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Non-Commutative Complex Projective Space

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§0. Introduction

The concept of quantized manifolds has much interest from a geometrical point of view. In fact, quantum groups [6] and non-commutative tori [4] [12] are typical examples in this spirit. One approach to constructing quantized manifolds is based on the deformation quantization introduced by Bayen et al [1]. This is the deformation of the Poisson algebra of functions on a symplectic manifold via a star product.

However, deformation quantization providing only an algebraic description does not seem to describe the "underlying space" adequately. From the geometric point of view, we want to construct something like non-commutative manifolds which just represent the quantum state space.

For this purpose, we introduced the notion of Weyl manifolds [10], [11] as a prototype of non-commutative manifolds. A Weyl manifold W_M is defined as a certain algebra bundle over a symplectic manifold M with the formal Weyl algebra as the fiber. The star product given by the deformation quantization is realized on a certain class of sections on W_M, called *Weyl functions.* We present in this paper a *non-commutative complex projective space* $W_{P_n(\mathbf{C})}$ as an example of a Weyl manifold.

There are two ways of constructing star products on $P_n(\mathbf{C})$. The first is *intrinsic*, and was initiated by Berezin [2], who gave a covariant symbol calculus for certain operators acting on local holomorphic functions on the 2-sphere and on the Lobachevskii plane, and defined the star product on these spaces by using the symbol calculus. Moreno [9] and Cahen-Gutt-Rawnsley [3] extended these ideas to Kaehler symmetric spaces.

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The second construction, which is in fact the aim of this paper, is *extrinsic*. We shall regard the ring of Weyl functions on $P_n(\mathbf{C})$ as the subalgebra of all $\mathbf{C}^*$-invariant Weyl functions on $\mathbf{C}^{n+1} - \{0\}$, where one can define the star product and the Weyl manifold structure naturally. In a forthcoming paper, we shall show that the two star products are isomorphic by using the fact that $\dim H^2(P_n(\mathbf{C})) = 1$. However, in this paper we shall concentrate our attention to the extrinsic construction of star products and Weyl manifolds.

Throughout this paper, we use the following convention on multi-indices, unless otherwise stated: $\alpha, \beta, \gamma \cdots \in \mathbf{N}^{n+1}; \alpha = (\alpha_1, \cdots, \alpha_{n+1})$. Denote ∂_{z_i} by ∂_i and $\partial_{\bar{z}_i}$ by $\bar{\partial}_i$, and for $\alpha \in \mathbf{N}^{n+1}$, set $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_{n+1}^{\alpha_{n+1}}$ and $\bar{\partial}^\alpha = \bar{\partial}_1^{\alpha_1} \cdots \bar{\partial}_{n+1}^{\alpha_{n+1}}$, etc.

§1. Deformation quantization on $P_n(\mathbf{C})$

1.1. Deformation quantization

Let (M, ω) be a symplectic manifold, where ω is the symplectic 2-form on M. Let ν be a (formal) parameter and let $\mathbf{C}[[\nu]]$ denote the formal power series ring in ν. Let $C^\infty(M; \mathbf{C}[[\nu]])$ be the set of the $\mathbf{C}[[\nu]]$-valued smooth functions on M. Any $a \in C^\infty(M; \mathbf{C}[[\nu]])$ has a formal sum expansion

$$a = \sum_{l=0}^{\infty} a_l(p) \nu^l \tag{1.1}$$

where $a_l \in C^\infty(M; \mathbf{C})$. $a \in C^\infty(M; \mathbf{C}[[\nu]])$ of the form (1.1) will be denoted by $a = a(p; \nu)$. ν is called a deformation parameter. Following to Bayen et al [1], we introduce the star product $*$:

(D 1) $*$ is an associative product on $C^\infty(M; \mathbf{C}[[\nu]])$.

(D 2) $\qquad a * b = ab + \frac{\nu}{2}\{a, b\} \qquad (\operatorname{mod} \nu^2)$.

where $\{\,,\,\}$ is the Poisson bracket given by ω.

(M, ω) is called to be *deformation quantizable* if there exists a star product on $C^\infty(M; \mathbf{C}[[\nu]])$. It is known that there exists a star product for any symplectic manifold (M, ω) (cf. [10] and [5]), i.e. it is deformation quantizable.

1.2. The star product on $\mathbf{C}^{n+1}$

Let $\omega_0 = \frac{1}{2\sqrt{-1}} \sum_{l=1}^{n+1} dz_l \wedge d\bar{z}_l$ be the canonical symplectic structure on $\mathbf{C}^{n+1}$. To give a star product on $\mathbf{C}^{n+1}$, we introduce a following integral transformation involving a real parameter $h > 0$ acting on holomorphic functions $\tilde{s}(z)$ of $\mathbf{C}^{n+1}$ (cf. [2], [9]):

$$(1.2)\quad (H_{\tilde{a}}\tilde{s})(z) = \left(\frac{1}{4\pi h}\right)^{n+1} \int_{\mathbf{C}^{n+1}} \tilde{a}(z,\bar{z}')e^{\frac{1}{2h}(z-z')\bar{z}'}\tilde{s}(z')d\mu(z',\bar{z}'),$$

where $d\mu(z',\bar{z}')$ is the volume element on $\mathbf{C}^{n+1}$, and $\tilde{a}(z,\bar{z}) \in C^{\omega}(\mathbf{C}^{n+1})$ must be chosen so that (1.2) makes sense (e.g., $\tilde{a}$ is a polynomial) and $\tilde{a}(z,\bar{v})$ is the analytic continuation of $\tilde{a}$ from the diagonal of $\mathbf{C}^{n+1} \times \bar{\mathbf{C}}^{n+1}$.

The operator in (1.2) has various expressions via non-holomorphic coordinate transformations. For instance, (1.2) can be rewritten as

$$(H_{\tilde{a}}\tilde{s})(z) = \left(\frac{1}{4\pi h}\right)^{n+1} \int_{\mathbf{C}^{n+1}} \tilde{a}(z,\bar{z}')e^{\frac{-1}{2h}z'\bar{z}'}\tilde{s}(z+z')d\mu(z',\bar{z}').$$

To compute asymptotic expansions, the class of admissible symbol functions $\tilde{a} = \tilde{a}(z,\bar{z})$ should be enlarged to the so-called class of admissible symbols of the form $\tilde{a}(z,\bar{z};h) = \sum \tilde{a}_l(z,\bar{z})h^l$ (formal sum).

