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Tits Metric and Visibility Axiom

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§1. Introduction

An Hadamard manifold H or H^n, i.e., a complete connected simply-connected n-dimensional Riemannian manifold with non-positive sectional curvature is called *a visibility manifold* if the angles at a fixed point subtended by geodesics going far away are arbitrarily small enough no matter how long they are. This condition given by P. Eberlein and B. O'Neill [3] plays basic roles in the study of Hadamard manifolds. They also defined the concept of points at infinity, $H(\infty)$, and it is known that H is a visibility manifold if and only if any different two points at infinity $x_1, x_2 \in H(\infty)$ can be joined by a geodesic of H. This property is called *the axiom* 1. The next two theorems determining this condition are classical:

Theorem 1 ([2],[3]). *If the sectional curvature of H is bounded above by a negative constant, then H is a visibility manifold.*

Theorem 2 ([1]). *In the case of H^n being a surface H^2, it is a visibility surface if and only if for every sector S of H^2, the total curvature of S,*

$$\iint_S K\,dv = -\infty$$

holds, where K is the Gaussian curvature and a sector S is a piece of surface which is cut off by two different rays starting a common point.

Theorem 1 is proved in [2] Lemma 9-10, and also in [3] Proposition 5-9 with an extended form using the idea of curvature order. These proofs in any cases depend essentially on the so-called Gauss-Bonnet theorem on surfaces. Similarly, using the Gauss-Bonnet theorem, we can prove easily Theorem 2 (cf. [1] page 57). Paying attention to the polar coordinate expression around a point in Theorem 2, K. Uesu [5]

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succeeded in generalizing Theorem 2 to higher dimensional case which is stated in terms of the growth rate of the length of Jacobi vector field, and proved directly the relation with the visibility axiom by estimating the angular length:

Theorem 3 ([5]). *H^n is a visibility manifold if and only if there exists a point p of H^n such that for every Lipschitz curve $c: [0.1] \to S(p)$ with non-zero length,*

$$\lim_{r\to\infty} \int_0^1 \|Y_t\|'(r)\, dt = \infty$$

holds, where $S(p)$ is the unit tangent sphere at p with natural metric and Y_t is the Jacobi vector field along the ray $[0,\infty) \ni r \mapsto \exp_p rc(t) \in H^n$ such that $Y_t(0) = 0$, $Y_t'(0)$ (= the covariant derivative of Y_t at $r = 0$) $= \dot{c}(t)$ with natural identification.

The next special case is useful.

Theorem 4. *Assume that there exists a point p of H^n such that*

$$\lim_{r\to\infty} \|Y_{v,w}\|'(r) = \infty$$

holds for any orthonormal vectors $\{v, w\}$ of H_p, where $Y_{v,w}$ is the Jacobi vector field along the ray $[0,\infty) \ni r \mapsto \exp_p rv \in H^n$ such that $Y_{v,w}(0) = 0$, $Y'_{v,w}(0) = w$. Then H^n is a visibility manifold.

On the other hand, in [1] lecture 1, W. Ballmann, M. Gromov and V. Schroeder investigated under the transparent idea the fundamental properties of Hadamard manifolds and derived the importance of notion of Tits metric $Td(x_1, x_2)$ in $H(\infty)$. We note in particular that the following view points of their arguments are essential. (1) They discussed elementarily, based only on the convexity of distance function which means $K \leq 0$, the law of cosine in the constant-negatively curved manifold and on the Rauch and Toponogov comparison theorems. (2) As a consequence, they showed that

$$Td(x_1,\, x_2) = \lim_{r\to\infty} \frac{1}{r}\, d_r(\,\gamma_{x_1}(r),\, \gamma_{x_2}(r)\,)$$

holds for any x_1, $x_2 \in H(\infty)$, where d_r is the interior metric of the distance sphere $S_r(p)$ of radius r at p and two γ_{x_i} are the rays directed towards x_i with a common starting point p, and that H is a visibility manifold if and only if $Td(x_1, x_2) = \infty$ for all distinct x_1, $x_2 \in H(\infty)$

as well as other equivalent properties. At a glance we see that Theorem 3 is similar to (2) and that Theorem 2 is proved without Gauss-Bonnet theorem as a special case of Theorem 3. As a matter of fact the converse of Theorem 4 is not generally true (cf. example 5-10 of [3]). Theorem 4 is proved directly and more simply than Theorem 3 is, and we may say that the proof of Theorem 4 gives the essential part of one of Theorem 3. Moreover, if H satisfies a sort of "*symmetry*" with respect to directions, (such a manifold is studied in detail and called *model* in [4],) the visibility axiom is determined completely by Theorem 4, namely, we have the following:

Theorem 5. *Let H be an Hadamard manifold with the following condition: there exist a point $p \in H$ and a continuous function $k : [0, \infty) \to [0, \infty)$ such that for every ray $\gamma : [0, \infty) \to H$ starting at $p = \gamma(0)$, $t \geq 0$ and for every section σ containing $\dot\gamma(t)$, the sectional curvature $K_\sigma = -k(t)$ holds. Then H is a visibility manifold if and only if*

$$\lim_{r \to \infty} \|Y_{v,w}\|'(r) = \infty$$

holds for any orthonormal vectors $\{v, w\}$ of H_p.

In this paper we prove these theorems systematically from the point of view of Tits metric, i.e., of the above (1) and (2) without employing Gauss-Bonnet theorem.

§2. Notations and preliminaries

In the following, let H be a complete simply-connected n-dimensional Riemannian manifold with sectional curvature $K \leq 0$ which is called *an Hadamard manifold.* H is diffeomorphic to $\mathbf{R}^n$ and any geodesics of H are minimal. We assume geodesics are always parametrized by arc-lengths if not stated otherwise. A geodesic $\gamma : [0, \infty) \to H$ ($\mathbf{R} \to H$) is called *a ray* (*line* , respectively) and two rays γ_1, γ_2 are said to be *asymptotic* if $\lim_{r\to\infty} d(\gamma_1(r), \gamma_2(r)) < \infty$, or equivalently, if the function $r \mapsto d(\gamma_1(r), \gamma_2(r))$ is monotone non-increasing on $[0, \infty)$, where $d(p_1, p_2)$ is the distance between p_1 and p_2 of H. This is an equivalent relation and the equivalent class of γ is called *a point at infinity* and denoted by $\gamma(\infty)$. The set of all $\gamma(\infty)$ of rays γ is called *the ideal boundary of H* and denoted by $H(\infty)$. For every $p \in H$ and $q \in H$ ($H(\infty)$) there exists a unique geodesic (ray, resp.) γ_{pq} from p to q. For any q_1, $q_2 \in H \cup H(\infty)$ different from $p \in H$, the angle $\angle(\dot\gamma_{pq_1}(0), \dot\gamma_{pq_2}(0))$ is called *the angle subtended by q_1, q_2 at p* and denoted by $\angle_p(q_1, q_2)$.

An Hadamard manifold H is said to satisfy *the visibility axiom* or simply to be *a visibility manifold* if for a point $p \in H$ and any $\varepsilon > 0$ there exists $r = r(p, \varepsilon) > 0$ such that for every geodesic $\gamma : [a, b] \to H$ satisfying $d(p, \gamma) \geq r$, $\angle_p(\gamma(a), \gamma(b)) \leq \varepsilon$ holds. It must be conscious that the choice of $p \in H$ in this definition may be arbitrarily fixed and moreover this property is equivalent to *the axiom* 1 , that is to say, for any distinct $x_1, x_2 \in H(\infty)$ there exists a line γ in H such that $\gamma(-\infty) := \gamma_-(\infty) = x_1$ and $\gamma(\infty) = x_2$, γ_- being the line with $\gamma_-(r) := \gamma(-r)$. For a point $p \in H$ and $r > 0$, the distance sphere centered at p with radius r, $S_r(p) := \{ q \in H \mid d(p, q) = r \}$ is a compact hypersurface of H. Let d_r be the distance function of $S_r(p)$ naturally induced by the metric. For each $x_1, x_2 \in H(\infty)$ the function $(0, \infty) \ni r \mapsto \frac{1}{r} d_r(\gamma_{px_1}(r), \gamma_{px_2}(r))$ is monotone non-decreasing and we call $Td(x_1, x_2) := \lim_{r \to \infty} \frac{1}{r} d_r(\gamma_{px_1}(r), \gamma_{px_2}(r)) \in \mathbf{R} \cup \{\infty\}$ *the Tits distance.* It must be conscious too that this definition does not depend on the choosed point p and that H is a visibility manifold if and only if $Td(x_1, x_2) = \infty$ holds for every distinct $x_1, x_2 \in H(\infty)$ or equivalently $Td(x_1, x_2) \geq a$, $a > 0$ being a constant. (cf. [1], [3])

