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On the L^2 Cohomology Groups of Isolated Singularities

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Dedicated to Professor Noboru Tanaka on his 60th birthday

Introduction

Let (V, x) be a (complex) n-dimensional isolated singularity. Given a Hermitian metric on $V\backslash\{x\}$, say ds^2, the r-th L^2 cohomology group of V at x is defined as the inductive limit of the L^2 de Rham cohomology groups $H^r_{(2)}(U\backslash\{x\}, ds^2)$, where U runs through the neighbourhoods of x. Recently, L. Saper [10] established a remarkable result that there exist Kähler metrics on $V\backslash\{x\}$, complete near x, for which the r-th L^2 cohomology groups of V at x are zero whenever $r \geq n$. It implies an important fact that the intersection cohomology group of a Kähler variety with isolated singularities carries a canonical Hodge structure. Relying on Saper's result, the author could show that the L^2 cohomology vanishing as above is also true with respect to the restriction of the euclidean metric associated to any holomorphic embedding $(V, x) \hookrightarrow (\mathbf{C}^N, 0)$ (cf. [7]). The purpose of the present article is to complement these works by giving a self-contained version of the latter work. Namely we shall first establish an abstract vanishing theorem as a consequence of a new L^2 estimate with respect to a certain family of metrics and weights which seems to be of interest in itself. Then we shall proceed to apply it to prove a vanishing theorem of Saper type with respect to a certain class of complete Kähler metrics which is actually wider than Saper's ones. Hopefully our method will be available to investigate the L^2 cohomology of spaces with non-isolated singularities. Next we shall give a new proof of our previous result mentioned above. The argument here is essentially the same except that we do not appeal to the existence of a projective variety containing (V, x) and tried to make the argument more transparent. Therefore some part of the proof will be only sketchy.

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§1. Notation and basic facts

We shall first prepare notations and state without proofs several known facts that we use afterwards.

Let (X, ds^2) be a Hermitian manifold of dimension n, and let $C_0(X)$ be the set of compactly supported $\mathbf{C}$-valued C^∞ differential forms on X. We set

$$C_0^r(X) := \{u \in C_0(X); \deg u = r\}$$

and

$$C_0^{p,q}(X) := \{u \in C_0^{p+q}(X); u \text{ is of type } (p,q)\}.$$

Let φ be any real-valued C^∞ function on X. We set

$$(u,v)_\varphi := \int_X e^{-\varphi} u \wedge \overline{*v} \quad \text{for} \quad u, v \in C_0(X),$$

where $*(= *_{ds^2})$ denotes the Hodge's star operator and $\overline{*v}$ the complex conjugate of $*v$. Then $C_0(X)$ is a pre-Hilbert space equipped with the above inner product. We define $L_\varphi(X)(= L_\varphi(X, ds^2))$ to be the completion of $C_0(X)$ with respect to the associated L^2 norm $\| \ \|_\varphi = \sqrt{(\ ,\ )_\varphi}$. We shall refer to φ as the weight of the L^2 norm. For any densely defined closed linear operator, say T, from $L_\varphi(X)$ into itself, we denote its domain, image and kernel by Dom T, Im T and Ker T, respectively. The adjoint of T will be denoted by T_φ^*. As usual φ will not be referred to if $\varphi \equiv 0$. By d we shall denote the exterior derivative, and by $\bar\partial$ (resp.∂) the $(0,1)$-component (resp. $(1,0)$-component) of d. Their maximal closed extensions will be denoted by the same symbol unless there is fear of confusion. By an abuse of language we often identify $\partial\bar\partial\varphi$ with the complex Hessian of φ.

Proposition 0. *Suppose that there exists a C^∞ function $\psi : X \to$ $\mathbf{R}$ such that*

1) $ds^2 = 2\partial\bar\partial\psi$

2) $|\partial\psi|$ *is bounded.*

Then

$$\|u\| \le C(\|\bar\partial u\| + \|\bar\partial^* u\|) \le C(\|du\| + \|d^* u\|)$$

for any $u \in C_0^r(X)$ with $r \ne n$. Here $C = 4 \sup |\partial\psi|$.

For the proof see [8].

We set

$$H^r_{(2)}(= H^r_{(2)}(X, ds^2)) := \operatorname{Ker} d \cap L^r(X)/\operatorname{Im} d \cap L^r(X)$$

$$H^{p,q}_{(2)}(X)(= H^{p,q}_{(2)}(X, ds^2)) := \operatorname{Ker} \bar{\partial} \cap L^{p,q}(X)/\operatorname{Im} \bar{\partial} \cap L^{p,q}(X).$$

One can deduce from Proposition 0 the following.

Proposition 1. *Let (X, ds^2) be a complete Kähler manifold equipped with ψ satisfying* 1) *and* 2). *Then $H^r_{(2)}(X)$ (resp. $H^{p,q}_{(2)}(X)$) is zero whenever $r \neq n$ (resp. $p+q \neq n$). Moreover $H^n_{(2)}(X)$ and $H^{p,n-p}_{(2)}(X)$ $(0 \leq p \leq n)$ are Hausdorff spaces with respect to the quotient topology.*

For the argument needed here, see [1] or [2].

Let V be a reduced irreducible complex space of dimension n which is properly embedded into $\mathbf{C}^N$ so that V contains the origin as the possibly unique singular point. Let $z = (z_1, \cdots, z_N)$ be the coordinate of $\mathbf{C}^N$ and let $\|z\| := (\sum_{i=1}^N |z_i|^2)^{1/2}$. We put $V' = V \backslash \{0\}$ and denote by $\|z\|_{V'}$ the restriction of the function $\|z\|$ to V'. Then $-\partial\bar{\partial} \log \log (\delta \|z\|^{-1}_{V'})$ defines a complete Kähler metric on $V'_\delta := \{z \in V'; \|z\| < \delta\}$. As a corollary of Proposition 1 we have

Proposition 2.

$$H^r_{(2)}(V'_\delta, -\partial\bar{\partial} \log \log (\delta \|z\|^{-1}_{V'})) = 0 \quad if \quad r \neq n$$

and

$$H^{p,q}_{(2)}(V'_\delta, -\partial\bar{\partial} \log \log (\delta \|z\|^{-1}_{V'})) = 0 \quad if \quad p+q \neq n.$$

Moreover
$H^n_{(2)}(V'_\delta, -\partial\bar{\partial} \log \log (\delta \|z\|^{-1}_{V'}))$ and $H^{p,n-p}_{(2)}(V'_\delta, -\partial\bar{\partial} \log \log (\delta \|z\|^{-1}_{V'}))$ are Hausdorff spaces.

