933 that the stability was proved for the case of space-independent solutions. Grad in a series of papers around 1963 proved the stability for a finite time for general solutions. Finally in 1974 Ukai proved the asymptotic stability, and hence the global existence of solutions near equilibrium, in the case of spatial periodicity. Then Nishida and Imai and Ukai in 1976 solved the problem without a periodicity assumption. Many others have made substantial contributions to the classical theory in the last 15 years. Here we announce the resolution of the relativistic problem with spatial periodicity.

Main Theorem. *Assume that the scattering cross-section σ satisfies $k_1 g(1+g)^{-1} \leq \sigma(g,\Theta) \leq k_2$ for some constants $k_1, k_2 > 0$. Let the initial distribution F^0 satisfy*

(i) $F^0(x,v) \geq 0$

(ii) F^0 *is continuous*

(iii) F^0 *is periodic in* x

(iv) $\int\int (a + b\cdot v - c\sqrt{1+|v|^2})\,[F^0(x,v) - \mu(v)]\,dv\,dx = 0$ *for all* a, b, c.

(v) $\left|F^0(x,v) - \mu(v)\right| \leq \varepsilon\sqrt{\mu(v)}\,(1+|v|)^{-\gamma-3/2}$

for some $\gamma > 0$ and for sufficiently small ε. Then there exists a global, continuous, x-periodic solution of (RB) *with $F(0,x,v) = F^0(x,v)$, and there exist $\delta > 0$ and $c_1 > 0$ such that*

$$|F(t,x,v) - \mu(v)| \leq c_1 \varepsilon e^{-\delta t}\sqrt{\mu(v)}$$

for $0 \leq t < \infty$.

This theorem is also true with C^k and H^k norms for arbitrarily large k. Hence there exist arbitrarily smooth solutions. It is also true under more general conditions on σ.

§3. Sketch of the proof of stability

We may normalize $\mu(v) = \exp(-\sqrt{1+|v|^2})$. Next we write the perturbation as $F-\mu=\sqrt{\mu}f$, so that f satisfies

$$\partial_t f + A f + K f = \tilde{Q}(f),$$

where

$A = \hat{v}\cdot\nabla_x + \alpha(v)$,

$K =$ a linear integral operator in v,

$\tilde{Q} =$ a quadratic term.

We wish to solve this equation globally with small initial data. To do this, we choose a space Y on which $\tilde{Q}$ is bounded:

$$\| \tilde{Q} f \|_Y \le c \| F \|_Y^2$$

and a similar Lipschitz property for $\tilde{Q}f - \tilde{Q}g$, together with decay of the linearized problem:

$$\int_0^\infty \| e^{-t(A+K)} \|_{\mathcal{L}(Y,Y)} dt < \infty.$$

It is a standard fact that these two properties imply the asymptotic stability.

To prove the Main Theorem, we choose the space Y of continuous functions $f(x,v)$, periodic in x, which satisfy

$$\int\int (a + b\cdot v + c\sqrt{1+|v|^2})\,\sqrt{\mu}\, f\, dv\, dx = 0$$

for all a, b, c, such that the norm

$$\| f \|_Y = \sup_{x,v} (1+|v|)^{\gamma+3/2}\, |f(x,v)|$$

is finite. We omit the proof of boundedness of $\tilde{Q}$ on this space in order to concentrate on the linearized problem.

The linearized entropy identity is

$$\langle Af + Kf, f\rangle$$

$$= \int\int\int\int V_M \sigma \mu(u)\mu(v) \left[\frac{f(v')}{\sqrt{\mu(v')}} + \frac{f(u')}{\sqrt{\mu(u')}} - \frac{f(\mu)}{\sqrt{\mu(u)}} - \frac{f(v)}{\sqrt{\mu(v)}} \right]^2$$

$$\times\, du\, d\Omega\, dv\, dx.$$

This expression is manifestly non-negative, and in fact is positive for all $f \neq 0$ in Y because of the orthogonality conditions. Thus properties (i) and (ii) from the beginning of this lecture are satisfied. Furthermore, $A = \hat{v} \cdot \nabla_x + \alpha(v)$ where

$$\alpha(v) = \frac{1}{2} \int\int V_M \sigma(g, \Theta) \mu(u) du \, d\Omega$$

is bounded above and below: $0 < \alpha_0 \leq \alpha(v) \leq \alpha_2 < \infty$. Hence $\langle Af, f \rangle \geq \alpha_0 \| f \|_{L^2}^2$.

In the classical case $\alpha(v)$ is like a constant times $|v|$, which means that the dissipation is large for large $|v|$. In the mid-1970's Shizuta showed how the concept of an A-smoothing operator can be applied to the classical Boltzmann equation. In fact, Grad showed in the 1960's that

$$Kf(t, x, v) = \int k(u, v) f(t, x, u) du$$

where $k(u, v) \leq c_1 |u - v|^{-1} \exp(-c_2 |u - v|^2)$, in the case of the hard sphere.

In the relativistic case the exponent is much weaker. Nevertheless we can improve the denominator to obtain

$$|k(u, v)| \leq c_1 \frac{e^{-c_2 |u - v|}}{|u - v| + |u \times v|}.$$

This estimate implies that

$$\sup_v \int \left(|k| + |k|^2 \right) du < \infty$$

and, for all $\beta \geq 0$,

$$\int (1 + |u|)^\beta \, |k| du \leq c \, (1 + |v|)^{-\beta - 1}.$$

Following Shizuta, we approximate the kernel as a sum

$$k(u, v) \sim \sum p_j(u) q_j(v)$$

with nice functions p_j and q_j. Therefore the A-smoothing property of K would follow from the compactness of the operator

$$e^{-t_1 A} QP e^{-t_2 A} QP e^{-t_3 A} QP \cdots e^{-t_\ell A} QP$$

where Q is multiplication by $q_j(v)$ and P is integration with $p_j(u)$. In this string of operators it suffices to prove the boundedness of the various factors and the compactness of one of the factors. In fact, one string of three factors is

$$\begin{aligned} Pe^{-tA}Qf(x) &= \int e^{-tA}[q(v)f(x)]p(v)dv \\ &= \int e^{-t\alpha(v)}q(v)f(x-t\hat{v})p(v)dv. \end{aligned}$$

We apply ∂_x to both sides of this identity. Inside the integral, ∂_x is converted to $t^{-1}\partial_{\hat{v}}$. A change of variables from v to $\hat{v}$ thus leads to the identity

$$\partial_x[Pe^{-tA}Qf] = \frac{1}{t}\int \frac{\partial}{\partial\hat{v}}(\text{a kernel}) \cdot f \cdot dv.$$

Thus we gain regularity in x and therefore $Pe^{-tA}Q$ is compact. For details, see [3].

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R.T. Glassey
Department of Mathematics
Indiana University
Bloomington, IN 47405-4301
U.S.A.

W.A. Strauss
Department of Mathematics
Brown University
Providence, RI 02912
U.S.A.