For every $v \in S(p) := \{ v \in H_p \mid \|v\| = 1 \}$, $\gamma_v(r) := \exp_p rv$ is the ray of initial vector $\dot\gamma_v(0) = v$ where $r \geq 0$. For any $w \in H_p$ orthogonal to v, let $Y_{v,w}$ be the Jacobi vector field along γ_v such that $Y_{v,w}(0) = 0$, $Y'_{v,w}(0) = w$ which is expressed by

$$Y_{v,w}(r) = \exp_{p*} r\mathbf{I}_{rv} w$$

for any $r \geq 0$ where for each $u \in H_p$, $\mathbf{I}_u : H_p \to (H_p)_u$ is the natural isomorphism defined by $\mathbf{I}_u w := \dot c_{u,w}(0)$, $c_{u,w}(t) := u + tw$ for any $t \in \mathbf{R}$ and $w \in H_p$. According to $K \leq 0$, $\|Y_{v,w}\|'' \geq 0$ holds, namely, the function $\|Y_{v,w}\|$ is convex and $\|Y_{v,w}\|'$ is monotone non-decreasing on $[0, \infty)$. In particular, if H^n is of constant curvature $K = -c^2$, $c > 0$, $Y_{v,w}$ is expressed by $Y_{v,w}(r) = \frac{1}{c} \sinh(cr) \cdot X(r)$ where X is the parallel vector field along γ_v with $X(0) = w$.

For any $p \in H$ we set $G_p := \{ \sigma \mid \sigma$ is a 2-dimensional vector subspace of $H_p \}$. The well-known Rauch comparison theorem means the following: Let v, $w \in H^n{}_p$ be orthonormal and we take another triple $\{\tilde v, \tilde w, \tilde H^{\tilde n}\}$. Denoting the corresponding terms by $\tilde{\ }$, we assume $n \leq \tilde n$ and $K_\sigma \leq K_{\tilde\sigma}$ for any $r \geq 0$, $\dot\gamma_v(r) \in \sigma \in G_{\gamma_v(r)}$ and $\dot{\tilde\gamma}_{\tilde v}(r) \in \tilde\sigma \in G_{\tilde\gamma_{\tilde v}(r)}$. Then it follows that

$$\|Y_{v,w}\| \geq \|\tilde Y_{\tilde v, \tilde w}\|, \quad \frac{\|Y_{v,w}\|'}{\|Y_{v,w}\|} \geq \frac{\|\tilde Y_{\tilde v,\tilde w}\|'}{\|\tilde Y_{\tilde v,\tilde w}\|}$$

on $(0,\infty)$ and

$$\frac{\|Y_{v,w}\|(r_1)}{\|Y_{v,w}\|(r_2)} \le \frac{\|\tilde{Y}_{\tilde{v},\tilde{w}}\|(r_1)}{\|\tilde{Y}_{\tilde{v},\tilde{w}}\|(r_2)}$$

for all $r_2 > r_1 \geq 0$. (cf. [7])