Proposition 3.

$$\lim_{\delta \to 0} H^r_{(2)}(V'_\delta, \partial\bar{\partial}(-\log \log \|z\|^{-1}_{V'})) = 0 \quad if \quad r > n$$

and

$$\lim_{\delta \to 0} H^{p,q}_{(2)}(V'_\delta, \partial\bar{\partial}(-\log \log \|z\|^{-1}_{V'})) = 0 \quad if \quad p+q > n.$$

Furthermore the homomorphism

$$\lim_{\delta \to 0} H^r_{(2)}(V'_\delta, \partial\bar{\partial}(-\log \log \|z\|^{-1}_{V'})) \to \lim_{\delta \to 0} H^r(V'_\delta)$$

is bijective if $r < n-1$ *and injective if* $r = n-1$, *and the homomorphism*

$$\lim_{\delta\to 0} H^{p,q}_{(2)}(V'_\delta, \partial\bar\partial(-\log\log\|z\|^{-1}_{V'})) \to \lim_{\delta\to 0} H^{p,q}(V'_\delta)$$

is bijective if $p+q < n-1$ *and injective if* $p+q = n-1$. *Here* $H^r(\,\cdot\,)$ *and* $H^{p,q}(\,\cdot\,)$ *denote respectively the* r*-th de Rham cohomology group and the Dolbeault cohomology group of type* (p,q).

We put $V_\delta := \{z \in V; \|z\| < \delta\}$ and

$$H^r_{(2)}(V_\delta) := H^r_{(2)}(V'_\delta, \partial\bar\partial\|z\|^2_{V'})$$

$$H^{p,q}_{(2)}(V_\delta) := H^{p,q}_{(2)}(V'_\delta, \partial\bar\partial\|z\|^2_{V'})$$

by an abuse of notation.

Proposition 4.
(1) $\lim_{\delta\to 0} H^r_{(2)}(V_\delta) = \lim_{\delta\to 0} H^{p,q}_{(2)}(V_\delta) = 0$ *if* $r, p+q > n$.
(2) *The homomorphism*

$$\lim_{\delta\to 0} H^r_{(2)}(V_\delta) \to \lim_{\delta\to 0} H^r(V'_\delta)$$

is bijective if $r < n-1$ *and injective if* $r = n-1$, *and the homomorphism*

$$H^{p,q}_{(2)}(V_\delta) \to H^{p,q}(V'_\delta)$$

is bijective if $p+q < n-1$ *and injective* $p+q = n-1$.

We note that (1) follows from Proposition 3 via a singular perturbation (cf. [5] or [9]), whereas (2) is a consequence of direct application of Andreottei-Vesentini's vanishing theorem (cf. [5, Supplement]).

So far the results have quite straightforward and self-contained proofs. However, to proceed further we must rely on the following deep result.

Theorem (Hironaka [H]). *There exists a complex submanifold* $\tilde V \subset \mathbf{C}^N \times \mathbf{P}^{N'}$ *for some* N' *such that the projection* $\mathbf{C}^N \times \mathbf{P}^{N'} \to \mathbf{C}^N$ *induces a proper bimeromorphic morphism from* $\tilde V$ *onto* V, *say* π. *Moreover* $(\tilde V, \pi)$ *can be chosen so that*

i) $\pi|_{\tilde V \setminus \pi^{-1}(0)}$ *is bijective.*

ii) $\pi^{-1}(0)$ *is a divisor whose associated line bundle is isomorphic to the restriction of the pull-back, by the projection* $\mathbf{C}^N \times \mathbf{P}^{N'} \to \mathbf{P}^{N'}$, *of the dual of the hyperplane section bundle.*

iii) *The support of* $\pi^{-1}(0)$ *is a divisor of simple normal crossings.*

Once for all we fix a $(\tilde{V}, \pi)$ satisfying i)$\sim$iii). By iii) there exist nonsingular divisors $E_1, \cdots, E_m$ ($E_i \neq E_j$ if $i \neq j$) such that

$$\mathrm{supp}\pi^{-1}(0) = E_1 \cup \cdots \cup E_m.$$

By $(v, w) = (v_1, \cdots, v_k, w_1, \cdots, w_{n-k})$ we denote a coordinate around a k-ple point of $\mathrm{supp}\pi^{-1}(0)$ such that $v_1 \cdot \cdots \cdot v_k = 0$ is a local defining equation of $\mathrm{supp}\pi^{-1}(0)$. By ii) there exist positive integers $p_1, \cdots, p_m$ such that the sheaf $\otimes_{i=1}^m \mathcal{O}(-E_i)^{p_i}$ is very ample. Hence there exists a nonsingular integral $m \times m$ matrix (p_{ij}) with $p_{ij} > 0$ such that

1) $\otimes_{i=1}^m \mathcal{O}(-E_i)^{p_{ij}}$ are ample for all j.

2) Let $1 \leq i_1 < \cdots < i_k \leq m$ $(1 \leq k \leq m)$. Then $\det(p_{i_\alpha i_\beta})_{\alpha,\beta=1}^k \neq 0$ whenever $\bigcap_{\alpha=1}^k E_{i_\alpha} \neq \emptyset$.