As in the computation of Ψ.D.Ops, we have the product formula:

$$(1.3)\qquad H_{\tilde{a}}H_{\tilde{b}} = H_{\tilde{e}(\tilde{a},\tilde{b})}$$

where

(1.4)
$$\tilde{e}(\tilde{a},\tilde{b})(z,\bar{z}) = \left(\frac{1}{4\pi h}\right)^{n+1} \int_{\mathbf{C}^{n+1}} \tilde{a}(z,\bar{z}')\tilde{b}(z',\bar{z})e^{\frac{-1}{2h}|z-z'|^2}d\mu(z',\bar{z}').$$

Moreover, we may modify (1.2) to a so-called Weyl type integral transformation of $\tilde{s}(z)$:

(1.5)
$$(H^w_{\tilde{a}}\tilde{s})(z) = \left(\frac{\sqrt{-1}}{4\pi\tilde{\nu}}\right)^{n+1} \int_{\mathbf{C}^{n+1}} \tilde{a}(\frac{z+z'}{2},\bar{z}')e^{\frac{\sqrt{-1}}{2\tilde{\nu}}(z-z')\bar{z}'}\tilde{s}(z')d\mu(z',\bar{z}'),$$

where $\tilde{\nu} = \sqrt{-1}h$. By a computation similar to (1.3), we have for suitable $\tilde{a},\tilde{b} \in C^{\infty}(\mathbf{C}^{n+1};\mathbf{C}[[\nu]])$,

$$(1.6)\qquad H^w_{\tilde{a}}H^w_{\tilde{b}} = H^w_{\tilde{e}^w(\tilde{a},\tilde{b})},$$

where after a non-holomorphic coordinate transformation (cf. Hörmander [7], p.374), we have

$$(1.7)\quad \tilde{e}^w(\tilde{a},\tilde{b})(z,\bar{z}) = \left(\frac{\sqrt{-1}}{2\pi\tilde{\nu}}\right)^{2(n+1)} \int_{\mathbf{C}^{2(n+1)}} \tilde{a}(z+u,\bar{z}+\bar{v})\tilde{b}(z+v,\bar{z}-\bar{u}) \times e^{-\frac{\sqrt{-1}}{\tilde{\nu}}(u\bar{u}+v\bar{v})} d\mu(u,\bar{u})d\mu(v,\bar{v}).$$

Note that (1.7) has the asymptotic expansion

$$(1.8)\qquad \tilde{e}^w(\tilde{a},\tilde{b}) \sim \sum_l \tilde{c}_l(\tilde{a},\tilde{b})\tilde{\nu}^l,$$

where

$$(1.9)\qquad \tilde{c}_l(\tilde{a},\tilde{b}) = \sum_{|\alpha|+|\beta|=l} \frac{(\sqrt{-1})^l}{\alpha!\beta!} \partial_z^\alpha \partial_{\bar{z}}^\beta \tilde{a} \cdot \partial_{\bar{z}}^\alpha (-\partial_z)^\beta \tilde{b},$$

so that $\tilde{e}^w(\tilde{a},\tilde{b})$ can be viewed as an element of $C^\infty(\mathbf{C}^{n+1};\mathbf{C}[[\tilde{\nu}]])$.

We now define a star product $\tilde{*}$ on $C^\infty(\mathbf{C}^{n+1};\mathbf{C}[[\tilde{\nu}]])$ as follows: For $\tilde{a},\tilde{b} \in C^\infty(\mathbf{C}^{n+1};\mathbf{C}[[\tilde{\nu}]])$, we put

$$(1.10)\qquad \tilde{a}\tilde{*}\tilde{b} = \sum \tilde{c}_l(\tilde{a},\tilde{b})\tilde{\nu}^l,$$

where $\tilde{c}_l(\tilde{a},\tilde{b})$ is given by (1.9). In fact, the formula (1.9) can be applied for any C^∞ functions $\tilde{a},\tilde{b}$ with the parameter $\tilde{\nu}$ viewed as a complex parameter. The restriction of $\tilde{*}$ to $C^\infty(\mathbf{C}^{n+1}-\{0\};\mathbf{C}[[\tilde{\nu}]])$ is denoted by the same symbol. In the following, we denote by $\tilde{\mathfrak{a}}[[\tilde{\nu}]]$ the topological vector space $C^\infty(\mathbf{C}^{n+1}-\{0\};\mathbf{C}[[\tilde{\nu}]])$ with the C^∞ topology. It has two products; one is the natural commutative product, and the other is the star product given above. It is a remarkable fact that the former $\cdot$ can be expressed in terms of the star product:

$$(1.11)\qquad \tilde{a}\cdot\tilde{b} = \sum_{l=0}^{\infty} \tilde{\nu}^l \sum_{|\alpha|+|\beta|=l} \frac{(\sqrt{-1})^l}{\alpha!\beta!} (-\partial_z^\alpha)(\partial_{\bar{z}}^\beta)\tilde{a}\tilde{*}(\partial_{\bar{z}}^\alpha)(\partial_z^\beta)\tilde{b}.$$

By (1.7), the both products on $\mathbf{C}^{n+1}$ are invariant under the parallel displacement and under the unitary group $U(n+1)$.

1.3. $\mathbf{C}^*$-action on $\tilde{\mathfrak{a}}[[\tilde{\nu}]]$

For $\lambda \in \mathbf{C}^* = \mathbf{C} - \{0\}$, we define an action $\rho(\lambda)$ on $\tilde{\mathfrak{a}}[[\tilde{\nu}]]$ as follows:

Definition 1.1. For $\lambda \in \mathbf{C}^*$, and $\tilde{a} \in \tilde{\mathfrak{a}}[[\tilde{\nu}]]$,

$$(\rho(\lambda)\tilde{a})(z, \bar{z}; \tilde{\nu}) = \tilde{a}(\lambda z, \bar{\lambda}\bar{z}; |\lambda|^2 \tilde{\nu}). \tag{1.12}$$

Set

$$\tilde{\mathfrak{a}}[[\tilde{\nu}]]^{\rho} = \{\tilde{a} \in \tilde{\mathfrak{a}}[[\tilde{\nu}]] \mid \rho(\lambda)\tilde{a} = \tilde{a} \text{ for all } \lambda \in \mathbf{C}^*\}. \tag{1.13}$$

It is obvious that $\rho(\lambda)$, $\lambda \in \mathbf{C}^*$, commutes with any $T \in U(n+1)$.

By (1.7), we have

Lemma 1.2. *For any $\tilde{a}, \tilde{b} \in \tilde{\mathfrak{a}}[[\tilde{\nu}]]$ and $\lambda \in \mathbf{C}^*$, we have*

$$\rho(\lambda)(\tilde{a} \tilde{*} \tilde{b}) = (\rho(\lambda)\tilde{a}) \tilde{*} (\rho(\lambda)\tilde{b}). \tag{1.14}$$

1.4. A deformation quantization on $P_n(\mathbf{C})$

In this section, using the product $\tilde{*}$ in 1.2, we construct a star product on $P_n(\mathbf{C})$ with the deformation parameter replaced by ν.

Let $P_n(\mathbf{C})$ be the n-dimensional complex projective space equipped with the standard symplectic structure ω (cf. [8], p. 160) and let $\pi : \mathbf{C}^{n+1} - \{0\} \to P_n(\mathbf{C})$ be the natural projection. Taking the deformation parameter ν, we put $\mathfrak{a}[[\nu]] = C^\infty(P_n(\mathbf{C}); \mathbf{C}[[\nu]])$. For $a \in \mathfrak{a}[[\nu]]$, we define a lift of a, denoting by $\pi^* a$ as an element of $\tilde{\mathfrak{a}}[[\tilde{\nu}]]$ by

$$(\pi^* a)(z, \bar{z}; \tilde{\nu}) = a(p; |z|^{-2}\tilde{\nu}), \qquad \pi(z) = p. \tag{1.15}$$

From Definition 1.1, we easily see that $\pi^* a \in \tilde{\mathfrak{a}}[[\tilde{\nu}]]^{\rho}$.