Given three distinct points $p_i \in H$ and geodesics $\gamma_i : [0,\, l_i] \to H\,(i = 0,1,2)$ such that $\gamma_i(l_i) = \gamma_{i+1}(0) = p_{i+2}$ (mod 3), the triple (p_0, p_1, p_2) or $(\gamma_0,\, \gamma_1,\, \gamma_2)$ is said to form *a geodesic triangle.* For each $i = 0,1,2$, $\theta_i := \pi - \angle(\dot{\gamma}_{i+1}(l_{i+1}), \dot{\gamma}_{i+2}(0))$ (mod 3) is called *the angle at* p_i. In the Hadamard manifold $H^n(-c^2)$ of constant negative curvature $K = -c^2$, $c > 0$, the law of cosine,

$$\cosh(cl_0) = \cosh(cl_1)\cdot\cosh(cl_2) - \sinh(cl_1)\cdot\sinh(cl_2)\cdot\cos\theta_0$$

holds for every geodesic triangle $(\gamma_0, \gamma_1, \gamma_2)$, and if $l_1 = l_2$, then we get

$$\sinh\frac{cl_0}{2} = \sinh(cl_1)\cdot\sin\frac{\theta_0}{2}.$$

§3. Proofs of Theorems 4, 1 and 5

Proof of Theorem 4. We show that

$$Td(x_1, x_2) = \lim_{r\to\infty}\frac{1}{r}d_r(\gamma_{px_1}(r), \gamma_{px_2}(r)) = \infty$$

holds for any $x_1,\, x_2 \in H(\infty)$, $x_1 \neq x_2$. Let $A := \{\,(v,w) \in H_p \times H_p \mid \|v\| = \|w\| = 1,\, \langle v,w\rangle = 0\,\}$. By the assumption, for any $(v,w) \in A$ and $M > 0$ there exists a $r(v,w) > 0$ such that $r \geq r(v,w)$ implies $\|Y_{v,w}\|'(r) > M$. Since $\|Y_{v,w}\|'(r)$ is continuous as to $(v,w) \in A$ and monotone non-decreasing relative to $r \in [0,\infty)$, there exists a neighbourhood $U = U(v,w)$ of (v,w) in A such that for any $(v',w') \in U$ and $r \geq r(v,w)$,

$$\|Y_{v',w'}\|'(r) \geq \|Y_{v',w'}\|'(\,r(v,w)\,) > M$$

holds. There exists a finite covering $\bigcup_{i=1}^{k} U(v_i,w_i) \supset A$ because A is compact. So we take $r_0 := \max\{r(v_i,w_i) \mid i = 1,\ldots,k\} > 0$. Hence for any $(v,w) \in A$ and $r \geq r_0$, we have

$$\|Y_{v,w}\|(r) \geq \|Y_{v,w}\|(r) - \|Y_{v,w}\|(r_0) \geq (r-r_0)\|Y_{v,w}\|'(r_0) \geq (r-r_0)M.$$

In every distance sphere $S_r(p)$ we take a minimal geodesic $c_r\colon [0,1] \to S_r(p)$ from $\gamma_{px_1}(r)$ to $\gamma_{px_2}(r)$ which is expressed by $c_r(t) = \exp_p r\tilde{c}_r(t)$

for all $t \in [0,1]$ where $\tilde{c}_r : [0,1] \to S(p) \subset H_p$ is a differentiable curve. Accordingly we have

$$d_r(\gamma_{px_1}(r), \gamma_{px_2}(r)) = L(c_r) = \int_0^1 \|\dot{c}_r\|(t)\,dt$$

and

$$\begin{aligned}\dot{c}_r(t) &= \exp_{p*}(r\tilde{c}_r)'(t)\\ &= \exp_{p*} r\mathbf{I}_{r\tilde{c}_r(t)} \cdot \mathbf{I}_{\tilde{c}_r(t)}{}^{-1} \cdot \dot{\tilde{c}}_r(t)\\ &= Y_{\tilde{c}_r(t), w_r(t)}(r)\\ &= \|\dot{\tilde{c}}_r\|(t) \cdot Y_{\tilde{c}_r(t), w_r(t)/\|w_r(t)\|}(r)\end{aligned}$$

where $w_r(t) := \mathbf{I}_{\tilde{c}_r(t)}{}^{-1} \cdot \dot{\tilde{c}}_r(t) \in H_p$. Therefore, for any $r > 2r_0$ we have

$$\begin{aligned}\frac{1}{r} d_r(\gamma_{px_1}(r), \gamma_{px_2}(r)) &\geq \frac{1}{r}\int_0^1 \|\dot{\tilde{c}}_r\|(t) \cdot (r - r_0) M\,dt\\ &\geq M(1 - \frac{r_0}{r}) L(\tilde{c}_r)\\ &\geq \frac{1}{2} M \angle_p(x_1, x_2)\end{aligned}$$

and get $Td(x_1, x_2) = \infty$.