Therefore we can find C^∞ metrics along the fibers of $\otimes_{i=1}^m \mathcal{O}(-E_i)^{p_{ij}}$, say a_j, whose curvature form is positive. Let $s_i \in \Gamma(\tilde{V}, \mathcal{O}(E_i))$ be so chosen that $E_i = \{y \in \tilde{V}; s_i(y) = 0\}$, and let σ_j be the length of $s_1^{p_{1j}} \cdot \cdots \cdot s_m^{p_{mj}}$ with respect to a_j. Then $-\log \log \sigma_j^{-1}$ is a plurisubharmonic function on a neighbourhood of $\mathrm{supp}\pi^{-1}(0)$, say U. We set

$$d\sigma^2 := -\partial\bar{\partial} \sum_{j=1}^m \log \log \sigma_j^{-1} \quad \text{on} \quad U \backslash \mathrm{supp}\pi^{-1}(0).$$

Then $d\sigma^2$ may well be identified via π with a Kähler metric on $V'_\delta := V_\delta \backslash \{0\}$ for sufficiently small δ. We shall refer to $d\sigma^2$ as a Saper metric afterwards. We note that, around any k-ple point of $\mathrm{supp}\pi^{-1}(0)$,

$$\text{(3)} \qquad d\sigma^2 \sim \sum_{i=1}^k \frac{dv_i d\overline{v_i}}{|v_i|^2 \log^2 |v_i \cdot \cdots \cdot v_k|^{-1}} + \frac{1}{\log |v_1 \cdot \cdots \cdot v_k|^{-1}} \Big(\sum_{i=1}^k dv_i d\overline{v_i} + \sum_{j=1}^{n-k} dw_j d\overline{w_j} \Big),$$

where $A \sim B$ means that there exists a $c \in (0, \infty)$ such that $c^{-1}A \leq B \leq cA$.

The following is also an immediate consequence of Proposition 3.

Proposition 5. *For sufficiently small δ and a Saper metric $d\sigma^2$ on V'_δ,*

1) $H^r_{(2)}(V'_\delta, d\sigma^2) = H^{p,q}_{(2)}(V'_\delta, d\sigma^2) = 0$ *if* $r, p+q > n$.

2) *The canonical homomorphisms*

$$H^r_{(2)}(V'_\delta, d\sigma^2) \to H^r(V'_\delta)$$

are bijective if $r < n-1$ and injective if $r = n-1$.

3) *The canonical homomorphisms*

$$H^{p,q}_{(2)}(V'_\delta, d\sigma^2) \to H^{p,q}(V'_\delta)$$

are bijective if $p+q < n-1$ and injective if $p+q = n-1$.

We call a Saper metric $d\sigma^2$ dominating if $d\sigma^2 \gtrsim -\partial\bar\partial \log \log \|z\|^{-1}_{V'}$. Here $A \gtrsim B$ means that $cA \geq B$ for some $c \in (0, \infty)$. Existence of a dominating Saper metric is assured also by Hironaka's theorem. Namely, applying Hironaka's desingularization theorem in a more precise form, we can find $(\tilde V, \pi)$ so that the maximal ideal of 0 is pulled-back by π to an invertible sheaf (cf. [H]). For such $\tilde V$ it is clear that $d\sigma^2 \gtrsim -\partial\bar\partial \log \log \|z\|^{-1}_{V'}$.

§2. An abstract L^2 vanishing theorem

In what follows we assume that X admits a C^∞ negative plurisubharmonic function φ such that $-\log(-\varphi)$ is strictly plurisubharmonic, and derive an L^2 estimate for the $\bar\partial$-operator with respect to the metrics $d\sigma^2_\varepsilon := 2(-\partial\bar\partial \log(-\varphi) + \varepsilon\partial\bar\partial\varphi)$ $(\varepsilon \geq 0)$ and weights $-\varepsilon\varphi$.

For simplicity we set

$$L_\varepsilon(X) := L_{-\varepsilon\varphi}(X, d\sigma^2_\varepsilon)$$

$$(u, v)_\varepsilon := \int_X e^{\varepsilon\varphi} u \wedge \overline{*_\varepsilon v},$$

where $*_\varepsilon$ denotes the Hodge's star operator with respect to $d\sigma^2_\varepsilon$, and $\|u\|_\varepsilon := \sqrt{(u, u)_\varepsilon}$.

Note that $L_\varepsilon(X) \supset L_\delta(X)$ if $\varepsilon > \delta$.

The adjoint of an operator T with respect to $(\ ,\ )_\varepsilon$ will be denoted by T^*_ε by an abuse of notation. For simplicity we set $\Lambda_\varepsilon := *_\varepsilon^{-1} e(\sqrt{-1}(\partial\bar\partial(-\log(-\varphi) + \varepsilon\varphi))) *_\varepsilon$, where $e(\,\cdot\,)$ stands for the exterior multiplication from the left hand side.

Proposition 6. *If $p+q<n$,*

$$\|u\|_\varepsilon^2 \le 8(\|\bar\partial u\|_\varepsilon^2 + \|\bar\partial_\varepsilon^* u\|_\varepsilon^2)$$

for any $u \in C_0^{p,q}(X)$ and $\varepsilon > 0$.

Proof. Since $|\partial \log(-\varphi)|_{d\sigma_\varepsilon^2} \le 1$ we have

$$\begin{aligned}&([\sqrt{-1}e(\partial\bar\partial \log(-\varphi)), \Lambda_\varepsilon]u, u)_\varepsilon \\ \le&\|u\|_\varepsilon(\|\bar\partial u\|_\varepsilon + \|\bar\partial_\varepsilon^* u\|_\varepsilon + \|\partial^* u\|_\varepsilon + \|\partial_\varepsilon u\|_\varepsilon).\end{aligned}$$

Here we put $\partial_\varepsilon := (\partial^*)_\varepsilon^*$. Hence for any $C \ge 1$ and $\sigma > 0$ we have

$$\begin{aligned}(4)\qquad &([\sqrt{-1}e(\partial\bar\partial \log(-\varphi)), \Lambda_\varepsilon]u, u)_\varepsilon \\ \le&2\sigma\|u\|_\varepsilon^2 + \frac{1}{2}C\sigma^{-1}(\|\bar\partial u\|_\varepsilon^2 + \|\bar\partial_\varepsilon^* u\|_\varepsilon^2 + \|\partial^* u\|_\varepsilon^2 + \|\partial_\varepsilon u\|_\varepsilon^2).\end{aligned}$$