For any $\tilde{a} \in \tilde{\mathfrak{a}}[[\tilde{\nu}]]^{\rho}$, we put

$$(\iota\tilde{a})(p; \nu) = \tilde{a}(z, \bar{z}; |z|^2 \nu), \quad \pi(z) = p. \tag{1.16}$$

(1.16) is independent of the choice of z.

Lemma 1.3.

$$\iota : \tilde{\mathfrak{a}}[[\tilde{\nu}]]^{\rho} \to \mathfrak{a}[[\nu]]$$

is an isomorphism with $\iota\pi^* = \mathrm{id}$.

By this lemma, we can identify $\tilde{\mathfrak{a}}[[\tilde{\nu}]]^{\rho}$ with $\mathfrak{a}[[\nu]]$. By Lemma 1.2 and Lemma 1.3, we can project the product $\tilde{*}$ onto $P_n(\mathbf{C})$. Namely, for any $a, b \in \mathfrak{a}[[\nu]]$, we put

$$a * b = \iota(\pi^* a \tilde{*} \pi^* b). \tag{1.17}$$

Consider the chart $U_{n+1} = \{p = \pi(z) \mid z_{n+1} \neq 0\}$ and the coordinate map $\phi_{n+1} : U_{n+1} \to \phi_{n+1}(U_{n+1}) = \mathbf{C}^n$, $\phi_{n+1}(p) = w = (w_1, \cdots, w_n)$, where $w_j = \frac{z_j}{z_{n+1}}$ $(j = 1, \cdots, n)$. Using these coordinates, the symplectic structure ω on $P_n(\mathbf{C})$ becomes (cf. [8] p. 160):

$$\begin{aligned} \omega \mid_{U_{n+1}} = \frac{1}{2\sqrt{-1}(1+|w|^2)^2}((1+|w|^2)\sum_{l=1}^{n} dw_l \wedge d\bar{w}_l \\ - \sum_{l,m=1}^{n} \bar{w}_l dw_l \wedge w_m d\bar{w}_m). \end{aligned} \tag{1.18}$$

By (1.18), in these coordinates, the Poisson bracket $\{a, b\}$ on $P_n(\mathbf{C})$ is

(1.19)

$$\begin{aligned} &\{a, b\}(w_1, \cdots, w_n) \\ &\quad = 2\sqrt{-1}(1+|w|^2)[\textstyle\sum_{l=1}^{n}(\partial_{w_l} a \cdot \partial_{\bar{w}_l} b - \partial_{\bar{w}_l} a \cdot \partial_{w_l} b) \\ &\qquad + \sum_{k,l}(w_k \partial_{w_k} a \cdot \bar{w}_l \partial_{\bar{w}_l} b - \bar{w}_k \partial_{\bar{w}_k} a \cdot w_l \partial_{w_l} b)] \end{aligned}$$

On the other hand, since $w_j = w_j(z_1, \cdots, z_{n+1})$, we have

(1.20)

$$\begin{aligned} \partial_{z_{n+1}} &= -\frac{1}{z_{n+1}} \sum_{i=1}^{n} w_l \partial_{w_l}, & \partial_{z_m} &= \frac{1}{z_{n+1}} \partial_{w_m} \quad (m = 1, \cdots, n), \\ \partial_{\bar{z}_{n+1}} &= -\frac{1}{\bar{z}_{n+1}} \sum_{l=1}^{n} \bar{w}_l \partial_{\bar{w}_l}, & \partial_{\bar{z}_m} &= \frac{1}{\bar{z}_{n+1}} \partial_{\bar{w}_m} \quad (m = 1, \cdots, n). \end{aligned}$$

By a direct computation using (1.20) and (1.10) and putting $z_{n+1} = 1$, $z_l = w_l (l = 1, \cdots, n)$, we have

Proposition 1.4. (1.17) *gives a star product* $*$ *on* $P_n(\mathbf{C})$, *i.e. for any* $a, b \in C^\infty(P_n(\mathbf{C}))$ *we have*

$$a * b = ab + \frac{\nu}{2}\{a, b\} \qquad (\text{mod } \nu^2). \tag{1.21}$$

§2. A Weyl manifold over $P_n(\mathbf{C})$

Using the notion of Weyl manifolds given in [10, 11], we describe the algebra $\mathfrak{a}[[\nu]]$ more geometrically.

2.1. The formal Weyl algebra

Let $\tilde{\mathbf{W}}'$ denote the algebra with $2n+3$ generators $\{\tilde{\nu}, Z_1, \cdots, Z_{n+1}, \bar{Z}_1, \cdots, \bar{Z}_{n+1}\}$ over $\mathbf{C}$ with the relations:

$$\begin{cases} [\tilde{\nu}, Z_i] = 0 \quad , \quad [\tilde{\nu}, \bar{Z}_i] = 0, \\ [Z_i, Z_j] = 0 \quad , \quad [\bar{Z}_i, \bar{Z}_j] = 0 \\ [Z_i, \bar{Z}_j] = 2\sqrt{-1}\nu\delta_{ij} \quad (1 \leq i, j \leq n+1), \end{cases} \tag{2.1}$$

where $[\ ,\ ]$ denotes the commutator $[a, b]{=}ab - ba$. For any $a, b \in \tilde{\mathbf{W}}'$, the product is denoted by $a * b$; for any α, $\beta \in \mathbf{N}^{n+1}$, we denote $Z_1^{\alpha_1} * \cdots * Z_{n+1}^{\alpha_{n+1}} * \bar{Z}_1^{\beta_1} * \cdots * \bar{Z}_{n+1}^{\beta_{n+1}}$, by $Z^\alpha * \bar{Z}^\beta$ where $Z_i^{\alpha_i} = \underbrace{Z_i * \cdots * Z_i}_{\alpha_i}$, $\bar{Z}_i^{\beta_i} = \underbrace{\bar{Z}_i * \cdots * \bar{Z}_i}_{\beta_i}$.

Define the degree of the generators by $d(\tilde{\nu}){=}2$, $d(Z_i) = d(\bar{Z}_i){=}1$ $(1 \leq i \leq n+1)$. For $l \geq 0$, let $\tilde{\mathbf{W}}(l)$ be the set of polynomials of degree l and $\tilde{\mathbf{W}}(0) = \mathbf{C}$. Then

$$\tilde{\mathbf{W}}' = \oplus_{l \geq 0} \tilde{\mathbf{W}}(l), \qquad (\text{direct sum}). \tag{2.2}$$

Any element $a \in \tilde{\mathbf{W}}'$ can be written as a finite sum $\sum a_l$, $a_l \in \tilde{\mathbf{W}}(l)$; a_l is called the *l-th component* of a.

