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Geometry of Laplace-Beltrami Operator on a Complete Riemannian Manifold

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§0. Introduction

This is a survey paper on recent developments of analytic and geometric aspects of the Laplace-Beltrami operator on a complete Riemannian manifold. Systematic treatments from a Riemannian geometric viewpoint have been already appreared in Berger, Gauduchon & Mazet ['71], Kotake, Maeda, Ozawa & Urakawa ['81], Bérard & Berger ['83], Bérard ['86], Chavel ['84], Gilkey ['84] and Sunada ['88]. But they are mainly concerned with compact case, except Chavel ['84]. In this paper, we shall focus on recent developments of spectral geometry of a *noncompact* complete Riemannian manifold. It seems that the materials may be divided into three parts:

(1) the distribution of the (essential) spectrum of the Laplacian,

(2) the heat kernel of a complete Riemannian manifold, and

(3) harmonic functions, and Green functions on such a manifold.

More precisely,

(1) in §3, we treat mainly results on estimates of the bottom of the (essential) spectrum of the Laplacian of a noncompact complete Riemannian manifold.

(2) In §4, following Ito ['88], Dodziuk ['83], we construct the (minimal) heat kernel of a noncompact complete Riemannian manifold, and show results on uniqueness and estimates of such heat kernel, under certain curvature conditions.

(3) In §5, we will treat positive harmonic functions, the Martin boundary, and Liouville type theorems for harmonic functions on complete manifolds.

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§1. Preliminaries

1.1. The Laplace-Beltrami operator

In this section, we prepare some basic materials about spectral theory of selfadjoint operators, and the Laplace-Beltrami operator on a noncompact complete Riemannian manifold.

All Riemannian manifolds we consider in this paper will be C^∞ connected noncompact complete Riemannian manifolds without boundary (unless otherwise stated).

Let (M, g) be a complete Riemannian manifold without boundary. Define the Laplace-Beltrami operator (we call it the Laplacian briefly hereafter) Δ_g acting on the space $C^\infty(M)$ of all C^∞ real valued functions on M by

$$\begin{aligned}\Delta_g f &= \delta d f = - \operatorname{div} \operatorname{grad} f \\ &= -\frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} \left(\sqrt{g} g^{ij} \tfrac{\partial f}{\partial x^j}\right) \\ &= -\sum_{i,j=1}^{n} g^{ij} \left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \sum_{k=1}^{n} \Gamma_{ij}^k \frac{\partial f}{\partial x^k} \right) \\ &= -\sum_{i=1}^{n} \left\{ e_i (e_i f) - (\nabla_{e_i} e_i) f \right\},\end{aligned} \tag{1.1}$$

where $\sqrt{g} = \sqrt{\det(g_{ij})}$, $(g^{ij}) = (g_{ij})^{-1}$ (the inverse matrix), $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$, and Γ_{ij}^k is Christoffel's symbol of g for a local coordinate $(x^1, \dots, x^n)$, and $\{e_1, \dots, e_n\}$ is a locally defined orthonormal frame field on M. Moreover, we denote by $\mathcal{X}(M)$ the space of all C^∞ vector fields on M, and define the divergence $\mathrm{div}(X)$ of $X \in \mathcal{X}(M)$ by

$$\mathrm{div}(X) = \frac{1}{\sqrt{g}} \sum_{i=1}^{n} \frac{\partial}{\partial x^i} \left(\sqrt{g}\, X^i\right),$$

where $X = \sum_i X^i \dfrac{\partial}{\partial x^i}$. The gradient vector field X = grad $f \in \mathcal{X}(M)$ is defined by $g(Y, X) = df(Y) = Yf$, $Y \in \mathcal{X}(M)$, i.e.,

$$\mathrm{grad}(f) = \sum_{i=1}^{n} e_i(f)\, e_i = \sum_{i,j=1}^{n} g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}.$$

We denote by $L^2(M)$ the space of all square integrable real valued functions on M. We define the inner product (,) on $L^2(M)$ by

$$(f_1, f_2) = \int_M f_1(x) f_2(x)\, v_g, \qquad f_1, f_2 \in L^2(M),$$

and put $\| f \| = \sqrt{(f, f)}$, $f \in L^2(M)$. We also define the global inner product (,) for tensor fields α, β by

$$(\alpha, \beta) = \int_M \langle \alpha, \beta \rangle\, v_g,$$

where $\langle\, , \rangle$ denotes the pointwise inner product on $\otimes T_x M \otimes T_x^* M$, $x \in M$, and put $\| \alpha \| = \sqrt{(\alpha, \alpha)}$. Put

$$\begin{aligned} C_o^\infty(M) &= \{f \in C^\infty(M);\, \mathrm{supp}(f), \quad \text{compact}\}, \\ A_o^1(M) &= \left\{\omega \in A^1(M);\, \mathrm{supp}(\omega), \quad \text{compact}\right\}, \\ \mathcal{X}_o(M) &= \{X \in \mathcal{X}(M);\, \mathrm{supp}(X), \quad \text{compact}\}. \end{aligned}$$

Here $A^1(M)$ is the space of all smooth 1 forms on M. Then the following is well-known:

Proposition 1.2. *For all $f, f_1, f_2 \in C_o^\infty(M), \omega \in A_o^1(M)$, and $X \in \mathcal{X}_o(M)$, we get*

(i) $$(f, \mathrm{div}(X)) = -(\mathrm{grad}\ f, X), \quad (df, \omega) = (f, \delta\omega),$$

(ii) $$(\Delta_g f_1, f_2) = (\mathrm{grad}\ f_1, \mathrm{grad}\ f_2) = (f_1, \Delta_g f_2),$$

(iii) $$\int_M \mathrm{div}(X)\, v_g = 0,$$

(iv) $$(\Delta_g f, f) \geq 0,$$

where v_g is the canonical measure of (M, g) given locally by

$$v_g = \sqrt{\det(g_{ij})}\, dx^1 \cdots dx^n.$$

Corollary 1.3. *The Laplacian $\Delta_g : C_o^\infty(M) \longrightarrow C_o^\infty(M)$ can be extended to a symmetric,* i.e., *formally selfadjoint operator of $L^2(M)$ into itself* (*see the definition below Theorem* 1.5).

Definition 1.4. We define the following spaces which are called *domains* of the differential operators div, d, δ, and Δ by

$$\begin{aligned} D(\mathrm{div}) &= \{\text{measurable vector field } X \text{ on } M;\ \|X\| < \infty,\ \|\mathrm{div}(X)\| < \infty\}, \\ D(d) &= \{\text{measurable function } f \text{ on } M;\ \|f\| < \infty,\ \|df\| < \infty\}, \\ D(\delta) &= \{\text{measurable 1 form } \alpha \text{ on } M;\ \|\alpha\| < \infty,\ \|\delta\alpha\| < \infty\}, \\ D(\Delta) &= \{\text{measurable function } f \text{ on} M;\ f \in D(d), df \in D(\delta)\} \\ &= \{\text{measurable function } f \text{ on } M;\ \|f\| < \infty, \|df\| < \infty, \|\delta df\| < \infty\}. \end{aligned}$$

Then we have:

Theorem 1.5 (Gaffney ['51] $\sim$ ['55]). *Let (M, g) be a complete Riemannian manifold. Then:* (i) *if $X \in D(\mathrm{div})$, and $|X|$ and* $\mathrm{div}(X)$ *are integrable, then*

$$\int_M \mathrm{div}(X)\, v_g = 0.$$

(ii) *If $f \in D(d)$, $\omega \in D(\delta)$, and $X \in D(\mathrm{div})$, then*

$$(df, \omega) = (f, \delta\omega),$$
$$(f, \mathrm{div}(X)) = -(\mathrm{grad}\ f, X).$$

(iii) (*symmetry*) *For $f_1, f_2 \in D(\delta)$,*

$$(\Delta f_1, f_2) = (f_1, \Delta f_2).$$

(iv) (*positivity*) *For $f \in D(\Delta)$,*

$$(\Delta f, f) \geq 0.$$

(v) *The closure to $L^2(M)$ of $\Delta : C_o^\infty(M) \longrightarrow C_o^\infty(M)$ is selfadjoint.*

In general, an operator $A :\ D \subset H \longrightarrow H$ of a Hilbert space H which is defined on a dense subset D is said to be *symmetric* (formally selfadjoint)

$$(Au, v) = (u, Av),\ u, v \in D.$$

A symmetric operator A; $D \subset H \longrightarrow H$ is said to be *selfadjoint* if

$$(Au, v) = (u, v^*) \quad \forall u \in D \Longrightarrow v \in D \quad \& \quad v^* = Av.$$

Then it is well-known that:

Theorem 1.6 (Spectral Resolution). *Let $A : D \subset H \longrightarrow H$ be selfadjoint. Then:*

(1) *A has the following resolution:*

$$A = \int_{-\infty}^{\infty} \lambda \, dE(\lambda),$$

where $\{E(\lambda)\,; \lambda \in \mathbb{R}\}$ is a one parameter family of projections of H satisfying the following conditions (i), (ii), (iii):

(i) $$\lambda < \mu \Longrightarrow E(\lambda) \leq E(\mu),$$

(ii) $$E(\infty) = I \ (\textit{identity operator}), E(-\infty) = 0 \ (\textit{null operator}),$$

(iii) $$E(\lambda + 0) = E(\lambda).$$

(2) *The spectrum of A is contained in the set of real numbers:* $\mathrm{Spect}(A) \subset \mathbb{R}$.

Here let us recall the notions of resolvent, (essential-)spectrum, eigenvalues of a selfadjoint operator.

Definition 1.7. (i) The *resolvent* $\mathrm{Resolv}(A)$ of a selfadjoint operator A is the set of $\lambda \in \mathbb{C}$ satisfyng that $\mathrm{Ker}(A - \lambda I) = \{0\}, \mathrm{Range}(A - \lambda I) \subset H$ is dense, and $(A - \lambda I)^{-1}$ is a bounded operator. The *spectrum* of A, $\mathrm{Spect}(A)$, is by definition $\mathbb{C} \setminus \mathrm{Resolv}(A)$.

(ii) $\lambda \in C\,\mathrm{Spect}(A)$ (the *continuous spectrum*) if $\mathrm{Range}(A - \lambda I) \subset H$ is dense, but $(A - \lambda I)^{-1}$ is not a bounded operator.

(iii) A real number $\lambda \in \mathbb{R}$ is an *eigenvalue* of A if there exists a nonzero $u \in D(A)$ such that $Au = \lambda u$. $\mathrm{Ker}(A - \lambda I)$ is called the eigenspace, and $\dim \mathrm{Ker}(A - \lambda I)$ is called the multiplicity. Let $\mathrm{Spect}_o(A)$ be the set of all the eigenvalues which are isolated in $\mathrm{Spect}(A)$ and have finite multiplicities, and call $\mathrm{Ess\ Spect}(A) = \mathrm{closure}(\mathrm{Spect}(A) \setminus \mathrm{Spect}_o(A))$ the *essential spectrum*.

It is known that:

$$\lambda \in \mathrm{Spect}(A) \iff \exists 0 \neq f_n \in D(A)\ ; \quad \|Af_n - \lambda f_n\| \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty,$$

$$\lambda \in \mathrm{Ess\ Spect}(A) \iff \exists \{f_n\}_{n=1}^{\infty} \subset D(A) \text{ (noncompact set)}\ ; \quad \|Af_n - \lambda f_n\| \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty,$$

and

$$\lambda \in \mathrm{Ess\ Spect}(A) \iff \text{either } \lambda \in C\,\mathrm{Spect}(A) \text{ or the eigenvalue with infinite multiplicity.}$$

1.2. Discreteness of spectrum

We mainly deal with the following three types of the eigenvalue problems:

(1) (*Boundary Value Problem*) Let (M, g) be a complete Riemannian manifold without boundary, $\Omega \subset M$ a relatively compact domain in M.

(1-i) (*Dirichlet Eigenvalue Problem*):

$$\begin{cases} \Delta u = \lambda u & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

(1-ii) (*Neumann Eigenvalue Problem*):

$$\begin{cases} \Delta u = \lambda u & \text{in} \quad \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{a.e.} \quad \partial\Omega. \end{cases}$$

(2) (*Free Boundary Problem*) For a smooth function V on M,

$$(\Delta + V)u = \lambda u \quad \text{on} \quad M.$$

Then it is well-known that:

Theorem 1.8. *Let (M, g) be a complete Riemannian manifold, $\Omega \subset M$ a relatively compact domain. Then:*

(1) (1-i) *The spectrum of the Dirichlet eigenvalue problem has a discrete spectrum of eigenvalues with finite multiplicities.*

(1-ii) *If Ω satisfies, furthermore, the segment property* (cf. Agmon ['65, p.13], Reed & Simon ['78, p.256]), *i.e., Ω is a finite union of coordinate neighborhoods in $M, (U_i, \phi_i)$, $\phi_i(U_i) \subset \mathbb{R}^n, n = \dim(M)$, with the property that there exists $y_i \in U_i$ such that $\phi_i(x) + t\,\phi_i(y_i) \in \phi_i(U_i)$ $(0 < \forall t < 1, \forall x \in \partial\Omega \cap U_i)$. Then the Neumann problem has a discrete spectrum of eigenvalues with finite multiplicities.*

(2) *Let V be a smooth function on M satisfying the following exhaustion condition:*

$$\{x \in M \; ; \; V(x) \leq C\} \quad \textit{is compact, for all} \quad C > 0.$$

Then the free boundary problem for $\Delta + V$ has a discrete spectrum of eigenvalues with finite multiplicities.

Remark. (1) If M is compact, then $\Delta + V$ has a discrete spectrum for any smooth function V. (2) If Ω has a piecewise smooth boundary, then it satisfies the segment property.

Outline of Proof. To prove (1), we set $\mathcal{M}(\Omega)$ to be the set of all real valued measurable functions on Ω, and

$$L^2(\Omega) = \{u \in \mathcal{M}(\Omega) \; ; \; \int_\Omega |u(x)|^2 \; v_g \; < \infty\}.$$

The inner product $(\,,\,)$ on $L^2(\Omega)$ is given by $(u, v) = \int_\Omega u(x)\, v(x)\, v_g$, $u, v \in L^2(\Omega)$. We also set the Sobolev space

$$H^1(\Omega) = \left\{u \in L^2(\Omega) \; ; \; |du| \in L^2(\Omega)\right\},$$

and define the inner product $(\,,\,)_1$, and the norm $|| \; ||_1$ by

$$(u, v)_1 = \int_\Omega u\, v\, v_g + \int_\Omega \langle du, dv \rangle v_g, \quad u, v \in H^1(\Omega),$$

$$||u||_1 = \sqrt{(u, v)_1}\,.$$

Let $\overset{\circ}{H}{}^1(\Omega)$ be the closure of $C_o^\infty(\Omega)$ in $H^1(\Omega)$, i.e.,

$$\overset{\circ}{H}{}^1(\Omega) = \left\{u \in H^1(\Omega) \; ; \; \exists u_n \in C_o^\infty(\Omega), ||u_n - u||_1 \longrightarrow 0\, (n \longrightarrow \infty)\right\}.$$

Lemma 1.9 (Green). *For $u, v \in C^\infty(\bar{\Omega})$,*

$$\int_\Omega u\, \Delta v\, v_g - \int_\Omega \langle du, dv \rangle v_g = \int_{\partial\Omega} u \frac{\partial v}{\partial \nu}\, d\sigma\,,$$

where $\dfrac{\partial v}{\partial \nu}$ *is the derivative of* v *with respect to the inward unit normal at* $\partial\Omega$*, and* $d\sigma$ *is the area element of* $\partial\Omega$*. In particular, we get*

$$\int_\Omega \{(\Delta u)\, v - u\, (\Delta v)\}\ v_g = \int_{\partial\Omega} \left\{ u\, \frac{\partial v}{\partial \nu} - \frac{\partial u}{\partial \nu}\, v \right\} d\sigma.$$

(*Dirichlet Problem*). By Green's theorem (cf. Lemma 1.9), the operator $\Delta : C_o^\infty(\Omega) \longrightarrow C_o^\infty(\Omega)$ is symmetric. If we define

$$D(\Delta_D) = \overset{\circ}{H}{}^1(\Omega) \cap \left\{ u \in L^2(\Omega)\ ;\ \Delta u \in L^2(\Omega) \right\},$$

then Δ can be extended to a selfadjoint operator

$$\Delta_D : D(\Delta_D) \longrightarrow L^2(\Omega),$$

and $(\Delta_D\, u, u) \geq 0$, $\forall u \in D(\Delta_D)$. Each element in $\overset{\circ}{H}{}^1(\Omega)$ can be regarded as the one in $H^1(M)$ by defining to be zero outside Ω.