Since

$$\begin{aligned}&\|\partial^* u\|_\varepsilon^2 + \|\partial_\varepsilon u\|_\varepsilon^2 \\ =&\|\bar\partial u\|_\varepsilon^2 + \|\bar\partial_\varepsilon^* u\|_\varepsilon^2 + ([\sqrt{-1}e(\varepsilon\partial\bar\partial\varphi), \Lambda_\varepsilon]u, u)_\varepsilon,\end{aligned}$$

we have

$$\begin{aligned}&([\sqrt{-1}e(\partial\bar\partial \log(-\varphi) - \tfrac{\varepsilon C}{2\sigma}\partial\bar\partial\varphi), \Lambda_\varepsilon]u, u)_\varepsilon - 2\sigma\|u\|_\varphi^2 \\ \le&C\sigma^{-1}(\|\bar\partial u\|_\varepsilon^2 + \|\bar\partial_\varepsilon^* u\|_\varepsilon^2),\end{aligned}$$

so that

$$\begin{aligned}&((1 - \frac{C}{2\sigma})[\sqrt{-1}e(\varepsilon\partial\bar\partial\varphi), \Lambda_\varepsilon]u, u)_\varepsilon + (1-2\sigma)\|u\|_\varepsilon^2 \\ \le&C\sigma^{-1}(\|\bar\partial u\|_\varepsilon^2 + \|\bar\partial_\varepsilon^* u\|_\varepsilon^2).\end{aligned}$$

Since $\partial\bar\partial \log(-\varphi) = -\varphi^{-1}\partial\bar\partial\varphi + \varphi^{-2}\partial\varphi\bar\partial\varphi$,

$$([\sqrt{-1}e(\partial\bar\partial\varphi), \Lambda_\varepsilon]u, u)_\varepsilon \le 0$$

if $\deg u < n$. Hence, letting $\sigma = \dfrac{1}{4}$ and $C = 1$ we obtain

$$\|u\|_\varepsilon^2 \le 8(\|\bar\partial u\|_\varepsilon^2 + \|\bar\partial_\varepsilon^* u\|_\varepsilon^2)$$

for all $u \in C_0^{p,q}(X)$ with $p+q<n$. □

Now we can state our vanishing theorem.

Theorem 7. *Let X be a complex manifold of dimension n admitting a negative plurisubharmonic function φ such that $-\partial\bar{\partial}\log(-\varphi)$ is a complet Kähler metric. Take any $f \in L^{p,q}(X, -\partial\bar{\partial}\log(-\varphi))$ with $p+q \leq n$. Then $f \in \operatorname{Im}\bar{\partial}$ if and only if there exist $g_\varepsilon \in L_\varepsilon(X)$ for every $\varepsilon > 0$ such that $\bar{\partial}g_\varepsilon = f$.*

Proof. Since $L_\varepsilon(X) \supset L_0(X)$, 'only if' part is clear. To prove 'if' part, one has only to apply Proposition 6.

□

We note that

$$\partial_\varepsilon u = \partial u + \varepsilon\partial\varphi \wedge u.$$

Hence

$$\|\partial u\|_{2\varepsilon}^2 \leq \|\partial_\varepsilon u\|_\varepsilon^2 + 4e^{-2}\|u\|_\varepsilon^2,$$

since $\sup e^{\varepsilon\varphi}|\varepsilon\partial\varphi|_{d\sigma_0^2}^2 \leq \sup_{t\in(-\infty,0)} e^t \cdot t^2 = 4e^{-2}$.

Therefore we have

$$\|\partial g\|_{2\varepsilon}^2 \leq A(\|g\|_\varepsilon^2 + \|\bar{\partial}g\|_\varepsilon^2 + \|\bar{\partial}_\varepsilon^* g\|_\varepsilon^2) \tag{5}$$

for any $g \in \operatorname{Dom}(\bar{\partial} + \bar{\partial}_\varepsilon^*)$. Here we may choose $A = n \cdot 2^n + 4e^{-2}$. Thus we obtain the following version of Theorem 7.

Theorem 8. *Let X and φ be as above, and take any $f \in L^r(X, -\partial\bar{\partial}\log(-\varphi))$ with $r \leq n$. Then $f \in \operatorname{Im} d$ if and only if there exist $g_\varepsilon \in L_\varepsilon^{r-1}(X)$ for every $\varepsilon > 0$ such that $dg_\varepsilon = f$.*

§3. Application of a topological lemma

Let $(V, 0) \hookrightarrow (\mathbf{C}^N, 0)$ be as before, and let $\rho : W \to V$ be any proper holomorphic map such that $\rho|_{W\setminus\rho^{-1}(0)}$ is bijective and W is nonsingular. We set $W_\delta = \rho^{-1}(V_\delta)$ and $W_\delta' = W_\delta \setminus \rho^{-1}(0)$. The following fact, first pointed out in [4], is crucial for our purpose.

Lemma 9. *The canonical homomorphisms*

$$H^r(W_\delta) \to H^r(\partial W_\delta)$$

are surjective for $r < n$ if $0 < \delta \ll 1$. Here ∂W_δ denotes the boundary of W_δ.

For the proof, see [3] or [6].

Let ds^2 be a Hermitian metric on V'. We put

$$H^r_{(2),0}(W'_\delta, \rho^* ds^2) :=$$
$$\{u \in L^r(W'_\delta, \rho^* ds^2); du = 0 \quad \text{and} \quad \operatorname{supp} u \Subset W_\delta\}$$
$$/\{u \in L^r(W'_\delta, \rho^* ds^2); \exists v \in L^{r-1}(W'_\delta, \rho^* ds^2) \quad \text{such that}$$
$$\operatorname{supp} v \Subset W_\delta \quad \text{and} \quad dv = u\}.$$

Then Lemma 9 implies the following.