Give $\tilde{\mathbf{W}}' = \oplus_l \tilde{\mathbf{W}}(l)$ the direct product topology. Denote by $\tilde{\mathbf{W}}$ the completion of $\tilde{\mathbf{W}}'$; $\tilde{\mathbf{W}}$ is called the *formal Weyl algebra* with generators

$\{\tilde{\nu}, Z_1, \cdots, Z_{n+1}, \bar{Z}_1, \cdots, \bar{Z}_{n+1}\}$. The formal Weyl algebra $\tilde{\mathbf{W}}$ is isomorphic (as a vector space) to the formal power series ring $\mathbf{C}[[\tilde{\nu}, Z_1, \cdots, Z_{n+1}, \bar{Z}_1, \cdots, \bar{Z}_{n+1}]]$. If we replace $Z_i, \bar{Z}_i$ by $(X_i + \sqrt{-1}Y_i)$ and $(X_i - \sqrt{-1}Y_i)$ respectively, then the algebra $\tilde{\mathbf{W}}$ is exactly the same as in [10]. We also use the formal Weyl algebra $\mathbf{W}$ with $2n+1$ generators $\{\nu, Z_1, \cdots, Z_n, \bar{Z}_1, \cdots, \bar{Z}_n\}$.

2.2. Symmetric product

For $a, b \in \tilde{\mathbf{W}}$, define the symmetric product by

$$a \circ b = \frac{1}{2}(a * b + b * a).$$

The above product is *not* associative but $(\tilde{\mathbf{W}}, \circ)$ is a Jordan algebra. However, by the general formula

$$(a \circ b) \circ c - a \circ (b \circ c) = \frac{1}{4}[b, [a, c]], \tag{2.3}$$

and the fact that $[Z_i, \bar{Z}_j]$ is in the center of $\tilde{\mathbf{W}}$, we have

$$\hat{Z}_i \circ (\hat{Z}_j \circ a) = \hat{Z}_j \circ (\hat{Z}_i \circ a) \quad (1 \le i, j \le n+1), \tag{2.4}$$

where $\hat{Z}_i = Z_i$ or $\bar{Z}_i$. Thus, we may set

$$(\hat{Z}_i \circ)^l \cdot a = \underbrace{\hat{Z}_i \circ (\hat{Z}_i \circ \cdots (\hat{Z}_i \circ a) \cdots)}_{l \text{ times}},$$

and

$$\begin{aligned} &(Z\circ)^\alpha (\bar{Z}\circ)^\beta \cdot a \\ &= (Z_1\circ)^{\alpha_1} \cdots (Z_{n+1}\circ)^{\alpha_{n+1}} (\bar{Z}_1\circ)^{\beta_1} \cdots (\bar{Z}_{n+1}\circ)^{\beta_{n+1}} \cdot a, \end{aligned} \tag{2.5}$$

where the right hand side of (2.5) is independent of the order of the $Z_i\circ$'s, and $\bar{Z}_i\circ$'s. Obviously, $\{\tilde{\nu}^l (Z\circ)^\alpha (\bar{Z}\circ)^\beta \cdot 1\ ; \alpha, \beta \in \mathbf{N}^{n+1}\}$ forms a linear basis of $\tilde{\mathbf{W}}$. $\tilde{\mathbf{W}}(k)$ is spanned by $\{\tilde{\nu}^l (Z\circ)^\alpha (\bar{Z}\circ)^\beta \cdot 1 : 2l + |\alpha| + |\beta| = k\}$ (cf. [10], Lemma 1.2).

By the above fact, we may introduce a new product $\odot$ defined by

$$(\hat{Z}\circ)^\alpha \cdot 1 \odot (\hat{Z}\circ)^\beta \cdot 1 = (\hat{Z}\circ)^{\alpha+\beta} \cdot 1, \qquad \alpha, \beta \in \mathbf{N}^{n+1}.$$

We denote $\hat{Z}_i \circ \hat{Z}_j$ and $(\hat{Z}\circ)^\alpha \cdot 1$ by $\hat{Z}_i \odot \hat{Z}_j$ and $(\hat{Z}\odot)^\alpha$ respectively. The following are easily seen:

(a) $(\tilde{\mathbf{W}}, \odot)$ is a commutative, associative topological algebra over $\mathbf{C}$.

(b) $(\tilde{\mathbf{W}}, \odot)$ is isomorphic to the algebra $\mathbf{C}[[\tilde{\nu}, Z_1, \cdots, Z_{n+1}, \bar{Z}_1, \cdots, \bar{Z}_{n+1}]]$.

2.3. Localization of the algebras $\tilde{\mathfrak{a}}[[\tilde{\nu}]]$ and $\mathfrak{a}[[\nu]]$

Let $\tilde{U}$ and U be open sets of $\mathbf{C}^{n+1} - \{0\}$ and $P_n(\mathbf{C})$ respectively. By formula (1.8) and Definition (1.17), the $\tilde{*}$(resp. $*$)-product can be restricted on $\tilde{U}$ (resp. U) and then extended to $C^\infty(\tilde{U}; \mathbf{C}[[\tilde{\nu}]])$ (resp. $C^\infty(U; \mathbf{C}[[\nu]])$). If $\pi(\tilde{U}) = U$, then π^* and ι given in (1.15) and (1.16) can be also restricted on U and $\tilde{U}$, wich are denoted by $\pi_U^*, \iota_{\tilde{U}}$ respectively. In particular, for any $a, b \in \mathfrak{a}_U[[\nu]]$,

$$a * b = \iota_{\tilde{U}}(\pi_U^* a \tilde{*} \pi_U^* b). \tag{2.6}$$

The algebra $(C^\infty(\tilde{U}; \mathbf{C}[[\tilde{\nu}]]), \tilde{*})$ (resp. $(C^\infty(U; \mathbf{C}[[\nu]]), *)$) with the C^∞-topology is denoted by $\tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]]$ (resp. $\mathfrak{a}_U[[\nu]]$).

Given an open set $\tilde{U} \subset \mathbf{C}^{n+1} - \{0\}$, we consider the trivial bundle $W_{\tilde{U}} = \tilde{U} \times \tilde{\mathbf{W}} \xrightarrow{\pi} \tilde{U}$. Define $2n+2$ smooth sections ζ_i, $\bar{\zeta}_i$ on $W_{\tilde{U}}$ by:

$$\zeta_i(z, \bar{z}) = z_i + Z_i, \quad \bar{\zeta}_i(z, \bar{z}) = \bar{z}_i + \bar{Z}_i, \quad (i = 1, \cdots, n+1). \tag{2.7}$$

For $f \in \tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]]$, we define a section $f^\#(\zeta, \bar{\zeta}) \in \Gamma(W_{\tilde{U}})$ by

$$f^\#(\zeta, \bar{\zeta})(z, \bar{z}) = \sum \frac{1}{\alpha! \beta!} (\partial^\alpha \bar{\partial}^\beta f)(z, \bar{z}) \cdot Z^\alpha \odot \bar{Z}^\beta, \quad \alpha, \beta \in \mathbf{N}^{n+1}. \tag{2.8}$$

$f^\#$ is called the *Weyl continuation* of $f \in \tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]]$. Let $\mathcal{F}(W_{\tilde{U}})$ be the algebra of $f^\#$ for $f \in \tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]]$ where the product is defined pointwisely on $\tilde{\mathbf{W}}$.