Proof of Theorem 1. We prove Theorem 1 using Theorem 4. We assume $K \leq -c^2$ for a positive constant c and have only to show $\lim_{r\to\infty} \|Y_{v.w}\|'(r) = \infty$ for an arbitrary orthonormal pair $\{v, w\}$ of TH. We take a geodesic $\tilde{\gamma}$ and a Jacobi vector field $\tilde{Y}$ along $\tilde{\gamma}$ in the Hadamard manifold $H^n(-c^2)$ of constant negative curvature $-c^2$ such that $\langle \tilde{Y}, \dot{\tilde{\gamma}} \rangle = 0$, $\tilde{Y}(0) = 0$ and $\|\tilde{Y}'(0)\| = 1$. Then we have the expression $\tilde{Y}(r) = \frac{1}{c}\sinh(cr) \cdot \tilde{X}(r)$ where $\tilde{X}$ is the parallel vector field along $\tilde{\gamma}$ with $\tilde{X}(0) = \tilde{Y}'(0)$. Hence, applying Rauch comparison theorem, we get

$$\|Y_{v,w}\|'(r) \geq \|\tilde{Y}\|'(r)\frac{\|Y_{v,w}\|(r)}{\|\tilde{Y}\|(r)} \geq \|\tilde{Y}\|'(r) = \cosh(cr)$$

for any $r > 0$, and consequently

$$\lim_{r \to \infty} \|Y_{v,w}\|'(r) = \infty.$$

Remark (1). It is also possible to prove Theorem 1 directly by the law of cosine without using Theorem 4 such as following: We take a fixed point $p \in H$ and any different $x_1, x_2 \in H(\infty)$. Using Rauch-Alexandrov comparison theorem (cf. [6]) and the law of cosine, we have

$$d(\gamma_{px_1}(r), \gamma_{px_2}(r)) \geq \frac{2}{c} \sinh^{-1}\left(\sin\frac{1}{2}\angle_p(x_1, x_2) \cdot \sinh(cr)\right)$$

hence

$$\begin{aligned} Td(x_1, x_2) &= \lim_{r \to \infty} \frac{1}{r} d_r(\gamma_{px_1}(r), \gamma_{px_2}(r)) \\ &\geq \lim_{r \to \infty} \frac{1}{r} d(\gamma_{px_1}(r), \gamma_{px_2}(r)) \\ &\geq \lim_{r \to \infty} \frac{2}{cr} \sinh^{-1}\left(\sin\frac{1}{2}\angle_p(x_1, x_2) \cdot \sinh(cr)\right) \\ &= \lim_{r \to \infty} \frac{2}{c} \frac{\sin\frac{1}{2}\angle_p(x_1, x_2) \cdot c \cdot \cosh(cr)}{\left(1 + \left(\sin\frac{1}{2}\angle_p(x_1, x_2) \cdot \sinh(cr)\right)^2\right)^{\frac{1}{2}}} \\ &= 2. \end{aligned}$$

This implies that H is a visibility manifold.

Remark (2). In general, an Hadamard manifold with smaller curvature than one of a visibility manifold satisfies the visibility axiom too. That is to say more precisely, the following assertion is obvious by Rauch comparison theorem.