Lemma 1.10 (Rellich). *If* S *is a bounded subset of* $H^1(M)$*, then* $\{ u\,|_\Omega\ ;\ u \in S\}$ *is relatively compact in* $L^2(\Omega)$*.*

Lemma 1.11. *If* S *is a bounded subset of* $L^2(\Omega)$*, then* $(\Delta_D + I)^{-1}(S) \subset \overset{\circ}{H}{}^1(\Omega)$ *is bounded.*

In fact, if $u = (\Delta_D + I)^{-1} f$, $f \in S$, then $u \in D(\Delta_D)$ and

$$\| u \|_1^2 = (\Delta u, u) + (u, u) = (f, u) \leq \| f \|\, \| u \| \leq \| f \|\, \| u \|_1 .$$

We get $\| u \|_1 \leq \| f \|$.

Therefore $(\Delta_D + I)^{-1}\ :\ L^2(\Omega) \longrightarrow D(\Delta_D) \subset L^2(\Omega)$ is a compact operator. In fact, if $S \subset L^2(\Omega)$ is bounded, then $(\Delta_D + I)^{-1}(S) \subset \overset{\circ}{H}{}^1(\Omega)$ is also bounded by Lemma 1.11. Then it is relatively compact in $L^2(\Omega)$ by Lemma 1.10.

Hence $\mathrm{Spect}(\Delta_D)$ is a discerete set of eigenvalues with finite multiplicities.

(*Neumann Problem*). For $u, v \in C^\infty(\bar{\Omega})$, satisfying $\dfrac{\partial u}{\partial \nu} = 0$, $\dfrac{\partial v}{\partial \nu} = 0$ on $\partial\Omega$, we get by Green's theorem (cf. Lemma 1.9),

$$(\Delta u, v) = (u, \Delta v), \qquad (\Delta u, u) = \int_\Omega \| \operatorname{grad} u \|^2\ v_g \geq 0.$$

We now set

$$D(\Delta_N) = \{u \in L^2(\Omega) \; ; \; \Delta u \in L^2(\Omega) \text{ and } u \text{ satisfies } (N)\},$$

where $u \in H^1(\Omega)$ is said to satisfy the condition (N) if

$$(\Delta u, v) - (\text{grad } u, \text{grad } v) = 0 \quad \forall v \in H^1(\Omega).$$

Note that, due to Green's theorem, the left hand side coincides with $\displaystyle\int_{\partial\Omega} \frac{\partial u}{\partial \nu} v \, d\sigma$ for smooth functions u, v.

Then Δ can be extended to a selfadjoint operator

$$\Delta_N \quad : \quad D(\Delta_N) \longrightarrow L^2(\Omega)$$

which satisfies $(\Delta_N u, u) \geq 0$, $\forall u \in D(\Delta_N)$, and $D(\Delta_N)$ is dense in $H^1(\Omega)$. Moreover, if Ω satisfies the *segment property*, then S is relatively compact in $L^2(\Omega)$ for all bounded $S \subset H^1(\Omega)$. Also, if $S \subset L^2(\Omega)$ is bounded, then $(\Delta_N + I)^{-1} \subset H^1(\Omega)$ is bounded. Therefore the operator

$$(\Delta_N + I)^{-1} \quad : \quad L^2(\Omega) \longrightarrow D(\Delta_N) \subset L^2(\Omega)$$

is a compact operator. Hence $\text{Spect}(\Delta_N)$ is a discrete set of eigenvalues with finite multiplicities.

(*Free Boundary Problem*). We assume that V is a function on M with the property that every $\{x \in M \; ; \; V(x) \leq C\}$ is compact. Then

$$\gamma = \min_{x \in M} V(x) < \infty.$$

So we put $H_o = \Delta + V : C_o^\infty(M) \longrightarrow C_o^\infty(M)$, which satisfies

$$(H_o u, u) \geq \gamma\,(u,u), \qquad (H_o u, v) = (u, H_o v), \quad u, v \in C_o^\infty(M).$$

Therefore H_o can be uniquely extended to a selfadjoint operator

$$H : \quad D(H) \longrightarrow L^2(M),$$

where

$$D(H) = H^1(M) \cap \{u \in L^2(M) \; ; \; \Delta u \in L^2(M)\}, \quad \text{and}$$
$$H^1(M) = \{u \in L^2(M) \; ; \; |du| \in L^2(M)\}.$$

Lemma 1.12. *For $u, v \in D(H)$,*

$$\sqrt{V + |\gamma|}\, u \in L^2(M),$$
$$\big((H + |\gamma|)\, u, v\big) = (\text{grad } u, \text{grad } v) + \big(\sqrt{V + |\gamma|}\, u, \sqrt{V + |\gamma|}\, v\big).$$

In particular,

$$\big((H+|\gamma|)\,u,u\big)=\|\ \mathrm{grad}\ u\ \|^2+\|\ \sqrt{V+|\gamma|}\,u\ \|^2\ .$$

Proof. If $u\in D(H)$, there exists $u_n\in C_o^\infty(M)$ such that $u_n\longrightarrow u$ and $(\Delta+V)\,u_n\longrightarrow Hu$ as $n\longrightarrow\infty$. Since $u_n\in C_o^\infty(M)$,

$$\big((\Delta+V+|\gamma|)\,u_n\,,u_n\big)=\|\ \mathrm{grad}\ u_n\ \|^2+\|\ \sqrt{V+|\gamma|}\,u_n\ \|^2,$$

where the left hand side converges to $\big((H+|\gamma|)\,u,u\big)$. Therefore $\{u_n\}_{n=1}^\infty$ is a Cauchy sequence in $H^1(M)$, and $u_n\longrightarrow u$ in $H^1(M)$. This u satisfies

$$\sqrt{V+\gamma}\in L^2(M),\quad \text{and}$$
$$\big((H+|\gamma|\,u),u\big)=\|\ \mathrm{grad}\ u\ \|^2+\|\ \sqrt{V+|\gamma|}\,u\ \|^2\ .$$

The rest of the statement can be proved in a similar way. □

Lemma 1.13. *Assume that S is a subset of $L^2(M)$ which satisfies $\|\ f\ \|\le C$ for all $f\in S$. Then, for all $u\in(H+|\gamma|+1)^{-1}(S)$,*

$$\|\ u\ \|_1\le C\qquad \textit{and}\qquad \|\ \sqrt{V+|\gamma|}\,u\ \|\le C.$$

Proof. Since $|\gamma|+1\in\mathrm{Resolv}(H)$, $\mathrm{Range}(H+|\gamma|+1)=L^2(M)$, and hence $u=(H+|\gamma|+1)^{-1}f\in D(H)$. Then $u\in H^1(M)$ and

$$\big((H+|\gamma|)\,u,u\big)=\|\ \mathrm{grad}\ u\ \|^2+\|\ \sqrt{V+|\gamma|}\,u\ \|^2\ .$$

Then we get

$$\begin{aligned}\|\ u\ \|_1^2&=\|\ \mathrm{grad}\ u\ \|^2+\|\ u\ \|^2\\&=\big((H+|\gamma|)\,u,u\big)-\|\ \sqrt{V+|\gamma|}\ \|^2+\|\ u\ \|^2\\&\le\big((H+|\gamma|)\,u,u\big)+\|\ u\ \|^2\\&=(f,u)\le\|\ f\ \|\,\|\ u\ \|_1,\end{aligned}$$

we get $\|\ u\ \|_1\le\|\ f\ \|\le C$. We get also the second inequality in a similar way. □

Lemma 1.14. (1) *We put $u_n=(H+|\gamma|+1)^{-1}\,f_n$ for any sequence $\{f_n\}_{n=1}^\infty$ in S. Then there exists a subsequence $\{u_k\}$ such that for every*

relatively compact domain $\Omega \subset M$, $\{u_k|_\Omega\}$ *is convergent in* $L^2(\Omega)$.
(2) *The sequence* $\{u_k\}$ *is convergent strongly in* $L^2(M)$.

Proof. (1) Take a sequence $0 < R_1 < R_2 < \cdots < R_j \longrightarrow \infty$, and a point $x_o \in M$. Put

$$K_j = \{x \in M \ ; \ d(x, x_o) \geq R_j\}.$$

By Rellich's theorem (cf. Lemma 1.10), there exist a subsequence $\{u_{1,k}\}$ of $\{u_n\}$ which is convergent strongly in $L^2(K_1)$, and a subsequence $\{u_{2,k}\}$ of $\{u_{1,k}\}$ which is convergent strongly in $L^2(K_2), \ldots$, and inductively subsequences $\{u_{j,k}\}$ which is strongly convergent in $L^2(K_j)$ for each j. Then, putting $u_k = u_{k,k}$, we get the desired subsequence $\{u_k\}$. (2) In the case M is compact, taking $\Omega = M$, (1) implies (2). When M is noncompact, due to the assumption of V, for all $N > 0$, there exists $R(N) > 0$ such that

$$d(x, x_o) \geq R(N) \quad \Longrightarrow \quad V(x) \geq N.$$

Then we get

$$\begin{aligned}
\int_{d(x,x_o)\geq R(N)} |u_k|^2 v_g &= \int_{d(x,x_o)\geq R(N)} |\sqrt{V + |\gamma|}\, u_k|^2 \, (V + |\gamma|)^{-1}\, v_g \\
&\leq (N + |\gamma|)^{-1} \int_{d(x,x_o)\geq R(N)} |\sqrt{V + |\gamma|}\, u_k|^2 v_g \\
&\leq C/N,
\end{aligned}$$

by Lemma 1.13. Then, for every $\epsilon > 0$, there exist $N > 0$ and $R(N) > 0$ such that for all k,

$$\int_{d(x,x_o)\geq R(N)} |u_k|^2\, v_g \ \leq \ \epsilon/3.$$

By (1) in Lemma 1.14, there exists $k_o = k_o(\epsilon) > 0$ such that

$$\|u_\ell - u_k\|^2_{B_{R(N)}} \ \leq \ \epsilon/3, \qquad \forall k, \ell \ \geq \ k_o(\epsilon),$$

where $B_{R(N)} = \{x \in M \ ; \ d(x, x_o) \leq R(N)\}$. Therefore for all $k, \ell \geq k_o(\epsilon)$,

$$\begin{aligned}
\|u_\ell - u_k\|^2 &= \|u_\ell - u_k\|^2_{B_{R(N)}} + \int_{d(x,x_o)\geq R(N)} |u_\ell - u_k|^2\, v_g \\
&\leq \epsilon/3 + \int_{d(x,x_o)\geq R(N)} |u_\ell|^2\, v_g + \int_{d(x,x_o)\geq R(N)} |u_k|^2\, v_g \\
&\leq \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon,
\end{aligned}$$

whence $\{u_k\}$ is a Cauchy sequence. □

Therefore the operator $(H+|\gamma|+1)^{-1} : L^2(M) \longrightarrow L^2(M)$ is compact. Thus the spectrum of H is discrete, i.e., Spect$(\Delta+V)$ consists of only eigenvalues with finite multiplicities.

Example 1.15. (*Harmonic Oscillator*) (1) On the standard line $(\mathbb{R}, g_o)$, the eigenvalue problem

$$-\frac{d^2}{dx^2}u + x^2 u = \lambda u, \quad u \in L^2(\mathbb{R}) \cap C^\infty(\mathbb{R}),$$

has the following spectrum, for $m = 0, 1, 2, \cdots$,

$$\begin{cases} \text{eigenvalue} & : \quad \lambda_m = 2m+1, (\text{multiplicity } 1), \\ \text{eigenfunction} & : \quad \varphi_m(x) = C_m\, H_m(x)\, \exp(-\frac{x^2}{2}), \end{cases}$$

where $H_m(x)$ is the Hermite polynomial and $C_m = \sqrt{\pi}\, 2^m\, m!$.

(2) On the standard Euclidean space $(\mathbb{R}^n, g_o)$, the eigenvalue problem:

$$(\Delta + |x|^2)\, u = \lambda\, u, \quad u \in L^2(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n),$$

has also the following spectrum: for $\mathbf{m} = (m_1, \cdots, m_n) \in \mathbb{Z}_+ \times \cdots \times \mathbb{Z}_+$, where $\mathbb{Z}_+ = \{0, 1, 2, \cdots\}$,

$$\begin{cases} \text{eigenvalue} & : \quad \lambda_{\mathbf{m}} = 2(m_1 + \cdots + m_n) + n, \\ \text{eigenfunction} & : \quad \varphi_{\mathbf{m}}(x) = C_{\mathbf{m}}\, H_{m_1}(x_1) \cdots H_{m_n}(x_n)\, \exp(-\frac{|x|^2}{2}), \end{cases}$$

for $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, where $C_{\mathbf{m}} = C_{m_1} \cdots C_{m_n}$.

§2. Asymptotic distribution of discrete spectrum

2.1. Mini-Max Principle

In this section, we consider the following three eigenvalue problems: the Dirichlet problem, the Neumann problem, and the free boundary problem, which have the discrete spectra of the eigenvalues with finite multiplicities as in section 1.2.

Definition 2.1. In each eigenvalue problem, we count the eigenvalues with their multiplicities:

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \longrightarrow \infty.$$

Remark 2.2. In case of a compact manifold M, the eigenvalue $\lambda_1 = 0$ corresponds to the constant functions.

We will characterize the k-th eigenvalue λ_k of each problem by Mini-Max Principle:

Let A be a selfadjoint operator defined on a dense subspace $D(A)$ of the Hilbert space $L^2(A)$, namely, in each eigenvalue problem, we take A, $D(A)$, $L^2(A)$ as follows:

Case (1-i) (*Dirichlet Eigenvalue Problem*)

$$A = \Delta_D,$$
$$D(A) = D(\Delta_D) = \overset{\circ}{H}{}^1(\Omega) \cap \left\{u \in L^2(\Omega); \Delta u \in L^2(\Omega)\right\},$$

and

$$L^2(A) = L^2(\Omega).$$

Case (1-ii) (*Neumann Eigenvlaue Problem*)

$$A = \Delta_N,$$
$$D(A) = D(\Delta_N) = \left\{u \in L^2(\Omega); \Delta u \in L^2(\Omega) \text{ and } u \text{ staitfies } (N)\right\},$$

and

$$L^2(A) = L^2(\Omega).$$

Case (2) (*Free Boundary Problem*)

$$A = H, \text{ the selfadjoint extension of } H_o = \Delta + V,$$
$$D(A) = D(H) = H^1(M) \cap \left\{u \in L^2(M); \Delta u \in L^2(M)\right\},$$

and

$$L^2(A) = L^2(M).$$

We define the Rayleigh-Ritz quotient as follows: for $0 \neq f \in D(A)$,

$$R(f) = \int_\Omega |df|^2 \, v_g \Big/ \int_\Omega f^2 \, v_g, \text{ or } \quad \int_M |df|^2 \, v_g \Big/ \int_M f^2 \, v_g,$$

where $\int_\Omega f^2\, v_g \neq 0$ for Dirichlet and Neumann boundary problems or $\int_M f^2\, v_g \neq 0$ for free boundary problem, respectively. Then the k-th eigenvalue λ_k is obtained by the following Mini-Max Principle:

Theorem 2.2. *The k-th eigenvalue of each eigenvalue problems is given by*

$$\lambda_k = \sup\, \Lambda(L_{k-1}),$$

where L_{k-1} runs through all $(k-1)$-dimensional subspaces of $D(A)$, and $\Lambda(L_{k-1})$ is defined by

$$\Lambda(L_k) = \inf\,\{\, R(f)\,;\, D(A) \ni f \neq 0, f \text{ orthogonal to } L_{k-1}\}\,.$$

Here the orthogonality means that with respect to the inner product

$$(f_1, f_2) = \int_\Omega f_1(x) f_2(x)\, v_g \quad or \quad \int_M f_1(x) f_2(x)\, v_g.$$

Theorem 2.3. *The k-th eigenvalue of each eigenvalue problem is given also by*

$$\lambda_k = \inf\, \tilde{\Lambda}(L_k),$$

where L_k runs through all k-dimensional subspaces of $D(A)$, and $\tilde{\Lambda}(L_k)$ is

$$\tilde{\Lambda}(L_k) = \sup\,\{\, R(f)\,;\, D(A) \ni f \neq 0\}\,.$$

For proofs and applications, see Bérard ['86] or Bando & Urakawa ['83].