We have shown in [10]:

Proposition 2.1. *$\mathcal{F}(W_{\tilde{U}})$ is naturally isomorphic to $\tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]]$ as an algebra.*

2.4. Main results

We now introduce systems of local generators:

Definition 2.2. Let $\tilde{U}$ and $U = \pi(\tilde{U})$ be open sets of $\mathbf{C}^{n+1} - \{0\}$ and $P_n(\mathbf{C})$ respectively. A $(2n+3)$-tuple $\{\tilde{w}_0; \tilde{w}_1, \cdots, \tilde{w}_{2n+2}\}$ of $\tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]]$

(resp. $(2n+1)$-tuple $\{w_0; w_1, \cdots, w_{2n}\}$ of $\mathfrak{a}_U[[\nu]]$) is called a *system of local generators* for $\tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]]$ (resp. $\mathfrak{a}_U[[\nu]]$) if they satisfy

(L 1) $\tilde{w}_0$ (resp. w_0) is in the center of $\tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]]$ (resp. $\mathfrak{a}_U[[\nu]]$).

(L 2) The closure of the algebra generated by $\{\tilde{w}_0; \tilde{w}_1, \cdots, \tilde{w}_{2n+2}\}$ (resp. $\{w_0; w_1, \cdots, w_{2n}\}$) coincides with $\tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]]$ (resp. $\mathfrak{a}_U[[\nu]]$).

We now consider this definition on each chart (U_l, ϕ_l) of $P_n(\mathbf{C})$. Namely, for each $l = 1, 2, \cdots, n+1$, let $\tilde{U}_l = \{z = (z_1, \cdots, z_{n+1}) \in \mathbf{C}^{n+1} - \{0\} \mid z_l \neq 0\}$, $U_l = \pi(\tilde{U}_l)$, and $\phi_l : U_l \to \phi_l(U_l) = \mathbf{C}^n$. Then, $\phi_l(p) = (\frac{z_1}{z_l}, \cdots, \frac{\hat{z}_l}{z_l}, \cdots, \frac{z_{n+1}}{z_l})$ with $p = \pi(z)$ gives the local coordinate of $P_n(\mathbf{C})$. For simplicity, we set $\pi_l^* = \pi_{\tilde{U}_l}^*$ and $\iota_l = \iota_{\tilde{U}_l}$.

Definition 2.3. A collection of systems of local generators $\{w_0^{(l)}; u_1^{(l)}, \cdots, u_n^{(l)}, v_1^{(l)}, \cdots, v_n^{(l)}\}$ for $\mathfrak{a}_{U_l}[[\nu]]$ for each $l = 1, \cdots, n+1$ is called a (*system of*) *Weyl coordinates on* $P_n(\mathbf{C})$ *associated with* $\{(U_l, \phi_l)\}$ if for any $l, m = 1, \cdots, n+1$

(C 1) $\pi_l^* w_0^{(l)} = \pi_m^* w_0^{(m)}$ on $\tilde{\mathfrak{a}}_{\tilde{U}_l \cap \tilde{U}_m}[[\tilde{\nu}]]$ if $U_l \cap U_m \neq \emptyset$

(C 2)

$$\begin{cases} [w_0^{(l)}, u_i^{(l)}] = 0, \quad [w_0^{(l)}, v_i^{(l)}] = 0, \quad [u_i^{(l)}, u_j^{(l)}] = 0, \\ [v_i^{(l)}, v_j^{(l)}] = 0, \quad [u_i^{(l)}, v_j^{(l)}] = -w_0^{(l)} \delta_{ij}. \end{cases}$$

(C 3) On each $U_k \cap U_l (\neq \emptyset)$, $u_1^{(k)}, \cdots, u_n^{(k)}, v_1^{(k)}, \cdots, v_n^{(k)}$ mod ν are $\mathbf{R}$-valued C^∞ functions of $(u_1^{(l)}, \cdots, u_n^{(l)}, v_1^{(l)}, \cdots, v_n^{(l)})$.

In §3-4, we shall prove the following:

Theorem 2.4. *There exists a system of Weyl coordinates on* $P_n(\mathbf{C})$ *associated with* $\{(U_l, \phi_l)\}$. (cf. Theorem 4.5.)

By this theorem, we can construct an algebra bundle over $P_n(\mathbf{C})$ with the formal Weyl algebra $\mathbf{W}$ of $2n+1$ generators as fiber. Namely, on each U_l we consider a trivial algebra bundle $\pi_l : U_l \times \mathbf{W} \to U_l$. Since $\{w_0^{(l)}; u_1^{(l)}, \cdots, u_n^{(l)}, v_1^{(l)}, \cdots, v_n^{(l)},\}$ can be viewed as C^∞-sections of W_{U_l}, this trivializes the bundle W_{U_l}. Moreover, we can patch the W_{U_j} together. This gives a Weyl manifold over $P_n(\mathbf{C})$ introduced in [9, 10]. Using the notation of [9, 10] on Weyl manifolds, we have

Theorem 2.5. *The algebra* $(\mathfrak{a}[[\nu]], *) = (C^\infty(P_n(\mathbf{C}); \mathbf{C}[[\nu]]), *)$ *gives a Weyl manifold* $W_{P_n(\mathbf{C})}$ *over* $P_n(\mathbf{C})$. *In particular,* $\mathfrak{a}[[\nu]]$ *is isomorphic to* $\mathcal{F}(W_{P_n(\mathbf{C})})$, *where* $\mathcal{F}(W_{P_n(\mathbf{C})})$ *is the set of all Weyl functions on* $P_n(\mathbf{C})$.

§3. Properties for $\tilde{\mathfrak{a}}[[\tilde{\nu}]]^\rho$

3.1. Several operations on $\tilde{\mathfrak{a}}[[\tilde{\nu}]]$

Note that the natural product $\cdot$ can be defined on $\tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]]$ for any open set $\tilde{U} \subset \mathbf{C}^{n+1} - \{0\}$. We use the notation $(\tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]], \cdot)$ when we consider $\tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]]$ as a commutative algebra. We can introduce a partial derivative $\partial_{\tilde{\nu}}$ on $\tilde{\mathfrak{a}}[[\tilde{\nu}]]$ and $\tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]]$ as follows: for any element $a \in \tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]]$ with the form $a = \sum a_l \tilde{\nu}^l$ where $a_l = a_l(z, \bar{z})$ is C^∞,

$$\partial_i a = \sum (\partial_i a_l)\tilde{\nu}^l, \quad \bar{\partial}_i a = \sum (\bar{\partial}_i a_l)\tilde{\nu}^l, \quad \partial_{\tilde{\nu}} a = \sum l a_l \tilde{\nu}^{l-1}. \tag{3.1}$$

We introduce the differential operators L_0 and L_1 on $\tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]]$ by

$$L_0 \tilde{a} = 2\tilde{\nu}\partial_{\tilde{\nu}}\tilde{a} + \sum_i (z_i \cdot \partial_i + \bar{z}_i \cdot \bar{\partial}_i)\tilde{a} \tag{3.2}$$

and

$$L_1 \tilde{a} = \sum \sqrt{-1}(\bar{z}_i \cdot \partial_i - z_i \cdot \bar{\partial}_i)\tilde{a}, \tag{3.3}$$

for $\tilde{a} \in \tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]]$.

Lemma 3.1. L_0 *and* L_1 *are derivations of* $(\tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]], \cdot)$: *i.e. for any* $\tilde{a}, \tilde{b} \in (\tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]], \cdot)$,

$$L_k(\tilde{a} \cdot \tilde{b}) = L_k(\tilde{a}) \cdot \tilde{b} + \tilde{a} \cdot L_k(\tilde{b}) \quad (k = 0, 1). \tag{3.4}$$

Note that L_1 can be rewritten as

$$L_1 \tilde{a} = -\frac{1}{\nu}[r, \tilde{a}] \ (= -\frac{1}{\nu} ad(r)\tilde{a}), \tag{3.5}$$

where $r = \frac{1}{2}|z|^2 = \frac{1}{2}\sum_{i=1}^{n+1} z_i \bar{z}_i$.