Let H^n, $\tilde{H}^{\tilde{n}}$ be two Hadamard manifolds, $p \in H^n$, $\tilde{p} \in \tilde{H}^{\tilde{n}}$, $n \leq \tilde{n}$ and $\iota : {H^n}_p \to {\tilde{H}^{\tilde{n}}}_{\tilde{p}}$ an isometric isomorphism. We assume that $K_\sigma \leq K_{\tilde{\sigma}}$ holds for every ray $\gamma : [0, \infty) \to H$ starting at p, $r \geq 0$ and every σ, $\tilde{\sigma}$ such that $\dot{\gamma}(r) \in \sigma \in G_{\gamma(r)}$, $\dot{\tilde{\gamma}}(r) \in \tilde{\sigma} \in G_{\tilde{\gamma}(r)}$, where $\tilde{\gamma} : [0, \infty) \to \tilde{H}^{\tilde{n}}$ is the ray starting at $\tilde{p}$ with $\dot{\tilde{\gamma}}(0) = \iota\dot{\gamma}(0)$. Then if $\tilde{H}$ is a visibility manifold, H is so too.

Proof of Theorem 5. We prove the converse of Theorem 4, that is, $\lim_{r\to\infty}\|Y_{v,w}\|'(r) = \infty$ for any two orthonormal vectors $\{v,w\}$ of H_p under the visibility condition. We take the curve $c\colon [0,\pi/2] \to S(p)$ and the variation $V : [0,\infty)\times[0,\pi/2] \to H$ by $c(t) := v\cos t + w\sin t$, $V(r,t) := \exp_p rc(t)$ for every $r\in[0,\infty)$ and $t\in[0,\pi/2]$. Then $V_*\frac{\partial}{\partial t}|_{(r,t)} = Y_{c(t),w(t)}(r)$ holds where $w(t) := -v\sin t + w\cos t = \mathbf{I}_{c(t)}{}^{-1}\dot c(t)$. Since clearly $Y_{v,w}(0) = 0 = Y_{c(t),w(t)}(0)$ and $\|Y'_{v,w}\|(0) = 1 = \|Y'_{c(t),w(t)}\|(0)$ are satisfied, we can apply Rauch comparison theorem to $Y_{v,w} = Y_{c(0),w(0)}$ and $Y_{c(t),w(t)}$ owing to the assumption on curvatures and get $\|Y_{v,w}\|(r) = \|Y_{c(t),w(t)}\|(r)$ for every $r\ge 0$ and $t\in[0,\pi/2]$. So we have

$$\|Y_{v,w}\|'(r) \ge \frac{1}{r}\|Y_{v,w}\|(r) = \frac{2}{\pi r}\int_0^{\frac{\pi}{2}} \|Y_{c(t),w(t)}\|(r)\,dt = \frac{2}{\pi r}L(\,V(r,\cdot)\,)$$

$$\ge \frac{2}{\pi r}d_r(\,V(r,0),\,V(r,\frac{\pi}{2})\,) = \frac{2}{\pi r}d_r(\gamma_v(r),\gamma_w(r)),$$

and $\lim_{r\to\infty}\|Y_{v,w}\|'(r) = \infty$ because $Td(\gamma_v(\infty),\gamma_w(\infty)) = \infty$.

§4. Proofs of Theorems 3 and 2

Proof of Theorem 3. We assume H is a visibility manifold and $p\in H$. For any Lipschitz curve $c\colon[0,1]\to S(p)$ with $L(c)\neq 0$ we take $0\le t_1<t_2\le 1$ such that $c(t_1)\neq c(t_2)$. Since $\|Y_t\|'(r)\ge \frac{1}{r}\|Y_t\|(r)$ holds for every $r>0$ and for almost all $t\in[0,1]$ because of $K\le 0$, we have

$$\int_0^1 \|Y_t\|'(r)\,dt \ge \int_{t_1}^{t_2}\frac{1}{r}\|Y_t\|(r)\,dt$$

$$= \frac{1}{r}L(\exp_p rc(\cdot)|_{[t_1,t_2]}\,)$$

$$\ge \frac{1}{r}d_r(\gamma_{c(t_1)}(r),\gamma_{c(t_2)}(r))$$

and

$$\lim_{r\to\infty}\int_0^1 \|Y_t\|'(r)\,dt \ge Td(\gamma_{c(t_1)}(\infty),\gamma_{c(t_2)}(\infty))$$