Proposition 10. *Let $r < n$. Suppose that the metric ds^2 enjoys a property that $C^r_0(W_\delta) \subset L^r(W'_\delta, \rho^* ds^2)$ for $\delta > 0$. Then the canonical homomorphism*

$$H^{r+1}_{(2),0}(W'_\delta, \rho^* ds^2) \to H^{r+1}_{(2)}(W'_\delta, \rho^* ds^2)$$

is injective for $0 < \delta \ll 1$.

Proof. Let $u \in L^{r+1}(W'_\delta, \rho^* ds^2)$, $\operatorname{supp} u \Subset W_\delta$ and $du = 0$. Assume that there exist a $v \in L^r(W'_\delta, \rho^* ds^2)$ satisfying $dv = u$. If δ is chosen so that $d\|z\|_{V'} \neq 0$ on $\partial V_{\delta'}$ for all $\delta' \in (0, \delta]$, from Lemma 9 there exists a measurable $r-1$ form g on W_δ with $\operatorname{supp} g \cap W_{\delta/2} = \emptyset$ such that g and dg are locally square integrable on W_δ and a locally square integrable d-closed r form w on W_δ, C^∞ on $W_{\delta/2}$, such that $v = w + dg$ outside a compact subset of W_δ. By assumption $v - w - dg \in L^r(W'_\delta, \rho^* ds^2)$. Since $\operatorname{supp}(v - w - dg) \Subset W_\delta$ and $d(v - w - dg) = u$, the assertion was proved. □

Corollary 11. *Under the above situation, suppose moreover that ds^2 is complete and $r = n-1$. Then the homomorphism*

$$H^n_{(2),0}(W'_\delta, \rho^* ds^2) \to H^n_{(2)}(W'_\delta, \rho^* ds^2) \quad (0 < \delta \ll 1)$$

has a dense image.

Proposition 12. *Let $d\sigma^2$ be a Saper metric on V'_δ associated to a desingularization $\pi : \tilde{V} \to V$, and let $\tilde{V}_\delta := \pi^{-1}(V_\delta)$. Then*

$$C^r_0(\tilde{V}_\delta) \subset L^r(\tilde{V}_\delta \backslash \operatorname{supp} \pi^{-1}(0), d\sigma^2).$$

Proof. Let $u \in C^r_0(\tilde{V}_\delta)$ be any element, and let D be a neighbourhood of a k-ple point of $\operatorname{supp}\pi^{-1}(0)$ with coordinate (v, w) as described before. Since $d\sigma^2$ satisfies (3), we have

$$|u|^2 \lesssim (\log |v_1 \cdot \cdots \cdot v_k|^{-1})^r$$

if $|v| < 1/2$. Let $dV_{(\sigma)}$ be the volume form of $d\sigma^2$. Then (3) implies that

$$dV_{(\sigma)} \sim |v_1 \cdot \cdots \cdot v_k|^{-2}(\log |v_1 \cdot \cdots \cdot v_k|^{-1})^{-n-k}.$$

Therefore, if $r < n$

$$\begin{aligned}
&\int_D |u|^2 dV_{(\sigma)} \\
&\lesssim \int_0^{1/2} \cdots \int_0^{1/2} (t_1 \cdot \cdots \cdot t_k)^{-1} \\
&\qquad \times (\log ((t_1 \cdot \cdots \cdot t_k)^{-1})^{-n-k+r} dt_1 \cdot \cdots \cdot dt_k \\
&\leq \int_0^{1/2} \cdots \int_0^{1/2} (t_1 \cdot \cdots \cdot t_k)^{-1} \\
&\qquad \times (\log ((t_1 \cdot \cdots \cdot t_k)^{-1})^{-k-1} dt_1 \cdot \cdots \cdot dt_k \\
&\leq (\int_0^{1/2} t^{-1}(\log t^{-1})^{-1-k^{-1}} dt)^k < \infty.
\end{aligned}$$

□

§4. A homotopy operator

Let $\pi : \widetilde{V}_{(\delta)} \to V$ be as before. Once for all we fix C^∞ metrics along the fibers of $\mathcal{O}(E_i)$ and denote by $|s_i|$ the length of the canonical section s_i of $\mathcal{O}(E_i)$. Then we put $s := \min_i |s_i|$ and $\widetilde{V}_{(\delta)} := \{y \in \tilde{V}; s < \sigma\}$. Note that $\widetilde{V}_{(\delta)}$ is a tubular neighbourhood of $\mathrm{supp}\pi^{-1}(0)$ if $0 < \sigma \ll 1$. We may choose δ so that $\partial\widetilde{V}_{(\delta)}$ is piecewise smooth and there exists a piecewise smooth retraction $r_\delta : \widetilde{V}_{(\delta)} \backslash \mathrm{supp}\pi^{-1}(0) \to \partial\widetilde{V}_{(\delta)}$ which is up to a local diffeomorphism of $\widetilde{V}_{(\delta)}$ of the form

$$(v, w) \to \big((\delta + |v_1| - |v_i|)e^{\arg v_1}, \cdots, \delta e^{\arg v_i}, \cdots, (\delta + |v_k| - |v_i|)e^{\arg v_k}, w\big)$$

on $\{y; |v_i(y)| = \min_{1\leq j\leq k} |v_j(y)|\}$. Note that any differential form f on $\widetilde{V}_{(\delta)} \backslash \mathrm{supp}\pi^{-1}(0)$ splits into the sum $ds \wedge f_0 + f_1$, where $f_i = g_i \cdot r_\delta^* h_i$ for some functions g_i and differential forms h_i on $\partial\widetilde{V}_{(\delta)}$ in the piecewise smooth sense. For any $u \in C_0(\widetilde{V}_{(\delta)})$, with a splitting $u = ds \wedge u_0 + u_1$ as above, we put

$$K_\delta u := \int_\delta^s u_0(t, \cdot\,)dt,$$

where t denotes the s-variable. Clearly K_δ is extendable by continuity to a linear operator on the space of locally square integrable forms on $\overline{\widetilde{V}_{(\delta)}}\backslash \mathrm{supp}\pi^{-1}(0)$, which shall be denoted also by K_δ. Note that $d(K_\delta u)$ if $du = 0$ and supp$u \Subset \widetilde{V}_{(\delta)}$.