Remark. In general, for $\tilde{a}, \tilde{b}, \tilde{c} \in \tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]]$, the equality

$$[\tilde{a}, \tilde{b} \cdot \tilde{c}] = [\tilde{a}, \tilde{b}] \cdot \tilde{c} + \tilde{b} \cdot [\tilde{a}, \tilde{c}]$$

does not hold.

Let $\tilde{U}$ be a conic open set in $\mathbf{C}^{n+1} - \{0\}$ and put $\tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]]^\rho = \tilde{\mathfrak{a}}[[\tilde{\nu}]]^\rho \cap \tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]]$. A characterization of $\tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]]^\rho$ by L_0 and r is given as follows:

Proposition 3.2. $\tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]]^\rho = \{\tilde{a} \in \tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]] \mid L_0\tilde{a} = 0, \quad [r, \tilde{a}] = 0\}$.

Proof. For a real parameter t and $\tilde{a} \in \tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]]$, consider curves $t \mapsto \rho(e^t)\tilde{a}$, $\rho(e^{\sqrt{-1}t})\tilde{a}$. Taking the derivatives at $t = 0$, we get

$$\frac{d}{dt}\rho(e^t)\tilde{a} \mid_{t=0} = L_0\tilde{a}, \tag{3.6}$$

$$\frac{d}{dt}\rho(e^{\sqrt{-1}t})\tilde{a} \mid_{t=0} = L_1\tilde{a}. \tag{3.7}$$

Since $L_0 r = 2r$ and $L_0\tilde{\nu} = 2\tilde{\nu}$, we have formally $L_0(\frac{1}{\tilde{\nu}}r) = 0$. This implies $[L_0, L_1] = 0$, which gives Proposition 3.2. Q.E.D.

Using Lemma 3.1 and Proposition 3.2, we have

Corollary 3.3. *Let $\tilde{U}$ be a conic open set in $\mathbf{C}^{n+1} - \{0\}$.*

(1) *$\tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]]^\rho$ is closed under the $\cdot$-product.*

(2) *For any $T \in U(n+1)$, we have*

(a) $T(r) = r, \qquad [T, L_0] = 0,$

(b) $T\tilde{\mathfrak{a}}_{T\tilde{U}}[[\tilde{\nu}]]^\rho = \tilde{\mathfrak{a}}_{\tilde{U}}[[\tilde{\nu}]]^\rho$.

3.2. Inverse of r

Since $r \neq 0$ on $\mathbf{C}^{n+1} - \{0\}$, it has the inverse $\frac{1}{r}$ for the $\cdot$-product. To obtain the inverse r^{-1} for the $\tilde{*}$-product, we first assume that r^{-1} is a function $f(r)$ of r and solve the equation $r\tilde{*}f(r) = 1$. By the product formulas (1.9) (1.10), we have

$$r\tilde{*}f(r) = rf(r) + \tilde{\nu}^2(\frac{n+1}{2}f'(r) + \frac{1}{2}f''(r)r) = 1.$$

Setting $f = \sum_{l=0}^{\infty} f_l \tilde{\nu}^l$, we have

$$
(3.8) \qquad \begin{cases} f_{2l}(t) = \left(-\frac{1}{2}\right)^l \left(\frac{d^2}{dt^2} + \frac{n+1}{t}\frac{d}{dt}\right)^l \left(\frac{1}{t}\right), \\ f_{2l+1} = 0. \end{cases}
$$

By (3.8), r^{-1} has the form

$$
(3.9) \qquad \begin{aligned} r^{-1} = \frac{1}{r}\{1 + \frac{n-1}{2}(\frac{\tilde{\nu}}{r})^2 - \frac{(n-1)}{2}\frac{3(n-3)}{2}(\frac{\tilde{\nu}}{r})^4 \\ + \frac{(n-1)}{2}\frac{3(n-3)}{2}\frac{5(n-5)}{2}(\frac{\tilde{\nu}}{r})^6 + \cdots\}. \end{aligned}
$$

On the other hand, $e_{\tilde{*}}^{t\tilde{\nu}r^{-1}} = \sum \frac{t^m}{m!}(\tilde{\nu}r^{-1}\tilde{*})^m$, $t \in \mathbf{R}$, in the $\tilde{*}$-product, satisfies the differential equation

$$
(3.10) \qquad \frac{d}{dt}g_t(r) = \tilde{\nu}r^{-1}\tilde{*}g_t(r), \quad g_0(r) = 1.
$$

Multiplying both sides of (3.10) by r, we have

$$
\frac{d}{dt}\{r \cdot g_t(r) + \tilde{\nu}^2(\frac{n+1}{2}g_t'(r) + \frac{1}{2}g_t''(r) \cdot r)\} = \tilde{\nu}g_t(r).
$$

By setting $g_t = \sum_{l=0}^{\infty} \tilde{\nu}^l g_t^{(l)}(r)$, we can compute $e_{\tilde{*}}^{t\tilde{\nu}r^{-1}}$ in the form $\sum_{l \geq k} a_{k,l} t^k (\frac{\tilde{\nu}}{r})^l$, where $a_{kk} = \frac{1}{k!}$. Comparing coefficients of t^k, we see that

$$
(3.11) \qquad (\tilde{\nu}r^{-1}\tilde{*})^m = \sum_{l=m}^{\infty} a_{m,l}(\frac{\tilde{\nu}}{r})^l \quad (m = 1, 2, \cdots).
$$

Since (3.11) can be solved conversely with respect to $(\frac{\tilde{\nu}}{r})^l$, we see that $\frac{\tilde{\nu}}{r}$ is written as a function of $\tilde{\nu}r^{-1}$.

3.3. The center of $\tilde{\mathfrak{a}}[[\tilde{\nu}]]^\rho$.

Put $\nu = \frac{\tilde{\nu}}{r} \in \tilde{\mathfrak{a}}[[\tilde{\nu}]]$. Then we have:

Proposition 3.4. *$\nu = \frac{\tilde{\nu}}{r}$ satisfies the following:*

(a) $\nu \in \tilde{\mathfrak{a}}[[\tilde{\nu}]]^\rho$,

(b) $[\nu, f] = 0$ *for any* $f \in \tilde{\mathfrak{a}}[[\tilde{\nu}]]^\rho$.

Proof. Since $[r, \tilde{\mathfrak{a}}[[\tilde{\nu}]]^\rho] = \{0\}$ by Proposition 3.2, we have $[r^{-1}, \tilde{\mathfrak{a}}[[\tilde{\nu}]]^\rho] = \{0\}$. Thus $[f(r^{-1}), \tilde{\mathfrak{a}}[[\tilde{\nu}]]^\rho] = \{0\}$. By Proposition 3.2, we obtain (b). Moreover, since $[\frac{\tilde{\nu}}{r}, r] = 0$ and $L_0 r = 2r$, we have $\frac{\tilde{\nu}}{r} \in \tilde{\mathfrak{a}}[[\tilde{\nu}]]^\rho$. Q.E.D.