2.2. Asymptotic distributions (I)

See also Protter ['87] for a survey of this topic.

Theorem 2.4 (Minakshisundaram-Pleijel's expansion). *Let Ω be a relatively compact domain in a complete Riemannian manifold (M, g). We assume the segment property of Ω for the Neumann problem. Then the zeta function*

$$Z(t) = \sum_{i=1}^{\infty} e^{-\lambda_i t}\,,\ t > 0$$

has the following asymptotic expansion:
(1) (*Boundary value problems*)

$$Z(t) \sim (4\pi t)^{-\frac{n}{2}} \left\{ a_0 + a_{\frac{1}{2}}\, t^{\frac{1}{2}} + a_1\, t + \cdots \right\} \quad as \ t \longrightarrow 0,$$

where the coefficients a_i are given by:

$$a_0 = \mathrm{Vol}(\Omega), a_{\frac{1}{2}} = \mp 4^{-1}\sqrt{4\pi}\,\mathrm{Vol}_{n-1}(\partial\Omega),$$

$$a_1 = 6^{-1}\left\{\int_\Omega \{\kappa_g + 6V\}\,v_g \;-\; 2\int_{\partial\Omega} J\,d\sigma\right\}, etc\cdots.$$

Here κ_g is the scalar curvature of (M,g), J the mean curvature of $\partial\Omega$ in (M,g), $d\sigma$ the $(n-1)$-dimensional area element of $\partial\Omega$ and the sign $-$ (resp. $+$) above corresponds to the Dirichlet (resp. Neumann) problem.
(2) (*Free boundary problem*) *If M is compact, then, for $\Delta + V$,*

$$Z(t) \sim (4\pi t)^{-\frac{n}{2}}\{a_0 + a_1\,t + \cdots\} \;\; as \;\; t \longrightarrow 0,$$

where the coefficients a_i are given as:

$$a_0 = \mathrm{Vol}(M,g),\; a_1 = 3^{-1}\int_M \{\kappa_g + 3V\}\,v_g,\;\; etc.\cdots.$$

For proofs, see McKean & Singer ['67], Branson & Gilkey ['90] for the boundary value problems of a relatively compact domain of a complete Riemannian manifold (M,g), and see also Minakshisundaram & Pleijel ['49], Berger ['68], Sakai ['71], Gilkey ['75-1], ['75-2] for the free boundary problem of $\Delta + V$ on a compact Riemannian manifold (M,g).

Corollary 2.5 (Weyl's formula). *Let Ω be a relatively compact domain in a complete Riemannian manifold (M,g). We assume the segment property of Ω for the Neumann problem. Let*

$$N(\lambda) = \sharp\{k;\; \lambda_k \le \lambda\}$$

be the (counting) number of the eigenvalues less than or equal to a positive real number λ. For the boundary problems or the free boundary problem of a compact Riemannian manifold (M,g), the asymptotic behavior of $N(\lambda)$ is given by:

$$N(\lambda) \sim \begin{cases} C_n\,\mathrm{Vol}(\Omega)\,\lambda^{\frac{n}{2}}, & (\textit{boundary value problems}),\\ C_n\,\mathrm{Vol}(M,g)\,\lambda^{\frac{n}{2}}, & (\textit{free boundary probelm}),\;\; as\;\lambda \longrightarrow \infty,\end{cases}$$

where

$$C_n = (2\sqrt{\pi})^{-n}\,\Gamma(\tfrac{n}{2}+1)^{-1} = (2\pi)^{-n}\,\mathrm{Vol}(B_1),$$

B_1 being the unit ball in $\mathbb{R}^n$.

Remark 2.6. Moreover, the following best possible estimates of the remainder term of $N(\lambda)$ hold for the Dirichlet problem (1-i) of any smooth bounded domain Ω in the standard Euclidean space $(\mathbb{R}^n, g_0)$, and for the free boundary problem (2) of $\Delta + V$ of a compact Riemannian manifold (M, g):

$$(1\text{-i}) \qquad N(\lambda) = (2\pi)^{-n}\ \mathrm{Vol}(B_1)\ \mathrm{Vol}(\Omega)\, \lambda^{\frac{n}{2}} + O(\lambda^{\frac{n-1}{2}}),$$

$$(2) \qquad N(\lambda) = (2\pi)^{-n}\ \mathrm{Vol}(B_1)\ \mathrm{Vol}(M)\, \lambda^{\frac{n}{2}} + O(\lambda^{\frac{n-1}{2}}).$$

For proofs, see Seeley ['78], Pham The Lai ['81] for (1-i), and Avakumovic ['56], Hörmander ['68] for (2).

For more precise asymptotic behavior of $N(\lambda)$ of the boundary problems, we have:

Theorem 2.7 (Weyl's conjectrure). *Let (M, g) be a complete Riemannian manifold, and Ω a relatively compact domain in M with smooth boundary $\partial\Omega$. We assume the following geodesic concave condition for the boundary $\partial\Omega$: the set of periodic points of the geodesic billiard, i.e., the union of the geodesic segments of (M, g) lying on the inside of Ω and reflecting 'normally' at the boundary $\partial\Omega$, has measure zero. Then the asymptotic behaviors of $N(\lambda)$ are given by:*

$$N(\lambda) = C_n\ \mathrm{Vol}(\Omega)\, \lambda^{\frac{n}{2}} \mp \frac{1}{4} C_{n-1}\ \mathrm{Vol}_{n-1}(\partial\Omega)\, \lambda^{\frac{n-1}{2}} + o(\lambda^{\frac{n-1}{2}}), \qquad \textit{as } \lambda \longrightarrow \infty,$$

where the sign $-$ (resp. $+$) of the right hand side corresponds to the Dirichlet (resp. Neumann) problem, and the constants C_n, C_{n-1} are those given in Corollary 2.5.

For proofs, see Ivrii ['80], Melrose ['80].

Remark 2.8. Bérard ['83] gave examples of domains Ω in S^2, for which $N(\lambda)$ has no asymptotic behavior such as

$$N(\lambda) = C\ \mathrm{Vol}(\Omega)\, \lambda^{\frac{n}{2}} \mp C'\ \mathrm{Vol}(\partial\Omega)\, \lambda^{\frac{n-1}{2}} + o(\lambda^{\frac{n-1}{2}}), \qquad \text{as } \lambda \longrightarrow \infty,$$

for some constants C, C'.

(*Polya's conjecture*). Due to Corollary 2.5, the asymptotic behavior of k-th eigenvalues of the eigenvalue problems for a domain Ω satisfies

$$\lambda_k \sim \left((2\pi)^{-n}\, \mathrm{Vol}(B_1)\, \mathrm{Vol}(\Omega)\right)^{-\frac{2}{n}} k^{\frac{2}{n}} \qquad \text{as } k \longrightarrow \infty.$$

Furthemore, Polya ['61] and Kellner ['66] conjectured the following inequalities: Let λ_k^D, (resp. λ_k^N) be the k-th eigenvalue of the Dirichlet, (resp. Neumann) boundary problems for a bounded domain Ω in $\mathbb{R}^n$. Then

$$\lambda_k^N \leq \left((2\pi)^{-n} \operatorname{Vol}(B_1) \operatorname{Vol}(\Omega)\right)^{-\frac{2}{n}} k^{\frac{2}{n}} \leq \lambda_k^D, \qquad \text{for all } k = 1, 2, \cdots .$$

They showed these inequalities for a tiling bounded domain Ω, i.e., an infinite number of non-overlapping domains which are congruent to Ω, cover $\mathbb{R}^n$ except a measure zero set.

Li & Yau ['83] showed:

$$\frac{n}{n+2}\left((2\pi)^{-n} \operatorname{Vol}(B_1) \operatorname{Vol}(\Omega)\right)^{-\frac{2}{n}} k^{\frac{2}{n}} \leq \lambda_k^D, \text{ for all } k = 1, 2, \cdots,$$

and Urakawa ['84] showed:

$$\delta(\Omega)^{\frac{2}{n}} \left((2\pi)^{-n} \operatorname{Vol}(B_1) \operatorname{Vol}(\Omega)\right)^{-\frac{2}{n}} k^{\frac{2}{n}} \leq \lambda_k^D, \text{ for all } k = 1, 2, \cdots.$$

Here the constant $\delta(\Omega)$ is the packing density of Ω, and $\delta(\Omega) = 1$ if Ω is a tiling domain.

2.3. Asymptotic distribution (II)

In this section, we are concerned with the free boundary problem of $\Delta + V$ on a noncompact complete Riemannian manifold with V satisfying the exhausion condition in Theorem 1.8.

We conjecture that, if a noncompact complete Riemannian manifold (M, g) has non-negative Ricci curvature Ric_M, and a function V on M satisfies the exhausion condition:

$$\{x \in M; V(x) \leq C\} \quad \text{is compact in } M, \text{ for all } C > 0,$$

then the counting number $N(\lambda) = \sharp\{\lambda_n; \lambda_n \leq \lambda\}$ would be asymptotically

$$N(\lambda) \sim C_n \int_M (\lambda - V(x))_+{}^{\frac{n}{2}} v_g, \qquad \text{as} \qquad \lambda \longrightarrow \infty,$$

where

$$C_n = (2\sqrt{\pi})^{-n} \Gamma(\frac{n}{2} + 1)^{-1} = (2\pi)^{-n} \operatorname{Vol}(B_1),$$
$$f(x)_+ = \max(f(x), 0).$$

In fact, it is believed to hold that:

"Theorem" 2.9. *Let V be a continuous function on $\mathbb{R}^n$ which satisfies the above exhausion condition and $V(x) \geq 1, \quad \forall x \in M$. Let $N(\lambda)$ be the counting function of the free boundary problem for $\Delta + V$ on $L^2(\mathbb{R}^n)$. Then*

$$\begin{aligned} N(\lambda) &\sim C_n \int_{\mathbb{R}^n} (\lambda - V(x))_+{}^{\frac{n}{2}}\, dx \\ &= (2\pi)^{-n} \operatorname{Vol}\left(\{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \,;\, |\xi|^2 + V(x) < \lambda\}\right). \end{aligned}$$

We follow the argument in Rosenbljum ['74] and show the precise statement of his theorem. See also Fefferman ['83], Tachizawa ['90] for these topics.

Let $Q = Q_d$ be a cube of edge d in $\mathbb{R}^n$. For each positive number a, let us consider the Dirichlet, Neumann eigenvalue problems for $\Delta + a$ on Q, and let us denote the counting numbers of the eigenvalue problems by $N_D(\lambda, a, Q)$, $N_N(\lambda, a, Q)$, respectively. Then due to Mini-Max Principle (cf. Theorems 2.2, 2.3), we get:

Lemma 2.10. *For all $0 < \epsilon < 1$, there exist positive constants $C_1(\epsilon)$ and $C_2(\epsilon)$ such that*

$$N_D(\lambda, a, Q) \geq (1-\epsilon)^{\frac{n}{2}}\, C_n \operatorname{Vol}(Q)\, \left((\lambda - a) - C_1(\epsilon)\, d^{-2}\right)_+{}^{\frac{n}{2}}, \tag{2.11}$$

$$N_N(\lambda, a, Q) \leq (1+\epsilon)^{\frac{n}{2}}\, C_n \operatorname{Vol}(Q)\, \left((\lambda - a) + C_2(\epsilon)\, d^{-2}\right)_+{}^{\frac{n}{2}}, \tag{2.12}$$

for all $\lambda > 0$.

Let Ξ be an arbitrary lattice of $\mathbb{R}^n$ defined by cubes of edge 1, and assume that V satisfies the following conditions:

(I) There exist a decreasing function ν on the interval $[1, \infty)$ satisfying $\nu(t) \longrightarrow 0$ as $t \longrightarrow \infty$, and $0 \leq \alpha \leq \dfrac{1}{2}$ such that

$$|V(x) - V(y)| \leq |x-y|^{2\alpha}\, V(x)^{1+\alpha}\, \nu(V(x)), \quad \text{for all} \quad x,\, y \in Q' \backslash \partial Q',$$

for any cube Q' of the lattice Ξ.

(II) Letting $\sigma(\lambda, V) = \operatorname{Vol}(\{x \in \mathbb{R}^n ; V(x) < \lambda\})$, there exists a constant $C_3 > 0$ such that

$$\sigma(2\lambda, V) \leq C_3\, \sigma(\lambda, V), \quad \text{for large} \quad \lambda \gg 1.$$

We say $V \in W_\alpha(\Xi)$ if V satisfies the conditions (I), (II).

Theorem 2.13. *Assume that $V \in W_\alpha(\Xi)$ for some Ξ and $0 < \alpha < 1$. Then we get:*

$$N(\lambda) \sim \Phi(\lambda, V),$$
$$\Phi(\lambda, V) = C_n \int_{\mathbb{R}^n} (\lambda - V(x))_+{}^{\frac{n}{2}}\, dx.$$

Proof. For arbitrarily fixed $\epsilon, \epsilon_1 > 0$, we choose $\epsilon_2 > 0$ in such a way that

$$C_1(\epsilon_1)\,\epsilon_2 < \epsilon, \quad C_2(\epsilon_1)\,\epsilon_2 < \epsilon \text{ and } \quad \lambda\,\epsilon_2 > 1.$$

We also choose a positive integer k such that

$$\epsilon_2\,\lambda \leq 4\,k^2 < 4\,\epsilon_2\,\lambda.$$

We divide all unit cubes of the lattice Ξ into cubes Q of edge $d = \frac{1}{k}$. Then by Mini-Max Principle (Theorems 2.2, 2.3), we get:

$$\sum_Q N_D(\lambda, V_Q^+, Q) \leq N(\lambda) \leq \sum_Q N_N(\lambda, V_Q^-, Q),$$

where Q in the sums run through all the above unit cubes of Ξ, and

$$V_Q^+ = \operatorname*{ess\,sup}_{x \in Q} V(x), \quad V_Q^- = \operatorname*{ess\,inf}_{x \in Q} V(x).$$

By Lemma 2.10, the right hand side of the above inequality is smaller than or equal to

$$(1+\epsilon_1)^{\frac{n}{2}}\, C_n \sum_Q \mathrm{Vol}(Q) \left(\lambda - V_Q^- + C_2(\epsilon_1) d^{-2}\right)_+^{\frac{n}{2}}$$
$$\leq (1+\epsilon_1)^{\frac{n}{2}}\, C_n \sum_Q \mathrm{Vol}(Q) \left(\lambda(1+\epsilon) - V_Q^-\right)_+^{\frac{n}{2}},$$

since $C_2(\epsilon_1)\,d^{-2} = C_2(\epsilon_1)\,k^2 < C_2(\epsilon_1)\,\epsilon_2\,\lambda < \epsilon\,\lambda$, by the choice of d, and ϵ_2. Here the cubes Q in the above sums run through, indeed, a finite number of the ones satisfying $\lambda(1+\epsilon) > V_Q^-$, divided from the unit cubes of the lattice Ξ. For $t > 0$, we denote by $\sum_{\mathrm{I}}$, the sum running over the Q's satisfying $V_Q^- < t$, and by $\sum_{\mathrm{II}}$, the sum running over the Q's satisfying $t \leq V_Q^- < \lambda(1+\epsilon)$.

Then we get

$$\sum_{\mathrm{I}} \mathrm{Vol}(Q) \left(\lambda(1+\epsilon) - V_Q^-\right)_+^{\frac{n}{2}} \le (\lambda(1+\epsilon))^{\frac{n}{2}} \, \sigma(t,V),$$

$$\begin{aligned}
&\sum_{\mathrm{II}} \mathrm{Vol}(Q) \left(\lambda(1+\epsilon) - V_Q^-\right)_+^{\frac{n}{2}} \\
&\quad \le \sum_{\mathrm{II}} \int_Q \left(\lambda(1+\epsilon) - V(x) + |V(x) - V_Q^-|\right)_+^{\frac{n}{2}} dx \\
&\quad \le \sum_{\mathrm{II}} \int_Q \left(\lambda(1+\epsilon) - V(x) + d^{2\alpha} \, (V_Q^-)^{1+\alpha} \, \nu(t)\right)_+^{\frac{n}{2}} dx,
\end{aligned}$$

since $V_Q^- \ge V(x) - |V(x) - V_Q^-|$ in the second inequality, and ν is decreasing in $t \le V_Q^-$ in the last inequality.