§5. L^2 vanishing theorems for isolated singularities

From now on we put $V'_{(\delta)} := \widetilde{V}_{(\delta)}\backslash \mathrm{supp}\pi^{-1}(0)$. Combining Theorem 8 with Proposition 2 and Corollary 11 we obtain the following.

Theorem 13. *Let φ be a C^∞ negative plurisubharmonic function on V'_{δ_0} $(0 < \delta_0 \ll 1)$ such that $ds^2 := 2\partial\bar\partial(-\log(-\varphi))$ is a complete Kähler metric on V'_{δ_0}. Suppose that the following conditions are satisfied.*

(a) $C_0^{n-1}(\widetilde{V}_{(\delta_0)}) \subset L^{n-1}(V'_{\delta_0}, ds^2)$

(b) *K_δ extends to a continuous linear map from $L^n(V'_{\delta_0}, ds^2)$ to $L^{n-1}_{-\varepsilon\varphi}(V'_{\delta_0}, ds^2 + 2\varepsilon\partial\bar\partial\varphi)$ if $\varepsilon > 0$ and $V'_{(\delta)} \Subset \widetilde{V}_{\delta_0}$.*

Then

$$\lim_{\delta\to 0} H^n_{(2)}(V'_\delta, ds^2) = 0.$$

Our next task is to apply Theorem 13 to prove that $\lim_{\delta\to 0} H^n_{(2)}(V'_\delta, d\sigma^2) = 0$ for any Saper metric $d\sigma^2$.

Lemma 14. *Let $d\sigma^2$ be a Saper metric on V'_δ. Then there exists a negative C^∞ plurisubharmonic function φ on V'_δ such that*

(i) $2\partial\bar\partial(-\log(-\varphi)) = d\sigma^2$ *on* $V'_{\delta/2}$.

(ii) $2\partial\bar\partial(-\log(-\varphi))$ *is a complete Kähler metric on* V'_δ.

Proof. Let σ_i be as in §1 and put

$$\varphi_\eta := -\prod_{i=1}^m (-\log\sigma_i)^\eta, \qquad \text{for} \quad \eta \in (0,1).$$

Then

$$\partial\bar\partial\varphi_\eta = (-\varphi_\eta)\left\{\sum_{i=1}^m (-\log\sigma_i)^{-\eta}\partial\bar\partial(-(-\log\sigma_i)^\eta) - \eta^2\sum_{i,j}\frac{\partial\log\sigma_i}{\log\sigma_i}\frac{\bar\partial\log\sigma_j}{\log\sigma_j}\right\}.$$

Since $\partial\bar\partial(-(-\log\sigma_i)^\eta) \geq \eta(1-\eta)(-\log\sigma_i)^{\eta-2}\partial\log\sigma_i\bar\partial\log\sigma_i$, we obtain $\partial\bar\partial\varphi_\eta \geq 0$ if $0 < \eta \leq 1/2$. Let λ be a C^∞ convex increasing function such that $\lambda(t) = -\frac{1}{2}\log 2$ on $(-\infty, -\log 2)$ and $\lambda(t) = t$ on $(-\frac{1}{2}\log 2, \infty)$. Then we put

$$\varphi = \varphi_{1/2} + \lambda\big(\log(\delta\|z\|_{V'}^{-1})\big).$$

Clearly φ satisfies (i) and (ii). □

From Proposition 12 it follows immediately that (a) is true for $d\sigma^2$ since so is it for $2\partial\bar{\partial}(-\log(-\varphi))$, where φ is as above. We are going to show that (b) is also true for this choice of φ.

Take any k-ple point $x \in \operatorname{supp}\pi^{-1}(0)$ and a neighbourhood $D \ni x$ with a local coordinate (v, w) around x as before. From the obvious asymptotics of $d\sigma^2$ and $\partial\bar{\partial}\varphi$ around x, the metric $d\sigma_\varepsilon^2 = d\sigma^2 + \varepsilon\partial\bar{\partial}\varphi$ is estimated as

$$d\sigma_\varepsilon^2 \gtrsim \sum_{i=1}^{k} \frac{dv_i d\bar{v}_i}{|v_i|^2(\log s)^2} + \frac{1}{-\log s}\sum_{j=1}^{n-k} dw_j d\bar{w}_j \tag{6}$$

and

$$d\sigma_\varepsilon^2 \lesssim \sum_{i=1}^{k} \frac{dv_i d\bar{v}_i}{|v_i|^2} + \sum_{j=1}^{n-k} dw_j d\bar{w}_j. \tag{7}$$

Let $D_i = \{y \in D; |v_i(y)| = \min_{1\le j\le k} |v_j(y)|\}$. We shall estimate $\|K_{\delta'}u\|_{\varepsilon, D_i}$ $(\delta' \ll \delta)$ for each i. Fixing i we set $t_j = |v_j| - |v_i|$ for $j \neq i$. Furthermore we put $\theta_j = \arg v_j$ for $1 \le j \le k$. Then (6) and (7) are rewritten in terms of a (piecewise smooth) local coordinate $(s, t_1, \cdots \overset{i}{\vee} \cdots, t_k, \theta_1, \cdots, \theta_k, w_1, \cdots, w_{n-k})$ as

$$\begin{aligned}\operatorname{Re} d\sigma_\varepsilon^2 \gtrsim & \frac{ds^2}{s^2(\log s)^2} + \sum_{j\neq i} \frac{dt_j^2}{(t_j+s)^2(\log s)^2} \\ & + \sum_{i=1}^{k} \frac{d\theta_i^2}{(\log s)^2} + \frac{1}{-\log s}\operatorname{Re}\sum_{j=1}^{n-k} dw_j d\bar{w}_j\end{aligned} \tag{8}$$

and

$$\text{(9)}\qquad \begin{aligned} \operatorname{Re} d\sigma_\varepsilon^2 \lesssim \frac{ds^2}{s^2} + \sum_{j\neq i} \frac{dt_j^2}{(t_j+s)^2} + \sum_{i=1}^{k} d\theta_i^2 \\ + \operatorname{Re} \sum_{j=i}^{n-k} dw_j d\bar{w}_j. \end{aligned}$$