By Proposition 3.4, we may use $\nu = \frac{\tilde{\nu}}{r}$ as a deformation parameter of $\mathfrak{a}[[\nu]]$. However, note that there is no general rule for determining deformation parameters as one may replace $\frac{\tilde{\nu}}{r}$ by $\tilde{\nu}r^{-1}$. If we choose $\tilde{\nu}r^{-1}$ as a deformation parameter, then the expression of $*$-product on $\mathfrak{a}[[\nu]]$ is changed.

§4. Manifold structures on $\mathfrak{a}[[\nu]]$

4.1. Local generators of $\mathfrak{a}[[\nu]]$

It is impossible to find generators of $\mathfrak{a}[[\nu]]$ with respect to which any element of $\mathfrak{a}[[\nu]]$ has a unique expression. Instead, we can localize $\mathfrak{a}[[\nu]]$ on open subsets to have convenient expressions for its elements. On the open set $\tilde{U}_{n+1} = \{z \in \mathbf{C}^{n+1} - \{0\} \mid z_{n+1} \neq 0\}$, consider

$$\tilde{\mathfrak{a}}_{\tilde{U}_{n+1}}[[\tilde{\nu}]]^\rho = \{\tilde{a} \in \tilde{\mathfrak{a}}_{\tilde{U}_{n+1}}[[\tilde{\nu}]] \mid \rho(\lambda)\tilde{a} = \tilde{a}, \lambda \in \mathbf{C}^*\}. \tag{4.1}$$

Note that on $\tilde{U}_{n+1}$, $\frac{1}{z_{n+1}}$ and $\frac{1}{\bar{z}_{n+1}}$ are well-defined. Thus, setting

$$\nu = \frac{\tilde{\nu}}{r}, \quad w_i = \frac{z_i}{z_{n+1}}, \quad \bar{w}_i = \frac{\bar{z}_i}{\bar{z}_{n+1}} \qquad (i = 1, \cdots, n), \tag{4.2}$$

we have $\nu, w_i, \bar{w}_i \in \tilde{\mathfrak{a}}_{\tilde{U}_{n+1}}[[\tilde{\nu}]]^\rho$. By Lemma 1.3, we can identify $\nu, w_i, \bar{w}_i$ with elements of $\mathfrak{a}_{U_{n+1}}[[\nu]]$.

For $\tilde{f} \in \tilde{\mathfrak{a}}_{\tilde{U}_{n+1}}[[\tilde{\nu}]]^\rho$, we may write $\tilde{f} = \sum_{l \geq 0} \tilde{f}_l(z, \bar{z})\tilde{\nu}^l$. Since $\tilde{f}$ is invariant under $\rho(\frac{1}{z_{n+1}})$, we have

$$
\begin{aligned}
\tilde{f}\,(z,\bar{z};\tilde{\nu}) &= \left(\rho\left(\frac{1}{z_{n+1}}\right)\tilde{f}\right)(z,\bar{z};\tilde{\nu}) \\
&= \tilde{f}\left(\frac{z}{z_{n+1}},\frac{\bar{z}}{\bar{z}_{n+1}};\frac{\tilde{\nu}}{|z_{n+1}|^2}\right) \\
&= \sum_l \tilde{f}_l\left(\frac{z}{z_{n+1}},\frac{\bar{z}}{\bar{z}_{n+1}}\right)\left(\frac{r}{|z_{n+1}|^2}\right)^l\left(\frac{\tilde{\nu}}{r}\right)^l \\
&= \sum_l f_l(w,\bar{w})\nu^l
\end{aligned}
\tag{4.3}
$$

where $f_l(w,\bar{w}) = \tilde{f}_l(w,\bar{w})(\frac{1}{2}(1+|w|^2))^l$. This gives:

Theorem 4.1. $\tilde{f} \in \tilde{\mathfrak{a}}_{\tilde{U}_{n+1}}[[\tilde{\nu}]]^{\rho}$ *if and only if there exists* $f \in C^{\infty}(U_{n+1};\mathbf{C}[[\nu]])$ *such that* $\tilde{f} = \pi^*_{\tilde{U}_{n+1}} f$.

4.2. Commutation relations for Weyl coordinates

We compute the commutation relations for $\{\tilde{\nu}, w_1, \cdots, w_n, \bar{w}_1, \cdots, \bar{w}_n\}$ on $\phi_{n+1}(U_{n+1})$. Using (1.9) and Proposition 3.4 (b), we easily have

Lemma 4.2. *For any* $i, j = 1, \cdots, n$,

$$
\begin{cases} [\nu, w_i] = [\nu, \bar{w}_i] = 0, \\ [w_i, w_j] = [\bar{w}_i, \bar{w}_j] = 0. \end{cases}
\tag{4.4}
$$

By Lemma 4.2 and the polynomial approximation theorem, the commutative algebra of the $\mathbf{C}[[\nu]]$-valued holomorphic functions on $\phi_{n+1}(U_{n+1})$ (resp. anti-holomorphic functions on $\phi_{n+1}(U_{n+1})$) is isomorphic to the subalgebra of $\mathcal{F}(W_{\phi_{n+1}(U_{n+1})})$ whose element has the form $f^{\#} = f(\nu, w_1, \cdots, w_n)^{\#}$ (resp. $f^{\#} = f(\nu, \bar{w}_1, \cdots, \bar{w}_n)^{\#}$).

By Theorem 4.1, we may call $\{\nu, w_1, \cdots, w_n, \bar{w}_1, \cdots, \bar{w}_n\}$ *the homogeneous complex Weyl coordinates* on $W_{\phi_{n+1}(U_{n+1})}$. By a careful computation, we have the following commutation relation:

Proposition 4.3.

(4.5)

$$
\begin{aligned}
[w_i, \bar{w}_j] =& \nu(1+\sum_{l=1}^{n} w_l\bar{w}_l)\cdot(\delta_{jk}+w_j\bar{w}_k) \\
&- (\nu(1+\sum_{l=1}^{n} w_l\bar{w}_l))^3\cdot(2!\delta_{jk}+3!w_j\bar{w}_k) \\
&+ (\nu(1+\sum_{l=1}^{n} w_l\bar{w}_l))^5(4!\delta_{jk}+5!w_j\bar{w}_k) - \cdots .
\end{aligned}
$$

4.3. Local trivialization on $\mathfrak{a}_{U_{n+1}}[[\nu]]$.

As seen in 4.2, it seems not so simple to write the commutation relations for $\{\nu, w_1, \cdots, w_n, \bar{w}_1, \cdots, \bar{w}_n\}$. By a change of generators, we can give a structure on $\mathfrak{a}_{U_{n+1}}[[\nu]]$ simpler than (4.5). However, we have to use a non-holomorphic transformation here.

Let $H = \frac{1}{\sqrt{1+\sum w_l\cdot\bar{w}_l}} \in \mathfrak{a}_{U_{n+1}}[[\nu]]$, where the square root is given in the $\cdot$-product.