Here we take a large t in such a way

$$2^{2\alpha} \, {\epsilon_2}^{-\alpha} \, (1+\epsilon)^{1+\epsilon} \, \nu(t) \le \epsilon.$$

Then

$$\begin{aligned}
d^{2\alpha} \, (V_Q^-)^{1+\alpha} \, \nu(t) &< d^{2\alpha} \, (\lambda(1+\epsilon))^{1+\alpha} \, \nu(t) \\
&= k^{-2\alpha} \, \lambda^{1+\alpha} \, (1+\epsilon)^{1+\alpha} \, \nu(t) \\
&\le \left(2^2 \, {\epsilon_2}^{-1} \, \lambda^{-1}\right)^{\alpha} \, \lambda^{1+\alpha} \, (1+\epsilon)^{1+\alpha} \, \nu(t) < \epsilon \, \lambda.
\end{aligned}$$

Therefore the right hand side of the last inequality is smaller than or equal to

$$\sum_{\mathrm{II}} \int_Q (\lambda(1+2\epsilon) - V(x))_+^{\frac{n}{2}} \, dx \le \int_{\mathbb{R}^n} (\lambda(1+2\epsilon) - V(x))_+^{\frac{n}{2}} \, dx.$$

Hence we have

$$\begin{aligned}
N(\lambda) &\le (1+\epsilon_1)^{\frac{n}{2}} \\
&\quad \times C_n \left\{ \int_{\mathbb{R}^n} (\lambda(1+2\epsilon) - V(x))_+^{\frac{n}{2}} \, dx + \lambda^{\frac{n}{2}} \, (1+\epsilon)^{\frac{n}{2}} \, \sigma(t,V) \right\}.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
&\limsup_{\lambda \longrightarrow \infty} \frac{N(\lambda)}{C_n \, \Phi(\lambda,V)} \\
&\le \limsup_{\lambda \longrightarrow \infty} \frac{(1+\epsilon_1)^{\frac{n}{2}} \int_{\mathbb{R}^n} (\lambda(1+2\epsilon) - V(x))_+^{\frac{n}{2}} \, dx}{\Phi(\lambda,V)},
\end{aligned}$$

since $\Phi(\lambda, V) = o(\lambda^{\frac{n}{2}})$ by definition of $\Phi(\lambda, V)$. Thus, letting $\epsilon \longrightarrow 0$ and then $\epsilon_1 \longrightarrow 0$, we obtain

$$\limsup_{\lambda \longrightarrow \infty} \frac{N(\lambda)}{C_n \, \Phi(\lambda, V)} \leq 1.$$

In a similar manner, we also have

$$\liminf_{\lambda \longrightarrow \infty} \frac{N(\lambda)}{C_n \, \Phi(\lambda, V)} \geq 1.$$

In consequence, we obtain Theorem 2.13. □

§3. The bottom of the (essential-)spectrum

3.1. Definitions of analytic and geometric quantities

In this section, we discuss the bottom of the spectrum of the Laplacian Δ of a *noncompact* complete Riemannian manifold (M, g). Namely, the following problems are considered:

(1) When are the Ess Spect(Δ), and the point-spectrum nonempty ?
(2) How to estimate the infima $\lambda_o(\Delta)$, $\lambda_o^{\text{ess}}(\Delta)$ of Spect(Δ), and Ess Spect(Δ) ?
(3) Compare such quantities to the other geometric ones.

Definition 3.1. For a noncompact complete Riemannian manifold (M, g), we define the following quantities:

(1) The bottom of the spectrum of the Laplacian Δ is

$$\begin{aligned} \lambda_1 &= \lambda_1(M, g) = \inf\ (\text{Spect}(\Delta)) \\ &= \inf \left\{ \frac{\|df\|^2}{\|f\|^2} ; 0 \neq f \in C_o^\infty(M) \right\}. \end{aligned}$$

(2) The bottom of the essential spectrum of the Laplacian Δ is

$$\begin{aligned} \lambda_1^{\text{ess}} &= \lambda_1^{\text{ess}}(M, g) = \inf\ (\text{Ess Spect}(\Delta)) \\ &= \sup \{\lambda_1(M \backslash K) \, ; \, K \subset M, \text{ compact}\}, \end{aligned}$$

where $\lambda_1(M \setminus K)$ is the bottom of the spectrum for $M \setminus K$ in (1).

(3) The exponential growth of volume of (M, g) is

$$\mu = \mu(M, g) = \limsup_{r \longrightarrow \infty} \frac{1}{r} \log V(r),$$

where $V(r) = \mathrm{Vol}(B(r))$ is the volume of the geodesic ball of radius r of some point $p \in M$. Note that the definition of μ does not depend on the choice of the point p.

(4) The Cheeger's constant of (M, g) is

$$h = h(M,g) = \inf\left\{ \frac{\mathrm{Vol}_{n-1}(\partial D)}{\mathrm{Vol}(D)}\,;\, D \subset M,\ \text{compact subdomain} \right\}.$$

(5) The isoperimetric growth of (M, g) is

$$\bar{h} = \bar{h}(M,g) = \limsup_{r\to\infty} \frac{S(r)}{V(r)},$$

where $V(r) = \mathrm{Vol}(B(r)), S(r) = \mathrm{Vol}_{n-1}(\partial B(r))$.

3.2. (Essential-)spectrum

In this section, we show results on the existence of the essential spectrum of the Laplacian Δ. In the next section, we will show results which compare the above quantities.

Theorem 3.2 (Donnelly['81-1]). *Let (M, g) be an n-dimensional noncompact complete Riemannian manifold with the Ricci curvature* $\mathrm{Ric}_M \geq -(n-1)c$, $c \geq 0$. *Then the essential spectrum appears* i.e.,

$$\text{Ess Spect}(\Delta) \cap \left[0, \frac{(n-1)^2}{4}c\right] \neq \emptyset.$$

Moreover, we know:

Theorem 3.3 (Donnelly['81-1]). *Let (M, g) be an n-dimensional simply connected complete Riemannian manifold with nonpositive sectional curvature. Let*

$$\phi(r) = \sup\{\,|K(x,\Pi) + c|\,;\, d(x,p) \geq r, \Pi \subset T_xM,\ \textit{plane},\ x \in M\},$$

where $p \in M$ is an arbitrarily fixed point and $c \geq 0$ is a constant, and $K(x, \Pi)$ is the sectional curvature of a plane Π. Assume that

$$\lim_{r\to\infty} \phi(r) = 0.$$

Then we get

$$\text{Ess Spect}(\Delta) = [\frac{(n-1)^2}{4}c, \infty).$$

Let (M, g) be a simply connected n-dimensional complete Riemannian manifold with nonpositive curvature. Fix $p \in M$. Let $\gamma(\omega, r)$ be a geodesic emanating from p, parametrized with distance r from p, and with unit direction ω at p, and let $K(\omega, r, \theta)$ be the curvature of the plane obtained by parallel translation along $\gamma(\omega, r)$ of (ω, θ) plane at p. We denote by $\|F\|$, the supremum of $F(\omega, r, \theta)$ where (ω, θ) run through $S^{n-1} \times S^{n-1}$, and by D the covariant derivative of the standard unit sphere (S^{n-1}, can). Then we have:

Theorem 3.4 (Donnely ['81-2], see also Pinsky ['78], ['81]). *Let (M, g) be as above. Suppose that the sectional curvature of (M, g) satisfies the following decay condition along geodesics emanating from a fixed point p:*

$$\text{i)} \quad \int_0^\infty r \, \|K + 1\| \, dr \, < \, d_1,$$

$$\text{ii)} \quad \int_0^\infty \|D_\omega K\| \, e^{2r} \, dr \, < \, d_2,$$

$$\text{iii)} \quad \int_0^\infty \|D_\omega{}^2 \, K\| \, e^{2r} \, dr \, < \, d_3 \,, \quad \textit{and}$$

$$\text{iv)} \quad \lim_{r \to \infty} r \, \|K + 1\| \, = \, 0,$$

for some positive constants d_1, d_2, d_3. Then:

(1) *Δ has no eigenvalue in* $\left(\dfrac{(n-1)^2}{4}, \infty \right)$.

(2) *Moreover, if the sectional curvature of (M, g) is bounded above by -1, then*

$$\mathrm{Spect}(\Delta) \, = \, C \; \mathrm{Spect}(\Delta) \, = \, [\frac{(n-1)^2}{4}, \infty).$$

Examples 3.5. As particular cases, we consider homogeneous spaces. Let G be a semi-simple Lie group, K be a maximal compact subgroup, and $\mathfrak{g}, \mathfrak{k}$ their Lie algebras. Let B be the Killing form of $\mathfrak{g}$, and define the positive definite inner product $\langle \, , \, \rangle$ on $\mathfrak{g}$ defined by

$$\langle X, Y \rangle \, = \, -B(X, Y), \quad X, Y \in \mathfrak{k} \quad ; \quad \langle X, Y \rangle \, = \, B(X, Y), \quad X, Y \in \mathfrak{p},$$

where $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to B. Let g_o be a G-invariant Riemannian metric on the symmetric space G/K corresponding to the inner product $\langle \, , \, \rangle$ on $\mathfrak{p}$, and g be the left invariant Riemannian metric on G corresponding to the inner product to $\langle \, , \, \rangle$ on $\mathfrak{g}$. Then:

(1) the spectrum of the Laplacian of the Euclidean space $(\mathbb{R}^n, \text{can})$ is

$$\text{Spect}(\Delta_{\text{can}}) = C\ \text{Spect}(\Delta_{\text{can}}) = [0, \infty).$$

(2) The spectrum $\text{Spect}(\Delta_{g_o})$ of the Laplacian of $(G/K, g_o)$ satisfies that

$$\text{Spect}(\Delta_{g_o}) = C\ \text{Spect}(\Delta_{g_o}) = [\,|\rho|^2, \infty),$$

where $|\rho|^2 = \langle \rho, \rho \rangle$ and ρ is half of the sum of all positive restricted root system of $(\mathfrak{g}, \mathfrak{k})$ (cf. Donnely ['79], Urakawa ['80]). And there is an interesting example, i.e.,

(3) if $G = SL(2, \mathbb{R})$, then the spectrum and the set of all eigenvalues of the Laplacian of (G, g) are given as follows (cf. Kobayashi, Ono & Sunada ['89]):

$$\text{Spect}(\Delta_g) = [\frac{1}{8}, \infty), \quad \text{and}$$

$$\text{the set of all eigenvalues of } \Delta_g = \{\frac{1}{8}(n^2 + 4nm + 2m^2 + 1)\,; \; n = 1, 2, 3, \cdots, m = 1, 3, 5, \cdots\} = \left\{1, \frac{15}{8}, 3, 4, \cdots\right\}.$$

On the other hand, in the following cases the essential spectrum does not appear:

Theorem 3.6 (cf. Donnelly & Li ['79]). *Let (M, g) be a noncompact complete Riemannian manifold. We denote by $K(x, \Pi)$, the sectional curvature of a plane Π in the tangent space $T_x(M), x \in M$, and fix $p \in M$, define*

$$\bar{K}(r) = \sup \{K(x, \Pi);\ d(x, p) \geq r, \Pi \subset T_x(M),\ \textit{plane},\ x \in M\}.$$

Assume that

$$\bar{K}(r) \longrightarrow -\infty, \qquad \textit{as } r \longrightarrow \infty.$$

(1) *If (M, g) is simply connected and has negative curvature, then*

$$\text{Spect}(\Delta) = \text{Spect}_o(\Delta), \ \textit{i.e.,}\ \text{Ess Spect}(\Delta) = \emptyset.$$

(2) *If $\dim(M) = 2$ and the fundamental group $\pi_1(M)$ is finitely generated, then we have the same conclusion as* (1).

3.3. Estimates of the bottom of the spectrum

In this section, we show results comparing several quantities defined in section 3.1.

Theorem 3.7 (McKean ['70]). *Let (M, g) be a complete simply connected Riemannian manifold whose sectional curvature K satisfies $K \leq -k^2 < 0$. Then the bottom of the spectrum, λ_1, satisfies:*

$$\lambda_1(M, g) \geq \frac{k^2}{4}.$$

Remark 3.8. The sectional curvature condition of Theorem 3.7 can be relaxed to some Ricci curvature one by Setti ['91].

Theorem 3.9 (Pinsky ['81]). *Let (M, g) be a simply connected complete Riemannian manifold with nonpositive sectional curvature K. Fix $p \in M$. Let $\psi(r)$ denote*

$$\sup\{|K(\gamma(r), \Pi) + c| \ ; \ \gamma(r) \text{ a geodesic emanating } p \text{ with tangent } \omega \in T_p(M), \|\omega\| = 1, \Pi \subset T_{\gamma(r)}(M)\},$$

where $c > 0$ is a constant. Then we have:

(i) *If either $\int_1^\infty \psi(r)\, dr < \infty$, or $K \equiv -c < 0$ outside a compact subset, then* $0 < \lambda_1(M, g) \leq \dfrac{(n-1)^2}{4} c.$

(ii) *If $\int_1^\infty \psi(r)\, dr < \infty$, and $K \leq -c < 0$ everywhere M, then* $\lambda_1(M, g) = \dfrac{(n-1)^2}{4} c.$

Moreover we get:

Theorem 3.10 (Osserman ['79]). (i) *Let (M, g) be a complete n-dimensional Riemannian manifold with a pole p,* i.e., *the exponential mapping* $\exp : T_p(M) \longrightarrow M$ *is an onto diffeomorphism. Assume that there exist constants C and $r_o > 0$ such that*

$$S'' \leq cV'', \quad \forall r \geq r_o.$$

Then we get: $\lambda_o(M, g) \leq \dfrac{1}{4} c^2.$

(ii) *Assume that* $\dim(M) = 2$ *and (M, g) has nonpositive curvature K. If $-c \leq K \leq -d < 0$ for some positive constants c, d, then* $\lambda_1(M, g) \leq \dfrac{1}{4} \dfrac{c}{d}.$

Theorem 3.11 (Brooks ['81]). *Let (M, g) be a noncompact complete Riemannian manifold. Assume that* $\mathrm{Vol}(M, g) = \infty$. *Then we* *have:*

$$\frac{1}{4}h^2 \le \lambda_o^{\mathrm{ess}} \le \frac{1}{4}\mu^2.$$

Theorem 3.12 (Urakawa ['89]). (i) *Let (M, g) be a simply connected complete Riemannian manifold without focal point (not necessarily, nonpositive curvature). Then we have:*

$$\frac{1}{4}m^2 \le \frac{1}{4}h^2 \le \lambda_1 \le \frac{1}{4}\mu^2 \le \frac{1}{4}\bar{h}^2,$$

where m is the infimum of the mean curvature of $\partial B(r), 0 < r < \infty$, and $B(r)$ is the geodesic ball of radius r of some fixed point.

(ii) *In particular, let (M, g) be a simply connected Riemannian symmetric space G/K of noncompact type whose metric comes from the Killing form of the Lie algebra $\mathfrak{g}$ of G. Then we have:*

$$\lambda_1 = \frac{1}{4}\mu^2 = \frac{1}{4}\bar{h}^2 = \|\rho\|^2; \quad m = \inf\left\{2\rho(H); H \in \mathfrak{a}^+, \|H\| = 1\right\},$$

where $\mathfrak{a}^+$ is a positive restricted Weyl chamber. Moreover, if M is rank one, i.e., $\dim(\mathfrak{a}) = 1$, *then*

$$\frac{1}{4}m^2 = \frac{1}{4}h^2 = \lambda_1 = \frac{1}{4}\mu^2 = \frac{1}{4}\bar{h}^2 = \|\rho\|^2.$$

There is the following striking result about the bottom of the spectrum for the Laplacian:

Theorem 3.13 (Brooks ['81-1]). *Let (M, g) be a compact Riemannian manifold and $(\tilde{M}, \tilde{g})$ the universal covering Riemannian manifold. Then:*

$\lambda_1(\tilde{M}, \tilde{g}) = 0 \Longleftrightarrow$ *the fundamental group $\pi_1(M)$ is an amenable group.*

Furthermore, Sunada ['89] clarifies the above Brooks' theorem as follows: Let $(X, \tilde{g}) \longrightarrow (M, g)$ be a normal Riemannian covering of a compact Riemannian manifold with covering transformation group G. For $\rho; G \longrightarrow U(V)$, a unitary representation of G, let E_ρ be a flat vector bundle over M associated to ρ, and Δ_ρ the Laplacian acting on the vector bundle E_ρ. Define

$$\lambda_1(\rho) = \inf(\mathrm{Spect}(\Delta_\rho)); \quad \delta(\rho, \mathbf{1}) = \inf_{v \in V, \|v\|=1} \sup_{\sigma \in A} \|\rho(\sigma)v - v\|,$$

where A is a finite set of generators of G. Then Theorem 3.13 follows from the following theorem:

Theorem 3.14 (cf. Sunada ['89]). (i) *There exist positive constants C_1, C_2 such that for all unitary representation ρ of G,*

$$C_1\,\delta(\rho,\mathbf{1})^2 \le \lambda_1(\rho) \le C_2\,\delta(\rho,\mathbf{1})^2.$$

(ii) *If ρ is the regular representation of G, then $\lambda_1(\rho) = \lambda_1(X,\tilde{g})$, and*

$$\delta(\rho,\mathbf{1}) = 0 \iff G \text{ is amenable.}$$

Ono ['88] showed:

Theorem 3.15. *Let (M,g) be a compact Riemannian spin manifold, and $(\tilde{M},\tilde{g})$ be its universal Riemannian covering. Assume that the A-roof genus of M does not vanish. Then we get:*

$$\lambda_1(\tilde{M},\tilde{g}) \le \frac{1}{4}\left(-\min_{x\in M}\kappa(x)\right),$$

where κ is the scalar curvature of (M,g).