Take any $\delta' > 0$ with $V'_{(\delta')} \Subset \widetilde{V}_\delta$ and let $u = ds \wedge u_0(s,\cdot) + u_1(s,\cdot) \in C_0^n(V'_{(\delta')})$, where u_0 and u_1 are determined as before. Then we put

$$\|u_0\|^2_{(\varepsilon),t} := \int_{\{y;s(y)=t\}} |u_0|^2_\varepsilon dV_{\varepsilon,t} \quad \text{for} \quad t < \sigma',$$

where $dV_{\varepsilon,t}$ denotes the volume form with respect to $d\sigma_\varepsilon^2|\{y; s(y) = t\}$ (in the piecewise smooth sense). Note that $s \lesssim |ds|_\varepsilon \lesssim 1$ and

$$\|u_0\|^2_{(\varepsilon),t} \lesssim (\log t^{-1})^{2n} \|u\|^2_{(0),t}$$

by (8) and (9). Therefore

$$\begin{aligned} &\|K_{\delta'}u\|^2_{\varepsilon,D_i} \\ &= \left\| \int_{\delta'}^{s} u_0(t,\cdot)dt \right\|^2_{\varepsilon,D_i} \\ &\lesssim \int_0^{\delta'} \left(\int_{\delta'}^{s} \|u_0\|^2_{(\varepsilon),s} |ds|_0^{-1} ds \int_{\delta'}^{s} |ds|_0 ds \right) s^{\varepsilon/2} |ds|_\varepsilon^{-1} ds \\ &\lesssim \int_0^{\delta'} \|u\|_0^2 s^{\varepsilon/2-1} (\log s^{-1})^{2n+1} ds \lesssim \|u\|_0^2 \end{aligned}$$

if $\varepsilon > 0$.

Thus we have verified (b) for φ. Consequently we obtain the following.

Theorem 15.

$$\lim_{\delta\to 0} H^n_{(2)}(V'_\delta, d\sigma^2) = 0$$

for any Saper metric $d\sigma^2$.

We now turn our attention to more general metrics. First we prepare a comparison lemma.

Lemma 16. *Let X be a complex manifold, let ds_i^2 $(i = 0, 1)$ be C^∞ Hermitian metrics on X satisfying $ds_0^2 \lesssim ds_1^2$, and let $\Omega \subset X$ be a domain whose boundary $\partial\Omega$ is compact. With respect to the metrics $ds_\varepsilon^2 := \varepsilon ds_0^2 + (1-\varepsilon) ds_1^2$, $\varepsilon \in [0,1]$, with associated L^2 norms $\| \ \|_\varepsilon$, suppose that ds_i^2 are complete and there exist a compact subset $K \subset \bar{\Omega}$ and a constant C independent of $\varepsilon \in [0,1]$ such that*

$$\|u\|_{\varepsilon,\Omega} \leq C(\|u\|_{\varepsilon,K} + \|du\|_{\varepsilon,\Omega} + \|d^*_{\varepsilon,\Omega} u\|_{\varepsilon,\Omega}) \tag{10}$$

for any $u \in \mathrm{Dom}(d + d^*_{\varepsilon,\Omega}) \cap L^{r\pm1}(\Omega, ds_\varepsilon^2)$. Here $d^*_{\varepsilon,\Omega}$ denotes the adjoint of d with respect to $\| \ \|_{\varepsilon,\Omega}$ and r is a nonnegative integer. Then $\dim H^{r\pm1}_{(2)}(\Omega, ds_i^2) < \infty$. *Moreover*

$$\dim H^r_{(2)}(\Omega, ds_0^2) \leq \dim H^r_{(2)}(\Omega, ds_1^2) \tag{11}$$

if

$$\dim H^{r+j}_{(2)}(\Omega, ds_0^2) \leq \dim H^{r+j}_{(2)}(\Omega, ds_\varepsilon^2) \tag{12}$$

hold for $j = \pm1$ and $\varepsilon \in [0,1]$.

Proof. That $\dim H^{r\pm1}_{(2)}(\Omega, ds_\varepsilon^2) < \infty$ follows from (10) is well known (cf. [2]). Suppose moreover that (12) holds. Then there must exist a constant C' such that

$$\|u\|_{\varepsilon,\Omega} \leq C'(\|du\|_{\varepsilon,\Omega} + \|d^*_{\varepsilon,\Omega} u\|_{\varepsilon,\Omega}) \tag{13}$$

if $u \in \mathrm{Dom}(d + d^*_{\varepsilon,\Omega}) \cap L^{r\pm1}(\Omega, ds_\varepsilon^2) \ominus \mathrm{Ker}(d + d^*_{\varepsilon,\Omega})$. (See [8] for the argument.) (13) shows that $\dim H^r_{(2)}(\Omega, ds_0^2) \leq \dim \mathrm{Ker}(d + d^*_{\varepsilon,\Omega}) = \dim H^r_{(2)}(\Omega, ds_\varepsilon^2)$. □

By Lemma 16, we have the following generalization of Theorem 13.

Proposition 17. *Let φ and V'_{δ_0} be as in Theorem* 13, *and let ψ be a C^∞ plurisubharmonic function on V'_{δ_0} such that*

1) *$\partial\bar{\partial}\psi$ is a complete Kähler metric*
2) *$|\partial\psi|_{\partial\bar{\partial}\psi}$ is bounded*
3) *$\partial\bar{\partial}\psi \lesssim \partial\bar{\partial}(-\log(-\varphi))$.*

Then $\lim\limits_{\delta\to0} H^n_{(2)}(V'_\delta, \partial\bar{\partial}\psi) = 0$.