$$=\infty.$$

Conversely, assume there exist two different $x_1, x_2 \in H(\infty)$ such that $Td(x_1, x_2) = \lim_{r\to\infty} \frac{1}{r} d_r(\gamma_{px_1}(r), \gamma_{px_2}(r)) < \infty$. We take a divergent monotone-increasing sequence (r_k) and a family of minimal curves $c_k : [0,1] \to S_{r_k}(p)$ from $\gamma_{px_1}(r_k)$ to $\gamma_{px_2}(r_k)$ in $S_{r_k}(p)$, where we parametrize so that each $\tilde{c}_k := \frac{1}{r_k} {\exp_p}^{-1} c_k : [0,1] \to S(p)$ is proportional to arc-length. Then we have

$$\dot{c}_k(t) = \exp_{p*} r_k \mathbf{I}_{r_k \tilde{c}_k(t)} \cdot {\mathbf{I}_{\tilde{c}_k(t)}}^{-1} \cdot \dot{\tilde{c}}_k(t) = Y_{\tilde{c}_k(t), w_k(t)}(r_k)$$

for any $t \in [0,1]$ where $w_k(t) := {\mathbf{I}_{\tilde{c}_k(t)}}^{-1} \cdot \dot{\tilde{c}}_k(t)$, and so

$$\|\dot{c}_k(t)\| \geq r_k \|w_k(t)\| = r_k \|\dot{\tilde{c}}_k(t)\|$$

by Rauch theorem, hence

$$\begin{aligned} L(\tilde{c}_k) &= \int_0^1 \|\dot{\tilde{c}}_k\|(t)\, dt \leq \frac{1}{r_k} L(c_k) = \frac{1}{r_k} d_{r_k}(\gamma_{px_1}(r_k), \gamma_{px_2}(r_k)) \\ &\leq Td(x_1, x_2) < \infty. \end{aligned}$$

Applying the Ascoli-Arzelà theorem we get a convergent subsequence of $(\tilde{c}_k)$ and also denote it by $(\tilde{c}_k)$ for simplicity. The limit curve $\tilde{c}_0 := \lim_{k\to\infty} \tilde{c}_k : [0,1] \to S(p)$ is a Lipschitz curve with $L(\tilde{c}_0) \neq 0$ as $x_1 \neq x_2$, whose convergence is uniform. Since $\lim_{r\to\infty} \int_0^1 \|Y_t\|'(r)\, dt = \infty$ holds by the assumption with $Y_t(r) := Y_{\tilde{c}_0(t), w(t)}(r)$ and $w(t) := {\mathbf{I}_{\tilde{c}_0(t)}}^{-1} \cdot \dot{\tilde{c}}_0(t)$, for a constant $M = 6Td(x_1, x_2) > 0$ there exists $r_0 > 0$ such that $r \geq r_0$ implies $\int_0^1 \|Y_t\|'(r)\, dt > M$.

Set $c_{k,r}(t) := \exp_p r\tilde{c}_k(t)$ for every $k \in \mathbf{N} \cup \{0\}$, $r \geq 0$ and $t \in [0,1]$, so it follows $c_k = c_{k,r_k}$ for every $k \in \mathbf{N}$. We choose $k_0 \in \mathbf{N}$ such that $r_{k_0} > 2r_0$. Since $L(c_{0,r}) \leq \liminf_{k\to\infty} L(c_{k,r})$ holds for each fixed $r \geq 0$, for $\varepsilon := \frac{1}{6} M r_{k_0} > 0$ there exists $k_1 = k_1(r_{k_0}) > k_0$ such that $k > k_1$ implies $L(c_{0,r_{k_0}}) - \varepsilon < L(c_{k,r_{k_0}})$. Using comparisom theorem, we have

$$\begin{aligned} L(c_{k,r_k}) &= \int_0^1 \|Y_{\tilde{c}_k(t), w_k(t)}\|(r_k)\, dt \\ &\geq \int_0^1 \|Y_{\tilde{c}_k(t), w_k(t)}\|(r_{k_0}) \frac{r_k}{r_{k_0}}\, dt \\ &= \frac{r_k}{r_{k_0}} L(c_{k,r_{k_0}}) \end{aligned}$$