§4. Heat kernel of a complete Riemannian manifold

4.1. Construction of heat kernel

In this section, we construct the heat kernel of a Riemannian manifold. This was done by Ito ['79], Dodziuk ['83], Strichartz ['83], and Yau ['78]. There are two ways to construct the heat kernel. The one is to apply an abstract semigroup theory of $e^{t\Delta}$ on L^2 space of a Riemannian manifold (M,g) which are due to Yau ['78] and Strichartz ['83]. The other is a more or less constructive way due to Ito ['79] and Dodziuk ['83], which takes the following steps:

(1) taking an exhausion sequence of relatively compact domains D_i of M,

(2) define

$$p(x,y,t) = \lim_{i\to\infty} p_i(x,y,t),$$

where $p_i(x,y,t)$ is the Dirichlet heat kernels of D_i.

(3) And show that $p(x,y,t)$ is the heat kernel on (M,g).

In the following, we show the latter way more precisely, following Dodziuk ['83].

Definition 4.1. Let (M,g) be an arbitrary Riemannian manifold, $T > 0$, and u_o a continuous function on M. Then a continuous function $u;\, M\times(0,T) \longrightarrow \mathbb{R}$ is said to be a *solution of the Cauchy problem* of the heat equation on $M \times [0,T)$ with the initial data u_o, if $u(x,t)$ is C^2 in x and C^1 in t, and satisfies

$$\begin{cases} \Delta_x u + \dfrac{\partial u}{\partial t} = 0 & \text{on} \quad M \times [0,T), \\ u(x,0) = u_o(x), & x \in M. \end{cases}$$

Definition 4.2. A continuous function $p(x,y,t)$ on $M \times M \times (0,\infty)$ is a *fundamental solution of the heat equation*, i.e., *heat kernel* on M if, for all bounded continuous function u_o on M,

$$u(x,t) = \begin{cases} \displaystyle\int_M p(x,y,t)\, u_o(y)\, v_g(y)\,, \; t > 0, \\ u_o(x), \qquad t = 0, \end{cases}$$

is a solution of the Cauchy problem of the initial data u_o.

In order to construct the heat kernel on M, we consider the heat kernel $p_D(x,y,t)$ on a relatively compact domain D in M with C^∞ boundary with Dirichlet condition. Then it satisfies that:

Proposition 4.3. *The function p_D is C^∞ on $\bar{D} \times \bar{D} \times (0,\infty)$, and $p_D(x,y,t) = 0$ if x or $y \in \partial D$. Moreover,*

(1) $\quad p_D(x,y,t) > 0, \quad p_D(x,y,t) = p_D(y,x,t),\ x,y \in D,\ t > 0.$

(2) $\quad (\Delta_x + \frac{\partial}{\partial t})\, p_D \equiv 0.$

(3) $\displaystyle\int_D p_D(x,z,t) p_D(z,y,s) v_g(z) = p_D(x,y,t+s),\ s,t > 0, x,y \in \bar{D}.$

(4) *For all relatively compact smooth domain $D \subset M$, there exists a C^∞ function Φ on $D \times D$ such that $\Phi(x,x) \equiv 1, x \in D$, and*

$$p_D(x,y,t) - (4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{d^2(x,y)}{4t}\right) \Phi(x,y)$$
$$= O\left(t^{-\frac{n}{2}+1} \exp\left(-\frac{d^2(x,y)}{4t}\right)\right), x,y \in D, \quad \text{as} \quad t \longrightarrow 0,$$

where the convergence in the right hand side is uniform on any compact subset of $D \times D$.

(5) *For relatively compact domains $D_1, D_2 \subset M$, let p_1, p_2 be the corresponding heat kernels. Then for all $x,y \in D_1 \cap D_2$, and for all $N > 0$,*

$$p_1(x,y,t) - p_2(x,y,t) = O(t^N) \quad \text{as} \quad t \longrightarrow 0,$$

where the estimate is uniform if x, y run on a compact subset of $D_1 \cap D_2$.

(6) *For all $x \in D, t > 0$,*

$$\int_D p_D(x, y, t)\, v_g(y) < 1.$$

(7) *For $D_1 \subset D_2$, we get*

$$p_{D_1}(x, y, t) \leq p_{D_2}(x, y, t), \quad x, y \in D_1, t > 0.$$

We omit its proof, but only note here that we need the following strong maximum principle to get (1), (6), and (7) in Proposition 4.3:

Lemma 4.4. (*strong maximum principle*) *Let $D \subset M$ be a relatively compact domain, u a bounded continuous function on $D \times [0, T]$ which is C^2 on $D \times (0, T)$ and satisfies*

$$\left(\Delta + \frac{\partial}{\partial t}\right) u \leq 0, \qquad \text{on } D \times (0, T).$$

Suppose that there exists $(x_o, t_o) \in D \times (0, T]$ such that

$$u(x_o, t_o) = \max_{(x,t) \in D \times [0,T]} u(x, t).$$

Then we get

$$u(x, t) = u(x_o, t_o) \quad \text{for all} \quad x \in D, t \leq t_o.$$

For a proof, see Nirenberg ['53, p.171].

The candidate of a heat kernel on M can be constructed as follows: We take an exhausion

$D_1 \subset D_2 \subset \cdots$; D_i are relatively compact domains with smooth ∂D_i.

I.e.,

$$\bar{D}_i \subset D_{i+1}, \qquad \bigcup_{i=1}^{\infty} D_i = M.$$

Definition 4.5. We define

$$p(x, y, t) = \lim_{i \to \infty} p_i(x, y, t),$$

where p_i is the heat kernel of D_i with the Dirichlet condition. Note that the limit allows infinity, but exists because of (7) in Proposition 4.3. Moreover we obtain:

Theorem 4.6. *The function $p(x,y,t)$ is C^∞ and a fundamental solution in the sense of Definition* 4.2. *Moreover,*

(1) $p(x,y,t) > 0, \quad p(x,y,t) = p(y,x,t), \quad t > 0,\ x,y \in M.$

(2) $$\left(\Delta_x + \frac{\partial}{\partial t}\right) p \equiv 0.$$

(3) $\displaystyle\int_M p(x,z,t)\,p(z,y,s)\,v_g(z) = p(x,y,t+s),\ t,s > 0,\ x,y \in M.$

(4) *$p(x,y,t)$ does not depend on the choice of an exhausion in its definition, and satisfies that*

$$p(x,y,t) = \sup_{D\subset M} p_D(x,y,t), \quad t > 0, x,y \in M,$$

where $D \subset M$ run over all relatively compact domains in M.

(5) *$p(x,y,t)$ is the smallest positive fundamental solution,* i.e., *for any $q(x,y,t)$ positive fundamental solution,*

$$p(x,y,t) \leq q(x,y,t).$$

Outline of Proof. We only show the covergence of p_i to p. Fix $y \in M$. Let us consider $u_i(x,t) = p_i(x,y,t)$, and show that $\{u_i\}_{i=1}^\infty$ converges uniformly to a C^∞ solution of the heat equation on $D\times[t_1,t_2]$ for a relatively compact domain $D \subset M$, and $0 < t_1 < t_2$. For this, we need:

Lemma 4.7. *Let (N,h) be a Riemannian manifold, $a,b \in \mathbb{R}$ with $0 < a < b < \infty$. Let $\{u_i\}_{i=1}^\infty$ be a nondecreasing sequence of solutions of the heat equations on $N \times (a,b)$. Assume that*

$$\int_N |u_i(x,t)|\,v_h(x) \leq C,$$

where C is a constant independent on $i, t \in (a,b)$. Then $u = \lim_{i\to\infty} u_i$ is a smooth solution of the heat equation, and the convergence of u_i to u is uniform with respect to the C^∞ topology on a relatively compact domains, and the derivatives of all orders converge.

In fact, let $D \subset N$ be a relatively compact smooth domain, $a < t_1 < t_2 < b$. Choose a function $h \in C_o^\infty(D)$, with $h \equiv 1$ on an open subset $V \subset D$. If $v(x,t)$ is a solution of the heat equation, then, for $x \in V, t \in (t_1, t_2)$, by Green's formula and Duhamel's principle, we get

$$\begin{aligned} v(x,t) &= \int_D v(y,t_1)\, h(y)\, p_D(x,y,t-t_1)\, v_h(y) \\ &+ \int_{t_1}^t ds \int_D v(y,t)\, \Delta h(y)\, p_D(x,y,t-s)\, v_h(y) \\ &+ 2\int_{t_1}^t ds \int_D v(y,s)\, \langle \nabla h(y), \nabla_y\, p_D(x,y,t-s)\rangle\, v_h(y). \end{aligned}$$

Since $\Delta h \equiv 0, \nabla h \equiv 0$ in a neighborhood of x, arbitrary large order derivatives of $v(x,t)$ with respect to x are estimated by the terms of the L^1-norm of v, and the same is true for all derivatives by means of $\frac{\partial u}{\partial t} = \Delta u$. Applying this to $\{u_i\}$, $\{u_i\}, \{\nabla u_i\}$ are locally bounded, by the assumption, and then $u = \lim_{i\to\infty} u_i$ is finite and continuous. By Dini's theorem the convergence is uniform on a compact subset. Repeating this to the differentials of $\{u_i\}$, u is C^∞ and satisfies the heat equation. □

(Proof of Theorem continued) For fixed $y \in M$, dut to Proposition 4.3 (6), the function

$$u_i(x,t) = p_i(x,y,t), x \in M,\ t > 0$$

satisfies the conditions of Lemma 4.7, and then the limit $p(x,y,t)$ satisfies the heat equation in the variable (x,t). Moreover, $p(x,y,t)$ is a C^∞ function on $M \times M \times (0,\infty)$: In fact, we consider the heat equation on $(M \times M) \times (0,\infty)$:

$$(*) \qquad \left(\Delta_x + \Delta_y + 2\frac{\partial}{\partial t}\right) v(x,y,t) = 0.$$

Fix a relatively compact domain $D \subset M$. Then, for large $i \gg 1$, $p_i(x,y,t)$ satisfies the equation $(*)$ on $D \times D \times (0,\infty)$, and by (6) of Proposition 4.3,

$$\int_{D\times D} p_i(x,y,t)\, v_g(x)\, v_g(y) \le Vol(D).$$

Thus by Lemma 4.7, $p(x,y,t) = \lim_{l\to\infty} p_i(x,y,t)$ satisfies a C^∞ solution of $(*)$.

Now we show the function $p(x,y,t)$ is a fundamental solution of the heat equation in the sense of Definition 4.2: (1) For a bounded continuous function u_o on M,

$$u(x,t) = \begin{cases} \int_M p(x,y,t)\, u_o(y)\, v_g(y), & t > 0, \\ u_o(x), & t = 0, \end{cases}$$

is bounded and continuous.

In fact, we first show, for any open subset $U \subset M$ and $x \in U$,

$$\text{(a)} \qquad \lim_{t\downarrow 0} \int_U p(x,y,t) v_g(y) = 1.$$

Because note that, by (4) of Proposition 4.3,

$$\text{(b)} \qquad \lim_{t\downarrow 0} \int_D p_D(x,y,t) v_g(y) = 1, \quad x \in D.$$

Then by (b), (6) of Proposition 4.3, and positivity of p, we get

$$\begin{aligned} 1 &\geq \liminf_{t\downarrow 0} \int_M p(x,y,t)\, v_g(y) \geq \liminf_{t\downarrow 0} \int_U p_D(x,y,t)\, v_g(y) \\ &\geq \lim_{t\downarrow 0} \int_D p_D(x,y,t)\, v_g(y) = 1, \quad x \in D, \end{aligned}$$

for a relatively compact smooth domain $D \subset U$. We get (a). Moreover, by (6) of Proposition 4.3, we get

$$\text{(c)} \qquad \int_M p(x,y,t)\, v_g(y) \leq 1, \quad x \in M.$$

By (a), (c) and positivity of p, we obtain (1).

(2) The function $u(x,t)$ is a solution of the heat equation. In fact, we may assume $u_o \geq 0$. Then the function

$$u(x,t) = \lim_{i\to\infty} \int_M p_i(x,y,t)\, u_o(y)\, v_g(y)$$

is a limit of nondecreasing sequence of solutions of the heat equation, and the L^1-norm of each function is bounded above by a constant independent on i. Therefore by Lemma 4.7, $u(x,t)$ satisfies the heat equation. Thus $p(x,y,t)$ is the heat kernel. The properties (4), (5) of Theorem 4.6 follow from the maximum principle. □

4.2. Uniqueness of solution of heat equation

In this section, we show uniqueness results on the heat equation. Namely, let (M, g) be a complete Riemannian manifold. For a continuous function $f(x)$ on M, let us consider the Cauchy problem:

$$(4.8) \qquad \begin{cases} \left(\Delta_x + \dfrac{\partial}{\partial t}\right) u = 0, & \text{on } M \times (0, \infty), \\ u(x, 0) = f(x), \; x \in M, & \end{cases}$$

where $u(x, t)$ is a continuous function on $M \times [0, \infty)$, and C^2 in x, and C^1 in t. Then we get:

Theorem 4.9 (cf. Dodziuk ['83]). *Let (M, g) be a complete Riemannian manifold with the Ricci curvature satisfying $\mathrm{Ric}_M \geq -C$, $C > 0$. Then a bounded solution of (4.8) is determined uniquely by the initial data f.*

For a proof, see Dodziuk ['83] or Chavel ['84].

Theorem 4.10 (cf. Donnelly ['83]). *Under the same assumption of Theorem 4.9, a non-negative solution of (4.8) is uniquely determined by the initial data.*

Theorem 4.11 (cf. Li ['84]). *Let (M, g) be a noncompact complete Riemannian manifold whose Ricci curvature satisfies*

$$\mathrm{Ric}_M(x) \geq -C\left(1 + r(x)^2\right), \quad \forall x \in M,$$

for some positive constant C, where $r(x) = d(x, p)$, $x \in M$, for some fixed point p. Then

(1) *any L^1-solution of (4.8) is uniquely determined by the initial data in $L^1(M)$.*

(2) *$1 < p < \infty$. Then any L^p-solution of (4.8) is uniquely determined by the initial data in $L^p(M)$.*

(3) (cf. Li & Yau ['86]) *any solution of (4.8) which is bounded below, is uniquely determined by the initial data.*

Theorem 4.12 (cf. Li & Karp ['91]). *Let (M, g) be a complete Riemannian manifold satisfying that there exist a point $p \in M$ and a constant C such that, either*

$$(1) \qquad \mathrm{Vol}(B_r(p)) \leq \exp(C r^2), \quad \forall r,$$

where $B_r(p)$ is the geodesic ball centered p with radius r, or

$$\text{(2)} \qquad \mathrm{Ric}_M(x) \geq -C\left(1+r^2(x)\right), \quad \forall x \in M,$$

where $r(x) = d(x,p)$, $x \in M$. Then any bounded solution of (4.8) *is determined by the initial data.*

Theorem 4.13 (cf. Nagasawa ['91]). *Let (M,g) be a complete Riemannian manifold, and $u(x,t)$ a continuous solution of* (4.8), *and assume that there exist $p \in M$ and $C > 0$ such that*

$$\int_{B_{r+1}(p)\setminus B_r(p)} |u(x,t)|^2\, v_g(x) \leq \exp\left(C(1+r^2)\right), \quad \forall r > 0.$$

Then $u(x,t) \equiv 0$, $\forall t > 0$, if $u(x,0) = f(x) \equiv 0$. In particular, let

$$K_p(r) = \inf_{x \in B_r(p)} \mathrm{Ric}_M(x), \text{ and } K_p^+(r) = \max\{K_p(r), 0\}$$
$$p \in M,\ r > 0.$$

Assume that there exist $p \in M$ and $C > 0$ such that

$$K_p^+ \leq C(1+r^2), \forall r.$$

Then any nonnegative continuous solution of (4.8) *is uniquely determined by the initial data.*

4.3. Estimates of the heat kernel

In this section, we show results on the asymptotic behavior, and upper and lower estimates of the heat kernel of a complete Riemannian manifold.