Proof. We put $ds_0^2 = \partial\bar{\partial}\psi$ and $ds_1^2 = \partial\bar{\partial}(-\log(-\varphi))$. Then we can apply Lemma 16 in virtue of Proposition 1. □

Thus the existence of dominating Saper metrics implies the following.

Corollary 18. $\lim_{\delta\to 0} H^n_{(2)}(V'_\delta, \partial\bar\partial(-\log\log\|z\|^{-1}_{V'})) = 0.$

Finally we shall prove the L^2 cohomology vanishing with respect to $\partial\bar\partial\|z\|^2_{V'}$. For that purpose we prepare another lemma.

Lemma 19. *Let Ω and ds^2_ε be as in Lemma* 16 *except that ds^2_0 is not necessarily complete and instead of* (10) *we assume the estimate*

$$(14) \qquad \|\eta_\varepsilon u\|_{\varepsilon,\Omega} \le C(\|u\|_{\varepsilon,K} + \|du\|_{\varepsilon,\Omega} + \|d^*_{\varepsilon,\Omega}u\|_{\varepsilon,\Omega})$$

for any $\varepsilon \in (0,1]$ and $u \in \mathrm{Dom}(d + d^*_{\varepsilon,\Omega}) \cap L^{r\pm 1}(\Omega, ds^2_\varepsilon)$. Here η_ε are continuous functions on Ω with values in $(1,\infty)$ such that

(15) $\eta_\varepsilon \to \eta_0$ *uniformly on compact subsets of* Ω.

(16) *There exists a sequence of C^∞ functions $\{\chi_\mu\}_{\mu=1}^\infty$ on $\bar\Omega$ satisfying*

i) $|d\chi_\mu|_{ds^2_0} \le \eta_0$

ii) $\mathrm{supp}\chi_\mu$ *is compact and* $\bigcup_{\mu=1}^\infty \mathrm{supp}\chi_\mu = \bar\Omega$

iii) $0 \le \chi_\mu \le 1$ *and* $\chi_\mu \equiv 1$ *on* $\mathrm{supp}\chi_{\mu-1}$.

Assume moreover that

$$(17) \qquad \dim H^{r\pm1}_{(2)}(\Omega, ds^2_0) \le \dim H^{r\pm1}_{(2)}(\Omega, ds^2_1).$$

Then $\dim H^r_{(2)}(\Omega, ds^2_1) \le \dim H^r_{(2)}(\Omega, ds^2_1)$.

Proof. To be precise, let $d_{\max}$ and $d_{\min}$ denote respectively the maximal and the minimal closed extensions of d on $L(\Omega, ds^2_0)$. By (16,i) we have

$$\mathrm{Dom}\, d_{\max} \cap \{u \in L(\Omega, ds^2_0);\, \|\eta_0 u\|_{0,\Omega} < \infty\} \subset \mathrm{Dom}\, d_{\min}.$$

Similarly $u \in \mathrm{Dom}\, d^*_{\max}$ if $\|\eta_0 u\|_{0,\Omega} < \infty$ and $\chi_\mu u \in \mathrm{Dom}\, d^*_{\max}$ for all μ. Suppose that $\dim H^r_{(2)}(\Omega, ds^2_0) > \dim H^r_{(2)}(\Omega, ds^2_1)$. Then there must exist a finite dimensional subspace $W \subset L^r(\Omega, ds^2_0) \cap \mathrm{Ker}\, d_{\max}$ consisting of 0 and non-$d_{\max}$-exact forms, and a sequence $f_\mu \in W$ $(\mu = 1, 2, \cdots)$ such that $\|f_\mu\|_0 = 1$ and $\chi_\mu f_\mu \perp \mathrm{Ker}\,(d + d^*_{1/\mu,\Omega})$ in $L^r(\Omega, ds^2_{1/\mu})$. Therefore, by (14) and (17) there must exist a constant $C', g_\mu \in L^{r-1}(\Omega, ds^2_{1/\mu})$ and $h_\mu \in L^{r+1}(\Omega, ds^2_{1/\mu})$ such that

$$\begin{cases} \chi_\mu f_\mu = dg_\mu + d^*_{1/\mu,\Omega} h_\mu \\ \|\varphi_{1/\mu} g_\mu\|_{1/\mu,\Omega} \le C' \\ \|\varphi_{1/\mu} h_\mu\|_{1/\mu,\Omega} \le C'. \end{cases}$$

Choosing weakly convergent subsequences of f_μ, g_μ and h_μ we thus obtain $f \in W$ $g \in \operatorname{Dom} d_{\min}$ and $h \in \operatorname{Dom} d^*_{\max}$ such that $f = d_{\min} g + d^*_{\max} h$. Since $f \in \operatorname{Ker} d_{\max}$, $d^*_{\max} h = 0$. Therefore $f = 0$. On the other hand $f \neq 0$ since $\|f_\mu\|_{1/\mu} = 1$ and W is finite dimensional. This is a contradiction.

□

Combining Corollary 18 and Lemma 19 we obtain the following.

Theorem 20.

$$\lim_{\delta \to 0} H^n_{(2)}(V'_\delta, \partial\bar{\partial}\|z\|^2_{V'}) = 0.$$

Proof. Put $ds_0^2 = \partial\bar{\partial}\|z\|^2_{V'}, ds_1^2 = \partial\bar{\partial}(-\log\log\|z\|^{-1}_{V'})$ and let η_ε be the smallest eigenvalue of $\partial\bar{\partial}(-\log\log\|z\|^{-1}_{V'})$ with respect to $(1-\varepsilon)ds_0^2+\varepsilon ds_1^2$. Since the other eigenvalues of $\partial\bar{\partial}(-\log\log\|z\|^{-1}_{V'})$ are equal to each other, we have the estimate (14) for $r = n$ (cf. [8]). (16) follows from the fact that $(t\log t)^{-1}$ is non-integrable on $(0, 1/2)$. (15) is trivial. (17) is a consequence of Corollary 18 together with Proposition 3 and Proposition 4. □

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