for any $k > k_1$ and get finally

$$\begin{aligned}
Td(x_1, x_2) &\geq \frac{1}{r_k} d_{r_k}(\gamma_{px_1}(r_k), \gamma_{px_2}(r_k)) = \frac{1}{r_k} L(c_k) = \frac{1}{r_k} L(c_{k,r_k}) \\
&\geq \frac{1}{r_{k_0}} L(c_{k,r_{k_0}}) \\
&> \frac{1}{r_{k_0}} (L(c_{0,r_{k_0}}) - \varepsilon) = \frac{1}{r_{k_0}} \left(\int_0^1 \|Y_t\|(r_{k_0})\, dt - \varepsilon \right) \\
&\geq \frac{1}{r_{k_0}} \left(\int_0^1 (\, (r_{k_0} - r_0)\|Y_t\|'(r_0) + \|Y_t\|(r_0)\,)\, dt - \varepsilon \right) \\
&> \frac{1}{r_{k_0}} (\, (r_{k_0} - r_0)M - \varepsilon\,) \\
&> \frac{1}{2}M - \frac{1}{6}M = \frac{1}{3}M \\
&= 2Td(x_1, x_2) > 0
\end{aligned}$$

which contradicts.

Proof of Theorem 2. Let $H^2 = (\,\mathbf{R}^2,\ ds^2 = dr^2 + f(r,\theta)^2 d\theta^2\,)$ be a geodesic polar coordinate around p, that is, the differentiable function $f\colon [0,\infty) \times S^1 \to [0,\infty)$ be assumed to satisfy $f(0,\theta) = 0$, $f_r(0,\theta) = 1$ and $f_{rr}(r,\theta) \geq 0$. Then for every sector $S = \{\,(r,\theta) \mid r \geq 0,\ a \leq \theta \leq b\,\}$ $(\,0 \leq a < b \leq 2\pi\,)$ with vertex p, we have

$$\begin{aligned}
\iint_S K\, dv &= \lim_{r\to\infty} \iint_{S_r(p)\cap S} K\, dv \\
&= \lim_{r\to\infty} \int_a^b d\theta \int_0^r \frac{f_{rr}}{-f} \cdot f|_{(t,\theta)}\, dt \\
&= -\lim_{r\to\infty} \int_a^b f_r(r,\theta)\, d\theta + (b-a).
\end{aligned}$$

We denote by γ_θ the ray from p with a direction θ and by X_θ the parallel vector field along γ_θ with $\|X_\theta\| = 1$ and $\langle X_\theta,\, \dot\gamma_\theta\rangle = 0$, then we have $Y_{\dot\gamma_\theta(0),X_\theta(0)}(r) = f(r,\theta)\cdot X_\theta(r)$ for all $r \geq 0$ and $\theta \in S^1$, hence $f_r(r,\theta) = \|Y_{\dot\gamma_\theta(0),X_\theta(0)}\|'(r)$. This fact and Theorem 3 imply Theorem 2.

Remark. If H is a surface H^2 in Theorem 5, namely, if the function $f(r,\theta)$ with $f(0,\theta) = 0$, $f_r(0,\theta) = 1$ and $f_{rr}(r,\theta) \geq 0$ which is adopted in the above proof depends only on r, then H^2 is a visibility surface if and only if $\lim_{r\to\infty} f'(r) = \infty$, and $K < 0$ is equivalent to $f'' > 0$ on $(0, \infty)$ and $\lim_{r\downarrow 0} \dfrac{f''(r)}{f(r)} \in (0, \infty)$. Therefore we gain easily an example of Hadamard surface with $K < 0$ which does not satisfy the visibility axiom. For example, for given $c > 0$ and $\varepsilon_2 > \varepsilon_1 > 0$ we are able to construct a C^∞-function $f(r)$ so as to satisfy $f(r) = \frac{1}{c}\sinh(cr)$ on $[0, \varepsilon_1]$, $f(r) = M_1 + M_2 \int_{\varepsilon_2}^{r} \tan^{-1} t\, dt$ on $[\varepsilon_2, \infty)$ and $f'' > 0$ on $[\varepsilon_1, \varepsilon_2]$ by choosing M_1, $M_2 > 0$ so large enough.

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