We first show the following asymptotic behavior of the heat kernel $p(x,y,t)$ of a complete Riemannian manifold (M,g), as t tends to zero.

Theorem 4.14 (cf. Cheng, Li, & Yau ['81, p.1040]). *Let (M,g) be an arbitrary complete Riemannian manifold, $p(x,y,t)$ be the heat kernel. Then we get:*

$$\lim_{t \downarrow 0} -4t \log p(x,y,t) = d^2(x,y), \quad \forall\, x,y \in M.$$

Remark 4.15. The above theorem was obtained by Varadhan ['67] when $(\mathbb{R}^n, g)$, where g satisfies the uniform Hölder condition and the uniform ellipticity condition. One can also see a proof of the above theorem in Chavel ['84, p.201], when (M, g) is a complete Riemannian manifold with Ricci curvature bounded from below.

On the other hand, the asymptotic behaviors of the heat kernel $p(x, y, t)$, as t tends to $+\infty$, are given as follows:

Theorem 4.16 (cf. Li ['86]). *Let (M, g) be an n-dimensional complete Riemannian manifold, $p(x, y, t)$ be the heat kernel of (M, g). Then*

$$(1) \qquad \lim_{t \to \infty} \frac{\log p(x, y, t)}{t} = -\lambda_1(M, g), \quad \forall\, x, y \in M.$$

(2) *Assume that (M, g) has nonnegative Ricci curvature $\mathrm{Ric}_M \geq 0$, and there exist a point p and a positive constant θ such that*

$$\liminf_{r \to \infty} \frac{\mathrm{Vol}(B_r(p))}{r^n} = \theta.$$

Then we have:

$$\lim_{t \to \infty} \mathrm{Vol}(B_{\sqrt{t}}(p))\, p(x, y, t) = \mathrm{Vol}_n(B_1)(4\pi)^{-\frac{n}{2}},$$

where B_1 is the unit ball in $\mathbb{R}^n$.

The lower and upper bounds of the heat kernel $p(x, y, t)$ are given as follows:

(Lower bounds) We first prepare some terminologies: For any Riemannian manifold (M, g), and a fixed point p, let $m(r, \theta)$ be the mean curvature function at point (r, θ), of $\partial B_r(p)$ with $\partial B_r(p) \cap C$ deleted. Here $\partial B_r(p)$ is the distance sphere centered with p, radius r, and C is the cut locus of p. Moreover, we call a Riemannian manifold $\mathcal{M}$ to be an *open model* if the following conditions hold:

1. For some point $z \in \mathcal{M}$ and $0 < R \leq \infty$, $\mathcal{M} = B_R(z)$ and the exponential map $\exp_z;\ B_R(0) \longrightarrow B_R(z)$ is a diffeomorphism.
2. For all $r < R$, the mean curvature of the distance sphere $\partial B_r(z)$ is constant on $\partial B_r(z)$, denoted by $m(r)$.

Then we get by definition:

Proposition 4.17. *Let $\mathcal{M}$ be an open model. Then its heat kernel $p(\tilde{x}, \tilde{y}, t) = p(d(\tilde{x}, \tilde{y}), t), \tilde{x}, \tilde{y} \in \mathcal{M}$, depends only on $r = d(\tilde{x}, \tilde{y})$, and t.*

Then the heat kernel $p(x, y, t)$ can be estimated as follows:

Theorem 4.18 (cf. Cheeger & Yau ['81]). *Let (M, g) be a complete Riemannian manifold, $\mathcal{M}$ an open model. Assume that*

$$m(r,\theta) \leq m(r), \quad \forall\, 0 < r \leq R.$$

Then we have:

$$p(d(x,y),t) \leq p(x,y,t), \quad \forall\, x, y \in M,\, t > 0,$$

and the equality holds if and only if (M, g) is isometric to $\mathcal{M}$ and $m(r,\theta) = m(r)$, $\forall\, r$.

Moreover, it is known that:

Theorem 4.19 (cf. Li & Yau ['86]). *Let (M, g) be a complete Riemannian manifold with nonnegative Ricci curvature* $\mathrm{Ric}_M \geq 0$. *Then for all $\epsilon > 0$, there exists a constant $C(\epsilon)$ such that*

$$p(x,y,t) \geq C(\epsilon)^{-1}\, \mathrm{Vol}\left(B_{\sqrt{t}}(x)\right)^{-1} \exp\left\{\frac{-d(x,y)^2}{(4+\epsilon)t}\right\},$$

$$p(x,y,t) \geq C(\epsilon)^{-1}\, \mathrm{Vol}\left(B_{\sqrt{t}}(x)\right)^{-\frac{1}{2}}\, \mathrm{Vol}\left(B_{\sqrt{t}}(y)\right)^{-\frac{1}{2}} \exp\left\{\frac{-d(x,y)^2}{(4+\epsilon)t}\right\},$$

where the constant $C(\epsilon)$ tends to $+\infty$ as $\epsilon \longrightarrow 0$.

(Upper bounds) In general, we obtain the following estimates:

Theorem 4.20 (cf. Cheng, Li & Yau ['81, p.1037]). *Let (M, g) be a complete Riemannian manifold. Then, for all $\beta > 1$, $T > 0$, and $x \in M$, there exists a constant $C = C(\beta, T, x)$ such that*

$$\int_{M\setminus B_R(x)} p(x,y,t)^2 v_g(y) \leq C\, t^{-\frac{n}{2}} \exp\left\{\frac{-R^2}{2\beta t}\right\},$$
$$\forall\, t \in [0,T],\ \forall\, R > 0,$$

where the constant C tends to $+\infty$ as $\beta \longrightarrow 0$.

In particular, we obtain:

Theorem 4.21 (cf. Cheng, Li & Yau ['81, p.1046]). *Let (M, g) be a complete Riemannian manifold with bounded curvature,* i.e., *whose sectional curvature is bounded. Then for all $\alpha > 4$, $T > 0$ and $x \in M$, there exists a constant $C' = C'(\alpha, T, x)$ such that*

$$p(x,y,t) \leq C'\, t^{-\frac{n}{2}} \exp\left\{-\frac{d(x,y)^2}{\alpha t}\right\}, \quad \forall\, t \in [0,T],\ \forall\, y \in M.$$

Theorem 4.22 (cf. Varopoulos ['84]). *Assume that (M,g) satisfies the same conditions of Theorem 4.21 and the injectivity radius is bounded below by a positive constant. Then the heat kernel satisfies that, for all $0 < \epsilon < 0.1$, there exist C_1, $C_2 > 0$ such that*

$$\sup_{x,y\in M} p(x,y,t) \leq \min\left\{C_1\, t^{-\frac{1}{2}+\epsilon},\, C_2\, t^{-\frac{1}{2}}\,(\log t)^{1+\epsilon}\right\}, \quad \forall t > 1.$$

Theorem 4.23 (cf. Li & Yau ['86 p.175]). *Let (M,g) be a complete Riemannian manifold with nonnegative Ricci curvature : $\mathrm{Ric}_M \geq 0$. Then, for $\forall\, 0 < \epsilon < 1$, there exists a constant $C(\epsilon)$ such that*

$$p(x,y,t) \leq C(\epsilon)\ \mathrm{Vol}\left(B_{\sqrt{t}}(x)\right)^{-1} \exp\left\{-\frac{d^2(x,y)}{(4+\epsilon)t}\right\},$$
$$\forall\, x,y \in M,\ \forall t > 0,$$

where the constant $C(\epsilon)$ tends to $+\infty$ as $\epsilon \longrightarrow 0$.

Theorem 4.24 (cf. Davies ['87]). *Let (M,g) be a Riemannian manifold whose heat kernel $p(x,y,t)$ satisfies*

$$p(x,y,t) \leq a\, t^{-\frac{n}{2}}, \quad \forall\, x,y \in M,\ t > 0,$$

for some positive constant a. Then, for all $\delta > 0$, there exists a constant $C(\delta)$ such that

$$p(x,y,t) \leq C(\delta)\, t^{-\frac{n}{2}} \exp\left\{-\frac{d(x,y)^2}{4(1+\delta)t}\right\}, \quad \forall\, x,y \in M, \forall t > 0.$$

Remark 4.25. The assumption of the heat kernel $p(x,y,t)$ in Theorem 4.24 is equivalent to the following:

$$\|f\|_{\frac{2n}{n-2}}{}^2 \leq a\,(\Delta f, f), \quad \forall\, 0 \leq f \in C_o^\infty(M),$$

which is satisfied, if the Ricci curvature of (M,g) is bounded below: $\mathrm{Ric}_M \geq -c$, $c > 0$, and the injectivity radius is bounded below by a positive constant.

Remark 4.26. Recently the following Lichnerowicz conjecture is solved negatively by E. Damek and F. Ricci ['91]: A noncompact complete Riemannian manifold whose heat kernel $p(x,y,t)$ depends only on the distance $r(x,y)$ and t, is the Euclidean space or a symmetric space of rank one.

§5. Harmonic functions

5.1. Green functions

In this section, we are concerned with Green function on a relatively compact domain $\Omega \subset M$ of a complete Riemannian manifold (M, g).

Definition 5.1. Let $\Omega_D = \{(x,x) \in \Omega \times \Omega; x \in \Omega\}$. Then a function $G_\Omega; \bar{\Omega} \times \bar{\Omega} \backslash \Omega_D \longrightarrow \mathbb{R}$ is said to be a *Green function* of Ω if

(1) it is C^2 function on $\Omega \times \Omega \setminus \Omega_D$,
(2) $\Delta_y G_\Omega = 0, \quad \forall x,\, y \in \Omega,\, x \neq y$,
(3) $G_\Omega(x,y) = 0, \quad x \in \Omega,\, y \in \Gamma = \partial\Omega$,
(4) G_Ω can be written in a neighborhood of Ω_D by $G_\Omega(x,y) = \psi(x,y) + h(x,y)$, where $h \in C^o(\bar{\Omega} \times \bar{\Omega}) \cap C^2(\Omega \times \Omega)$, and

$$\psi(x,y) = \begin{cases} \dfrac{1}{C_{n-1}} \dfrac{d(x,y)^{2-n}}{n-2}, & n > 2, \\ \dfrac{1}{2\pi} (-\log d(x,y)), & n = 2, \end{cases}$$

$d(x,y)$, x, $y \in M$ being the geodesic distance in (M,g), and $C_{n-1} = \dfrac{2\,\pi^{n/2}}{\Gamma(n/2)}$, the $(n-1)$-volume of the unit sphere in $\mathbb{R}^n$.

For the existence of such a function G_Ω, see John ['82], for example.

Definition 5.2 (cf. Aomoto ['66]). Let $M_D = \{(x,x) \in M \times M; x \in M\}$ for a Riemannian manifold (M,g). A C^2 function $G; M \times M \backslash M_D \longrightarrow \mathbb{R}$ is called a *Green function* of (M,g) if the following hold:

(1) $\Delta_y G = 0, \quad \forall x,\, y \in M,\, x \neq y$,
(2) G can be written in a neighborhood of M_D by $G(x,y) = \psi(x) + h(x,y)$, where $h \in C^2(M \times M)$, and $\psi(x,y)$ satisfies the same properties as (4) in Definition 5.1.
(3) For all $y \in M$, there exists $\delta > 0$ such that $G(x,y)$ is a bounded function in x on $M_\delta = \{x \in M\,;\, d(x,y) > \delta\}$.

Definition 5.3. A function f on (M,g) is said to be *superharmonic* if the following hold:

(1) $-\infty < f(x) \leq \infty$, and f does not vanish identically on M.
(2) f is lower semi continuous on M.
(3) Let $\Omega \subset M$ be a relatively compact smooth domain. If a function w which is continuous on $\bar{\Omega}$, and harmonic on Ω, satisfies that $w(x) \leq f(x), \quad \forall\, x \in \partial\Omega$, then $w(x) \leq f(x), \quad \forall\, x \in \Omega$.

Note that a C^2 function f on (M, g) is superharmonic if and only if $\Delta f \geq 0$ everywhere on M. Here notice that our Laplacian is $\Delta = \delta\, d$. Then it is known that:

Theorem 5.4 (cf. Ito ['64-1], ['64-2]). *Let (M, g) be a Riemannian manifold, $p(x, y, t)$ be the heat kernel defined in §4. Define*

$$G(x, y) = \int_0^\infty p(x, y, t)\, dt,\ x,\ y \in M.$$

Then $G(x, y)$ gives a Green function of (M, g) if and only if there exists a nonconstant positive superharmonic function on (M, g).

Definition 5.5. A Riemannian manifold (M, g) is said to be *hyperbolic* if it has a nonconstant positive superharmonic function, *parabolic* otherwise. For these examples, see section 5.3.

5.2. The Martin boundary

In this section, we introduce the notion of the Martin boundary. To do this, we first prepare the Harnack inequality, the Harnack principle, and the maximum principle:

Theorem 5.6 (Harnack inequality) (cf. Moser ['61]). *Let $\Omega \subset M$ be a relatively compact domain in a complete Riemannian manifold (M, g). Let $\Omega' \Subset \Omega$ be a domain whose closure is contained in Ω. Let u be a positive harmonic function on Ω. Then we get:*

$$\sup_{x \in \Omega'} u(x) \leq C \inf_{x \in \Omega'} u(x),$$

where C is a positive constant which depends only on Ω, Ω', and the curvature of (M, g).

Theorem 5.7 (Harnack principle). *Let Ω be a relatively compact domain in a complete Riemannian manifold (M, g). Let $\{u_n\}_{n=1}^\infty$ be a sequence of harmonic functions on Ω. Assume that there exists a positive constant K such that $|u_n| \leq K$, $n = 1, 2, \cdots$. Then $\{u_n\}_{n=1}^\infty$ is a normal family,* i.e., *there exists a subsequence which is convergent to a harmonic function on Ω and the convergence is uniform on each compact subset of Ω.*

For a proof, see Tsuji ['59], Kishi ['74], Doob ['83].

Theorem 5.8 (Maximum principle). *Let $\Omega \subset M$ be a relatively compact domain in a complete Riemannian manifold (M, g). Assume that*

$$\Delta u = 0 \quad \text{on } \Omega, \text{ and } u \leq 0 \text{ on } \partial\Omega.$$

Then we get $u \leq 0$ on Ω.

For a proof, see Protter & Weinberger ['84].

Assume that (M, g) is hyperbolic, i.e., it has a nonconstant positive superharmonic function. Let $o \in M$ be a fixed point. For x, $y \in M$, let

$$K_y(x) = K(y,x) = \begin{cases} G(y,x)/G(y,o), & y \neq o, \\ 0, & y = o,\, x \neq o, \\ 1, & x = y = o, \end{cases}$$

then the function K satisfies the following:

(1) For each fixed $y \in M$, K_y is a nonnegative harmonic function in $x, x \neq y$,

(2) $K_y(o) = 1$, and

(3) for each fixed $x \in M$, $K(y,x)$ is a continuous function in $y, y \neq x$.

Assume that $\{y_n\}_{n=1}^{\infty}$ is a sequence in M which has no accumulation point in M. By the Harnack principle (cf. Theorem 5.7), a sequence $\{K_{y_n}|_{\Omega}\}_{n=1}^{\infty}$ has a subsequence which converges to a harmonic function on Ω, for every relatively compact domain $\Omega \subset M$. Take an exhausion $\Omega_1 \subset \Omega_2 \subset \cdots \subset M$, $\cup_i \Omega_i = M$, and use the diagonal method as in the proof of Lemma 1.14 to get a subsequence $K_{y_{n_k}}$ of K_{y_n} which converges to a harmonic function on M, say K.

Definition 5.9. A sequence $\{y_n\}$ in M is said to be *fundamental* if K_{y_n} converges to a harmonic function K on M.

By the above argument, we get:

Lemma 5.10. *Assume that a complete Riemannian manifold (M, g) is hyperbolic. Then any sequence in M which has no accumulation point, has a fundamental subsequence.*

Definition 5.11. Let (M, g) be a hyperbolic Riemannian manifold. Then two fundamental sequences in (M, g) are *equivalent* if the corresponding limit harmonic functions in Lemma 5.10 coincides each other. The *Martin boundary* or *ideal boundary* $\mathcal{M}$ of (M, g) is the equivalence classes of all fundamental sequences of (M, g).

Note that, for $[Y] \in \mathcal{M}$,

$$K_Y(x) = \lim_{i \to \infty} K_{y_i}(x),\ x \in M, \tag{5.12}$$

where $\{y_i\}$ is a fundamental sequence associated to $[Y] \in \mathcal{M}$, and K_Y is a positive harmonic function satisfying $K_Y(o) = 1$. Therefore each $[Y] \in \mathcal{M}$ corresponds to a unique positive harmonic function K_Y on M with $K_Y(o) = 1$.

Definition 5.13. Put $\tilde{M} = M \cup \mathcal{M}$, and define the following metric ρ on $\tilde{M}$:

$$\rho(Y, Y') = \int_{B_1(o)} \frac{|K_Y(x) - K_{Y'}(x)|}{1 + |K_Y(x) - K_{Y'}(x)|} v_g(x),\ Y,\ Y' \in \tilde{M},$$

$$\left(\text{or} \quad \sup_{x \in B_1(o)} |K_Y(x) - K_{Y'}(x)| \right),$$

where $B_1(o)$ is the geodesic ball centered at o with radius 1 in (M, g).

Proposition 5.14 (cf. Martin ['41]). *This ρ is actually a complete metric on $\tilde{M}$, and $(\tilde{M}, \rho)$ is compact, M is open in $\tilde{M}$, and $\mathcal{M}$ is the boundary of $\tilde{M}$. The relative topology of M with respect to ρ coincides with the original topology of M. Moreover, for each $x \in M$, the mapping $Y \longmapsto K_Y(x)$ is continuous on $\tilde{M} \setminus \{x\}$ with respect to ρ.*

Then Martin showed

Theorem 5.15 (Representation theorem) (cf. Martin ['41]). *For each nonnegative harmonic function u on (M, g), there exists a Borel measure μ on $\mathcal{M}$ such that*

$$u(x) = \int_{\mathcal{M}} K_Y(x)\, d\mu(Y),\ x \in M. \tag{5.16}$$

Conversely, for any Borel measure μ on $\mathcal{M}$, (5.16) *gives a nonnegative harmonic function on (M, g), and $\mu(\mathcal{M}) = u(o)$.*

Definition 5.17. A positive harmonic function u on (M, g) is *minimal* if any positive harmonic function v with $v(x) \leq u(x)$, $\forall x \in M$ is a constant multiple of u.

Note that, if u is a positive minimal harmonic function, then there exists a positive constant C such that $u = C K_Y$ for some $Y \in \mathcal{M}$. Then we define:

Definition 5.18. Put $\mathcal{M}_1 = \{Y \in \mathcal{M};\ K_Y \text{ minimal}\}$, and $\mathcal{M}_o = \mathcal{M} \setminus \mathcal{M}_1$. A Borel measure μ on $\mathcal{M}$ is *canonical* if $\mu(\mathcal{M}_o) = 0$.

Theorem 5.19 (Canonical representation theorem) (cf. Martin ['41]). *For any nonnegative harmonic function u on (M, g), there exsists a unique canonical Borel measure μ on $\mathcal{M}$ such that*

$$u(x) = \int_{\mathcal{M}} K_Y(x)\, d\mu(Y),\ x \in M.$$

Moreover, Brelot ['56] showed:

Theorem 5.20 (Solvability of Dirichlet problem). *Let ν be a canonical Borel measure on $\mathcal{M}$ and f a continuous function on $\mathcal{M}$. Define a function $\mathcal{P}_f$ on M by*

$$\mathcal{P}_f(x) = \int_{\mathcal{M}} f(Y)\, K_Y(x)\, d\nu(Y),\ x \in M.$$

Then $\mathcal{P}_f$ is a harmonic function on (M, g), and satisfies that

$$\lim_{x \to Y'} \mathcal{P}_f(x) = f(Y'),\ Y' \in \mathcal{M}.$$

The function $\mathcal{P}_f$ is called a *Poisson integral* on the Martin boundary. For a proof, see Brelot ['56], Doob ['83, p.207, p.101], and Ito ['88].

5.3. Examples of hyperbolic Riemannian manifolds

In this section, we give examples of complete hyperbolic Riemannian manifolds (M, g) and realize their Martin boundaries.

Let G be a real semisimple Lie group with finite center, K a maximal compact subgroup, and $M = G/K$ a symmetric space of noncompact type as in Example 3.5. Let g_o be the Riemannian metric on M induced from the Killing form of the Lie algebra $\mathfrak{g}$ of G, and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, the Cartan decomposition of $\mathfrak{g}$. Then:

Theorem 5.21 (cf. Furstenberg ['63]). *Let (M, g_o) be as above. Then it is hyperbolic and its Martin boundary $\mathcal{M}$ coincides with the homogeneous space $K/Z_K(A) = G/B$. Here $Z_K(A)$ is the centralizer of A in K, A is the analytic subgroup of G corresponding to a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ contained in $\mathfrak{p}$ and B is a Borel subgroup of G. Moreover, the Poison integral coincides with the integral*

$$\mathcal{P}_f(x) = \int_K f(xk)\, dk, x \in G,$$

for a continuous function f on K which satisfies $f(km) = f(k), k \in K$, $m \in Z_K(A)$, and dk is the Haar measure on K.

For more interesting results about the Poisson integrals on symmetric spaces, see Korányi ['69], Helgason ['70-'76], Kashiwara, Kowata, Minemura, Okamoto, Oshima and Tanaka ['78].

In case of a general (not necessary homogeneous) Riemannian manifold, we get:

Theorem 5.22 (Aomoto ['66]). *Let (M, g) be an n-dimensional simply connected complete Riemannian manifold. Assume that the sectional curvature K of (M, g) is nonpositive in case of $n \geq 3$, and $K \leq -C, C > 0$ in case of $n = 2$. Then (M, g) is hyperbolic,* i.e., *it has nonconstant positive superharmonic function.*

Moreover, there are the following criteria telling which (M, g) is hyperbolic or parabolic, due to Kasue ['82]:

Let (M, g) be a complete noncompact connected Riemannian manifold. For $x \in M$, let $\sigma : [0, \infty) \longrightarrow M$, a geodesic emanating x with unit speed, and define functions R_x and $f_x : [0, \infty) \longrightarrow \mathbb{R}$, such that

$$R_x(t) \leq \frac{1}{n-1} \operatorname{Ric}_M(\dot{\sigma}(t)), \forall t \in [0, \infty),$$

where Ric_M is the Ricci curvature of (M, g), and

$$f_x{}'' + R_x\, f_x = 0, f_x(0) = 0 \text{ and } f_x{}'(0) = 1.$$

Then we get:

Theorem 5.23 (Kasue ['82]). *Assume that (M, g) has a positive Green function $G(x, y)$. Then the following holds:*

$$G(x, y) \geq \frac{1}{\omega_{n-1}} \int_{d(x,y)}^{\infty} f_x(t)^{1-n}\, dt,$$

where ω_{n-1} is the $(n-1)$-dimensional volume of the unit sphere of $\mathbb{R}^n$. In particular, if $\int^{\infty} f_x(t)^{1-n}\, dt = \infty$, then (M, g) has no positive Green function, therefore parabolic, i.e., *(M, g) does not have any nonconstant positive superharmonic function.*

On the contrary, let us define functions $\bar{K}_x$, $\bar{F}_x$; $[0, \infty) \longrightarrow \mathbb{R}$, by

$$\bar{K}_x(t) \geq K(\Pi), \forall\, \dot{\sigma}(t) \in \Pi \subset T_{\sigma(t)}M, \text{ plane},$$

where $K(\Pi)$ is the sectional curvature of the plane Π, and

$$\bar{F}_x{}'' + \bar{K}_x \bar{F}_x = 0, \; \bar{F}_x(0) = 0, \text{ and } \bar{F}_x{}'(0) = 1.$$

Then we get:

Theorem 5.24.

(1) (Kasue ['82]) *Let $i(x)$ be the injectivity radius of (M,g) at $x \in M$. Assume that $i(x) = \infty$, and $\int^{\infty} \bar{F}_x(t)^{1-n} dt < \infty$ for all $x \in M$. Then (M,g) admit a positive Green function G which satisfies*

$$G(x,y) \le \frac{1}{\omega_{n-1}} \int_{d(x,y)}^{\infty} \bar{F}_x(t)^{1-n}\, dt, \; \forall x \neq y \in M,$$

where ω_{n-1} is as in Theorem 5.23.

(2) (cf. Li & Tam ['87-1]) *Let (M,g) be a Riemannian manifold whose sectional curvature is nonnegative outside some compact subset. Assume that (M,g) has at least one large end (see section 5.5 for definition). Then (M,g) is hyperbolic.*

Example 5.25. Let $M = \mathbb{R} \times_f F$ be the warped product where F is an n-dimensional compact Riemannian manifold and f is a positive C^{∞} function on $\mathbb{R}$. Then:

(1) If $\displaystyle\int_{-\infty}^{\infty} f(t)^{-n}\, dt < \infty$, then M is hyperbolic and admit a non-constant harmonic function with finite Dirichlet integral.

(2) If $\displaystyle\int_{0}^{\infty} f(t)^{-n}\, dt = \infty$ and $\displaystyle\int_{-\infty}^{0} f(t)^{-n}\, dt = \infty$. Then M is parabolic.

Example 5.26. Let (M,g) be a complete Riemannian manifold with nonnegative Ricci curvature. Then a theorem of Cheeger and Gromoll says that $(M,g) = (N,h) \times (\mathbb{R}^k, g_o)$, the Riemannian product, and g_o is the standard metric on $\mathbb{R}^n$. Moreover, the following hold:

(1) If $k \ge 3$, then (M,g) is hyperbolic.
(2) If $k \le 2$ and N is compact, then (M,g) is parabolic.

For a proof of these examples, see also Kasue ['82].

On the other hand, Lyons & Sullivin ['84] studied a Riemannian manifold (M,g) which admits a positive Green function. Note that a Riemannian manifold (M,g) has the property (1) : (M,g) admits a positive Green function, is equivalent to (2) : (M, g) admits a nonconstant bounded subharmonic function, and also equivalent to (3) : the Brownian motion on (M, g) is transient.

They constructed (M, g) which does not admit a nonconstant positive harmonic function, but have a positive Green function, as a Corollary of the following theorems: Let $\Gamma \subset \mathrm{Isom}(M)$ be a discrete subgroup of the isometry group of (M, g), whose quotient space $N = M/\Gamma$ is smooth.

Definition 5.27. M is an *Abelian* (resp. *nilpotent*, *solvable*, *ω-nilpotent*) *cover* of N , if Γ is Abelian (resp. nilpotent, solvable, Γ is a infinite union of normal subgroups Z_i of Γ with Z_{i+1} contained in the center of Γ/Z_i).

Then they obtained:

Theorem 5.28.

(1) *Let (N, h) be a compact Riemannian manifold. Then any Riemannian nilpotent covering space of (N, h) admits no nonconstant positive harmonic function.*

(2) *Let (M, g) be a Riemannian ω-nilpotent cover of (N, h). Assume that (N, h) admits no positive Green function. Then a bounded harmonic function on (M, g) is always constant.*

(3) *Let (M, g) be a Riemannian non-amenable cover of (N, h). Then (M, g) admits a nonconstant bounded harmonic function.*

(4) *Let (M, g)be a Riemannian Abelian cover of (N, h). Then (M, g) admits a positive Green function if and only if the rank of Γ is bigger than or equal to* 3.

Corollary 5.29. *The universal Riemannian cover of a compact negatively curved manifold admits a nonconstant bounded harmonic function.*

They also extended a theorem of Kelvin, Nevanlinna, & Royden:

Theorem 5.30. *Let (M, g) be a complete Riemannian manifold. Then it admits a positive Green function if and only if there exists a vector field V on M such that*

$$\int_M |\,\mathrm{div}\ V|^2\, v_g < \infty, \int_M |V|^2\, v_g < \infty, \ \ \textit{and} \ \ \int_M \mathrm{div}\ V\, v_g \neq 0.$$

Corollary 5.31. *Let (X, g), (Y, h) be complete Riemannian manifolds. Assume that they are quasi-isometric. Then the one admits a positive Green function if and only if the other does so.*

(*Open Problem*). Under the assumption of Corollary 5.31, does the property that the one admits a bounded harmonic function if and only if the other does so, hold ?

5.4. The Martin boundary and the ideal boundary

We first, in this section, introduce the ideal boundary of a Riemannian manifold of nonpotive curvature following Eberlein & O'Neill ['73], and show results of Anderson ['83], Sullivan ['83], and Anderson & Schoen ['85]. In this section, we assume (M, g) is a complete Riemannian manifold of nonpositive curvature.

Definition 5.32. A geodesic ray $\gamma\,;\,[0,\infty) \longrightarrow M$ is a geodesic of (M, g) parametrized with arc length, and each of whose segment is minimal between its endpoints. Two geodesic rays γ_1, γ_2 are said to be *asymptotic* if $\sup_{0 \le t < \infty} d(\gamma_1(t), \gamma_2(t)) < \infty$. Let $S(\infty)$ be the set of all asymptotic classes of geodesic rays, which is called the *ideal boundary* or *geometric boundary*. For a geodesic ray γ, we denote by $\gamma(\infty)$ the asymptotic classes containing γ.

Note that for each $p \in M$ and $x \in S(\infty)$, there exists a unique geodesic ray γ_{px} such that $\gamma_{px}(0) = p$, and $\gamma_{px}(\infty) = x$.

Let $\bar{M} = M \cup S(\infty)$, and introduce the topology, called the *cone topology*, which is compatible to that of M, and with respect to which $S(\infty)$ is homeomorphic to the $(n-1)$-dimensional unit sphere S^{n-1}: Let $p \in M$, a, $b \in \bar{M}$, $p \neq a$, b. The *angle subtended* by a, b at $p \in M$, denoted by $\sphericalangle_p(a, b)$, is the angle $\sphericalangle(\gamma_{pa}{}'(o), \gamma_{pb}{}'(0))$ between the geodesics γ_{pa}, γ_{pb} at p. Then for $\pi > \epsilon > 0$, and $v \in S(p)$, the unit sphere in the tangent space T_pM, let us define the *cone* of vertex p, axis v and angle ϵ by

$$C(v, \epsilon) = \left\{b \in \bar{M}\,;\, \sphericalangle_p(\gamma_v(\infty), b) < \epsilon\right\}.$$

We can define the topology on $\bar{M}$ (called the *cone topology*) in such a way that, for each point $x \in S(\infty)$, a collection of the set

$$\{C(v, \epsilon)\,;\, x \in C(v, \epsilon), v \in S(p), p \in M, \pi > \epsilon > 0\}$$

is a neighborhood system of x in $\bar{M}$.

Propsition 5.33 (Eberlein & O'Neill ['73, p.54]). *Let* $B(p) = \{v \in T_pM; \|v\| < 1\}$, *and* $S(p) = \{v \in T_pM; \|v\| = 1\}$ *for* $p \in M$. *Let* $f : [0,1] \longrightarrow [0,\infty]$ *be a homeomorphism. Then* $\varphi\,;\, B(p) \ni v \longmapsto \exp(f(\|v\|)\,v) \in M$ *gives a homeomorphism of* $S(p)$ *onto* $S(\infty)$.

Theorem 5.34 (cf. Anderson ['83], Sullivan [83]). *Assume that (M, g) be a complete simply connected Riemannian manifold whose sectional curvature K satisfies $-\infty < -b^2 \leq K \leq -a^2 < 0$, for some positive constants a, b. Then, for every continuous function φ on $S(\infty)$, there exists a unique function $u \in C^\infty(M) \cap C^o(\bar{M})$ such that*

$$\begin{cases} \Delta u = 0, \\ u\,|_{S(\infty)} = \varphi. \end{cases}$$

Furthermore, Anderson & Schoen ['85] showed that $\mathcal{M} = S(\infty)$ under the same assumption of Theorem 5.34. Namely,

Theorem 5.35. *Under the same assumption of (M, g) in Theorem 5.34 there exists a homeomorphism Φ of $\mathcal{M}$ onto $S(\infty)$. Therefore, if we put, for a fixed point $o \in M$,*

$$K(x, Q) = \lim_{y \to Q} \frac{G(y, x)}{G(y, o)},\ Q \in S(\infty),\ x \in M,$$

each positive harmonic function u on (M, g) can be uniquely expressed by a finite positive Borel measure on $S(\infty)$ such that

$$u(x) = \int_{S(\infty)} K(x, Q)\, d\mu(Q),$$

and has nontangential limit at a.e. $Q \in S(\infty)$, i.e., *for every nontangential domian Ω at Q, $\lim_{\Omega \ni x \to Q} u(x)$ exists, and the limit is the absolutely continuous part of the Borel measure μ on $S(\infty)$ corresponding to u.*

Here, a domain $\Omega \subset M$ is a *nontangential* domain at $Q \in S(\infty)$ if $\Omega \cap S(\infty) = \{Q\}$, and there exists a neighborhood V of Q which is contained in a nontangential cone at Q. The *nontangential cone* at Q is, by definition, $T_c = \{x \in M\,;\, \rho(x, \gamma) < c\}$ for a positive constant c, where $\gamma : [0, \infty) \longrightarrow M$ is a geodesic ray in (M, g) with $\gamma(0) = o, \gamma(\infty) = Q$, and ρ is the metric of Definition 5.13 on $\bar{M}$ under the identification $S(\infty)$ and $\mathcal{M}$.

Remark 5.36. (1) Sasaki ['84] showed that if (M, g) is a simply connected complete Riemannian manifold whose curvature is negative and asymptotically constant curvature $-c^2$, $c > 0$, then $S(\infty)$ is homeomorphic to the Martin boundary $\mathcal{M}$ and $\mathcal{M} = \mathcal{M}_1$ (cf. Definition 5.17).

(2) Ancona ['87] extends Theorem 5.19 to a general elliptic operator (see also Ito ['64-2]).

(3) Arai ['87], ['89] studied Fatou type theorems of the boundary behavior of harmonic functions, and BMO on negatively curved manifolds.

5.5. Liouville type theorems for harmonic functions

One of the first remarkable results on existence of harmonic functions on a complete Riemannian manifold is the following theorem:

Theorem 5.37 (cf. Yau ['75]). *Let (M,g) be a complete Riemannian manifold with nonnegative Ricci curvature:* $\mathrm{Ric}_M \geq 0$.

(1) *Then any positive harmonic function on (M,g) must be a constant.*

(2) *Moreover assume that (M,g) has a point $p \in M$ whose cut locus is empty. Then any harmonic function f on (M,g) satisfying*

$$\inf_{x\in M} (f(x) + \gamma(x)^s) > -\infty, \quad \text{for some } 0 \leq s < 1,$$

must be a constant. Here $\gamma(x) = d(x,p), x \in M$.

Theorem 5.38 (cf. Yau ['76]). *Let (M,g) be a complete Riemannian manifold. Let $f \in C^\infty(M)$ satisfy $f\,\Delta f \leq 0$, where $\Delta = \delta d$. If $\int_M f^p\, v_g < \infty$, for some $p > 1$, then f is a constant.*

He also showed in the paper that

(1) There is no nonconstant holomorphic L^p functions on a complete Kähler manifold for some $p > 1$.

(2) Any L^2 harmonic 1 form on a complete Riemannian manifold with nonnegative Ricci curvature is parallel.

(3) As their applications, any noncompact complete Riemannian manifold (M,g) with nonnegative Ricci curvature has infinite volume $\mathrm{Vol}(M,g) = \infty$.

In the case of L^1 harmonic functions, the following is known:

Theorem 5.39 (cf. Li ['84]). *Let (M,g) be a noncompact complete Riemannian manifold whose Ricci curvature satisfies that*

$$\mathrm{Ric}_M(x) \geq -C\left(1+\gamma(x)^2\right), \forall\, x \in M,$$

where $\gamma(x) = d(x,p)$, $x \in M$ for some $p \in M$. Then any L^1 subharmonic function must be a constant.

The condition of nonnegativity of Ricci curvature can be relaxed as follows:

Theorem 5.40 (Li ['85]). *Let (M, g) be an n-dimensional complete Riemannian manifold.*

(1) *Assume that there exist $C > 0$, $\alpha > 0$ such that the Ricci curvature satisfies*

$$\mathrm{Ric}_M(x) \geq -C\left(1+\gamma(x)^2\right)\left\{\log\left(1+\gamma(x)^2\right)\right\}^{-\alpha}, \forall x \in M.$$

Then any L^1 nonnegative subharmonic function is constant.

(2) *Assume that there exist positive constants $C \leq \delta(n)$ depending only on n such that $\mathrm{Ric}_M(x) \geq -C\,\gamma(x)^{-2}$, $x \in M$. Then any L^p nonnegative subharmonic function is constant for all $0 < p < 1$.*

(3) *Assume that either (M, g) is simply connected and has nonpositive sectional curvature, or (M, g) satisfies $\mathrm{Ric}_M \geq -c$, for some $c > 0$, and $\mathrm{Vol}(B_1(x)) \geq d > 0$, for all $x \in M$. Then each nonnegative L^p subharmonic function is constant for all $0 < p < 1$.*

Kanai ['85] introduced the notions of rough isometry, rough isometric.

Definition 5.41. For two metric spaces $(X, d_X), (Y, d_Y)$, a map (not necessarily continuous) $\phi;\, X \longrightarrow Y$ is said to be *rough isometric* if

(1) image of ϕ is full in Y, i.e., the ϵ-ball of the image of ϕ coincides with Y, and

(2) there exists constsnts $a \geq 1$ and $b \geq 0$ such that

$$\begin{aligned} a^{-1}\, d_X(x_1, x_2) - b &\leq d_Y(\phi(x_1), \phi(x_2)) \\ &\leq a\, d_X(x_1, x_2) + b, \,\forall\, x_1,\, x_2 \in X. \end{aligned}$$

Then he showed the following:

Theorem 5.42 (Kanai ['85]). (1) *Let (M, g) be an n-dimensional complete Riemannian manifold whose Ricci curvature satisfies $\mathrm{Ric}_M \geq -c$, for some constant $c > 0$. Assume that (M, g) is rough isometric to the standard Euclidean space $(\mathbb{R}^m, g_o)$ with $m \geq n$. Then any positive harmonic function on (M, g) is constant.*

(2) *Let (M, g), (N, h) be complete Riemannian manifolds whose Ricci curvatures are bounded below. Assume that these Riemannian manifolds are rough isometric. Then (M, g) is parabolic if and only*

if (N, h) is parabolic. Here let us recall (M, g) is parabolic if all positive superharmonic function on (M, g) is constant.

He also introduced (cf. Kanai ['85]) the notion of a *parabolic net* of a Riemannian manifold and showed its relation to parabolicity of Riemannian manifold:

Definition 5.43. A countable set P of points of a Riemannian manifold (M, g) is called to be a *net* if there corresponds to $\{N_p\}_{p \in P}$ such that

(1) for all $p \in P$, N_p is a finite subset of P, and
(2) for all p, $q \in P$, $p \in N_q \iff q \in N_p$.

A sequence $\mathbf{P} = \{p_o, \ldots, p_s\}$ of a net P is said to be a *path* if $p_k \in N_{p_{k-1}}$ for all $k = 1, \ldots, s$. The net P is *connected* if each two point can be joined by a path. We define the Laplacian Δ_P acting on functions on P by

$$(\Delta_P f)(p) = -\frac{1}{\# N_p} \sum_{q \in N_p} f(q) + f(p), p \in P,$$

for a function f on P. Then a function f on P is said to be *superharmonic* if $\Delta_P f \geq 0$. A net P is said to be *parabolic* if each positive superharmonic function on P is constant. A subset P of M is *ϵ-separated* if $d(p,q) \geq \epsilon$, for all p, $q \in P$, $p \neq q$. If we take a maximal ϵ-separated subset P in M, it has a net structure, in fact, for each $p \in P$, we may set $N_p = \{q \in P\,;\, 0 < d(p,q) \leq 3\epsilon\}$. We call this P a ϵ-net in (M, g). Then

Theorem 5.44 (Kanai ['85]). *Let (M, g) be a complete Riemannian manifold whose Ricci curvature is bounded below. Then (M, g) is parabolic if and only if for $\forall \epsilon > 0$, any ϵ-net P is parabolic.*

Furthermore, Kanai ['85] defined the notion of Green function on a net P: For each $k = 0, 1, 2, \ldots$, define inductively $\pi_k\,;\, P \times P \longrightarrow \mathbb{R}$ by

$$\pi_o(p,q) = \begin{cases} 1, & p = q, \\ 0, & p \neq q, \end{cases}$$

$$\pi_{k+1}(p,q) = \sum_{r \in P} \pi_k(p,r)\, \pi(r,q),$$

where

$$\pi(p,q) = \begin{cases} \dfrac{1}{\# N_p}, & q \in N_p, \\ 0, & q \notin N_p. \end{cases}$$

Then the *Green function* G_P on a net P is defined by

$$G_P(p,q) = \sum_{k=0}^{\infty} \pi_k(p,q), \quad p,\, q \in P,$$

if the sum is convergent. Then

Theorem 5.45 (Kanai ['85]). *A net P of a complete Riemannian manifold (M,g) is hyperbolic if and only if*

$$G_P(p,q) < \infty, \forall p \neq q \in P.$$

See Gaveau & Okada ['91] for de Rham-Hodge theory and the heat kernels on graphs, and see also Dodziuk ['81], Bérard ['90] about the vanishing theorems of L^2 harmonic sections of a vector bundle.

5.6. Miscellaneous topics of harmonic functions

In this section, we treat with the problem which a complete Riemannian manifold admits a nonconstant bounded harmonic function.

We consider, in this section, a Riemannian manifold (M,g) whose sectional curvature K satisfies $K \geq 0$ outside some compact subset, following Li & Tam ['87-1], ['87-2], and Li ['90]. For such one (M,g), an end E is said to be *large* if, for a fixed point $p \in M$, we put $V_E(t) = \mathrm{Vol}(E \cap B_t(p))$, it holds that $\int_1^{\infty} \frac{t}{V_E(t)}\, dt < \infty$, and we call it *small* otherwise. Then:

Theorem 5.46 (Li & Tam ['87-2]). *Let (M,g) be as above.*

(1) *Let E be a large end. Then there exists a unique positive harmonic function f on (M,g) such that $\lim_{E \ni x \to \infty} f(x) = 1$, and $\lim_{D \ni x \to \infty} f(x) = 0$ for other large end D (if exists).*

(2) *(M,g) admits at least one large end and one small end, say E. Then there exists a unique (up to a positive constant multiple), positive harmonic function g such that $\lim_{E \ni x \to \infty} g(x) = \infty$, $\lim_{D \ni x \to \infty} g(x) = 0$, for any large end D, and g is bounded on the other small end if any.*

(3) *If (M, g) has only small ends, then it is parabolic.*

Theorem 5.47 (Li & Tam ['87-2]). *Let (M, g) be as above. We denote by $\mathcal{H}_\infty$ the space of all bounded harmonic functions on (M, g). Then:*

(1) *if (M, g) has only small ends. Then* $\dim \mathcal{H}_\infty = 1$, i.e., *any bounded harmonic function is constant.*

(2) *If (M, g) has large ends, say $\{E_i\,;\, i = 1, \ldots, k\}$. Then* $\dim \mathcal{H}_\infty$ $= k$. *Here we can take as a basis of $\mathcal{H}_\infty$, the unique positive harmonic function as in* (1) *of Theorem* 5.46, *$f_i, i = 1, \ldots, k$ on (M, g) satisfying $\lim_{E_i \ni x \to \infty} f_i(x) = 1$, and $\lim_{E_j \ni x \to \infty} f_i(x) = 0\,(\forall j \neq i)$.*

Theorem 5.48 (Li & Tam ['87-2]). *Let (M, g) be as above. We denote by $\mathcal{H}_+$, the positive cone of positive harmonic functions. Then:*

(1) *if (M, g) has only small ends, then $\mathcal{H}_+ = \{constant\ functions\}$.*

(2) *If (M, g) has only k large ends, then $\mathcal{H}_+ \subset \mathcal{H}_\infty$, and any positive harmonic function is a nonnegative linear combination of $\{f_i\,;\, i = 1, \ldots, k\}$ in* (2) *of Theorem* 5.47.

(3) *If (M, g) has k small ends and s large ends, then any positive harmonic function is a nonnegative linear combination of $\{f_i\,; i = 1, \ldots, k\}$ as in* (2) *of Theorem* 5.47 *and $\{g_j\,;\, j = 1, \ldots, s\}$ positive harmonic functions as in* (2) *of Theorem* 5.46 *corresponding to s small ends.*

Next we consider a noncompact complete Kähler manifold (M, g) whose sectional curvature is nonnegative outside some compact subset. Then one gets:

Theorem 5.49 (Li ['90]). *Let (M, g) be as above. Assume that (M, g) has $k(\geq 2)$ large ends $\{E_i\,;\, i = 1, \ldots, k\}$. Then there exists a unique bounded harmonic function h on (M, g) satisfying $\lim_{E_1 \ni x \to \infty} h(x)$ $= 1$, and $\lim_{E_i \ni x \to \infty} h(x) = 0$ for $i \neq 1$. At infinity of each small end, h is asymptotically a constant in the interval* $(0, 1)$, *and has a finite Dirichlet integral over M.*

Corollary 5.50 (Li ['90]). *Let (M, g) as above. Assume that (M, g) has at least* 2 *ends. Then its all ends are small, and there exists a compact subset $D \subset M$ such that $M \setminus D$ is isometrically product of a compact Kähler manifold with nonnegative sectional curvature and nonnegatively curved Riemann surface with boundary.*

5.7. Open problems

Finally we gather open problems about the Laplacian on a complete

Riemannian manifold:

(1) The first main problem is to determine the spectrum Spect($\Delta + V$) of a complete Riemannian manifold (M, g), The bottom of the (essential) spectrum for a noncompact Riemannian manifold is particularly interesting.

(1-1) The index (i.e., the number of negative eigenvalues of the second variation operator of the volume) of a complete minimal submanifold has been studied by many people. Then the essential spectrum, and the distribution of discrete spectrum of minimal submanifolds with infinite index must be studied next.

(1-2) Show the counting number $N(\lambda) = \#\{\lambda_n\,;\ \lambda_n \leq \lambda\}$ for $\Delta + V$ of (M, g) behaves asymptotically

$$N(\lambda) \sim C_n \int_M (\lambda - V(x))_+{}^{\frac{n}{2}} v_g, \quad \text{as} \quad \lambda \longrightarrow \infty,$$

under certain Ricci curvature condition of (M, g) and the exhausion one of N (cf. section 2.3).

(2) (due to T. Nagasawa) Extend to a complete Riemannian manifold, the following Täcklind's theorem on the Euclidean space for uniqueness of solution of the heat equation: For a positive measurable function on the interval $(0, \infty)$, the only solution of

$$\Delta u + \frac{\partial u}{\partial t} = 0 \quad \text{on} \quad \mathbb{R}^n \times (0, \infty),$$

satisfying $u(x, 0) \equiv 0$, and

$$|u(x,t)| \leq \exp\{|x| h(|x|)\},$$

is $u \equiv 0$ if and only if

$$\int_1^\infty \frac{dr}{h(r)} = \infty.$$

See Nagasawa ['91], for more detail.

(3) Extend theories (existence and Liouville type theorems, etc.) about harmonic functions on a complete Riemannian manifold to ones about harmonic maps between complete Riemannian manifolds. See for example Akutagawa ['89], ['90], Li & Tam ['91-2].

(4) For two quasi- or rough isometric Riemannian manifolds, does the property that the one admits a bounded harmonic function if and only if the other does so, hold ? See section 5.3.

(5) Study the Martin boundary of a Riemannian manifold of non positive curvature outside a compact set (cf. Sasaki ['84], Freire ['91]). Recently a remarkable progress on a study of the Martin boundary of a strictly pseudo-convex domain has been made by H. Arai ['91].

(6) Construct two isospectral bounded domains in the Euclidean space with *smooth* boundaries which are not isometric each other. Examples of isospectral plane domains with piecewise smooth boundaries like tangrams have been constructed by Gordon, Webb & Wolpert ['91].

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