Lemma 4.4. *For any $j, k = 1, \cdots, n$,*

$$
(4.6)\qquad \begin{cases} [H\cdot w_j, H\cdot w_k] = [H\cdot\bar{w}_j, H\cdot\bar{w}_k] = 0 & (\mathrm{mod}\ \nu^2), \\ [H\cdot w_j, H\cdot\bar{w}_k] = 2\sqrt{-1}\nu\delta_{ik} & (\mathrm{mod}\ \nu^2). \end{cases}
$$

Proof. By the product formula (1.9),

$$H\cdot w_j = H * w_j \quad (\mathrm{mod}\,\nu) \quad (\nu = \frac{\tilde{\nu}}{r}).$$

Hence

$$[H\cdot w_j, H\cdot\bar{w}_k] = [H * w_j, H * \bar{w}_k] \quad (\mathrm{mod}\,\nu^2), etc.$$

Thus

$$[H\cdot w_j, H\cdot\bar{w}_k] = H^2[w_j,\bar{w}_k] + H\cdot[w_j, H]\bar{w}_k + [H,\bar{w}_k]\cdot H\cdot w_j \quad (\mathrm{mod}\,\nu^2).$$

By these equalities and (1.11), we obtain the formulas (4.6). Q.E.D.

Setting

$$\xi_j'' = \frac{1}{2}(H \cdot w_j + H \cdot \bar{w}_j), \quad \eta_j'' = \frac{1}{2\sqrt{-1}}(H \cdot w_j - H \cdot \bar{w}_j) \quad (1 \le j \le n),$$

and using the last lemma yields

$$(4.7) \qquad \begin{cases} [\xi_j'', \xi_k''] & = [\eta_j'', \eta_k''] = 0 \pmod{\nu^2} \\ [\xi_j'', \eta_k''] & = -\nu\delta_{jk} \pmod{\nu^2}. \end{cases}$$

In particular, $\{\xi_j'', \xi_k''\}= \{\eta_j'', \eta_k''\}=0$, and $\{\xi_j'', \eta_k''\}=-\delta_{jk}$. The following theorem may be called a quantized Darboux theorem:

Theorem 4.5. *There exist $\xi_1, \cdots, \xi_n, \eta_1, \cdots, \eta_n \in \mathfrak{a}_{U_{n+1}}[[\nu]]$ such that*

$$[\xi_i, \xi_j] = [\eta_i, \eta_j] = 0$$
$$[\xi_i, \eta_j] = -\nu\delta_{ij}, \quad \text{where } \nu = \frac{\tilde{\nu}}{r}.$$

Proof. (cf. [11], 3.4 Lemma) Set

$$[\xi_i'', \xi_j''] = \nu^2 a_{ij}^{(2)} + \nu^3 a_{ij}^{(3)} + \cdots,$$
$$[\eta_i'', \eta_j''] = \nu^2 a_{n+i,n+j}^{(2)} + \nu^3 a_{n+i,n+j}^{(3)} + \cdots,$$
$$[\xi_i'', \eta_j''] = -\nu\delta_{ij} + \nu^2 a_{i,n+j}^{(2)} + \cdots.$$

By the Jacobi identity, we have

$$(4.8) \qquad \sum_{(i,j,k):cyclic} \{\zeta_i, a_{jk}^{(2)}\} = 0 \quad (1 \le i, j, k \le 2n),$$

where $(\zeta_1, \cdots, \zeta_{2n}) =(\xi_1'', \cdots, \xi_n'', \eta_1'', \cdots, \eta_n'')$. Define a 2-form ω' on $\phi_{U_{n+1}}$ as

$$\omega' = \frac{1}{2} \sum_{1 \le i,j \le n} (a_{n+i,n+j}^{(2)} dx_i \wedge dx_j - 2a_{n+i,j}^{(2)} dx_i \wedge dy_j + a_{ij}^{(2)} dy_i \wedge dy_j),$$

where $\xi_i'' = x_i + O(\nu)$, $\eta_i'' = y_i + O(\nu)$ and $x_1, \cdots, x_n, y_1, \cdots, y_n$ is a symplectic coordinate system on $\phi_{n+1}(U_{n+1})$. Then (4.8) implies $d\omega'$=0. Since $\phi_{n+1}(U_{n+1})= \mathbf{C}^n$ is 2-connected, there exists $\theta' = \sum_{s=1}^n (b_s dx_s + b_{n+s} dy_s)$ such that $\omega'=d\theta'$.

Consider

$$\begin{cases} \xi_i' &= \xi_i'' + \nu b_{n+i} \\ \eta_i' &= \eta_i'' - \nu b_i. \end{cases}$$

Replacing $(\xi_i'', \cdots, \xi_n'', \eta_1'', \cdots, \eta_n'')$ by $(\xi_1', \cdots, \xi_n', \eta_1', \cdots, \eta_n')$, we see that

$$\begin{cases} [\xi_i', \xi_j'] &= [\eta_i', \eta_j'] = 0 \mod \nu^3 \\ [\xi_i', \eta_j'] &= -\nu\delta_{ij} \mod \nu^3. \end{cases}$$

Repeating this procedure for ν^3,ν^4, $\cdots$ finishes the proof. Q.E.D.

Note that $(w_1, \cdots, w_n)$ in 4.3 is a complex local coordinate system of $P_n(\mathbf{C})$ and hence $(\xi_1'', \cdots, \xi_n'', \eta_1'', \cdots, \eta_n'')$ is a real local coordinate system of $P_n(\mathbf{C})$. Since $\xi_i = \xi_i''$, $\eta_i = \eta_i''$ mod ν in the above proof, Theorem 4.5 implies also Theorem 2.4.

Using ν,ξ_1,$\cdots$,ξ_n η_1,$\cdots$,η_n obtained above, we may define the $\odot$-product on $\mathfrak{a}_{U_{n+1}}[[\nu]]$ by the same manner as in 2.2. Let $B_{\xi,\eta}$ be the closure of the space of all polynomials of the form $\sum a_{\alpha\beta}\xi^\alpha \odot \eta^\beta$, $a_{\alpha\beta} \in \mathbf{R}$. It is a $\odot$-subalgebra over $\mathbf{R}$ of $(\mathfrak{a}_{U_{n+1}}[[\nu]], \odot)$, and $(B_{\xi,\eta}, \odot)$ is isomorphic to the algebra $(C^\infty(U_{n+1}; \mathbf{R}), \cdot)$. Via this isomorphism, we can regard ξ_1,$\cdots$,ξ_n, η_1,$\cdots$,η_n as coordinate functions on U_{n+1}.

Since $\phi_{n+1}(U_{n+1})$=$\mathbf{C}^n$, we have

Corollary 4.6. $(\mathfrak{a}_{U_{n+1}}[[\nu]], *) \cong \mathcal{F}(W_{\mathbf{C}^n})$

Since U_{n+1} can be replaced by any U_l, this result shows that $\mathfrak{a}[[\nu]]$ is obtained by patching $\mathcal{F}(W_{\mathbf{C}^n})$'s, and hence $\mathfrak{a}[[\nu]]$ can be regarded as the space of certain sections of a Weyl algebra bundle $W_{P_n(\mathbf{C})}$ over $P_n(\mathbf{C})$. The coordinate transformations are given by isomorphisms

$$\Psi_{k,l} : \mathcal{F}(W_{\mathbf{C}^n - \{k\}}) \longrightarrow \mathcal{F}(W_{\mathbf{C}^n - \{l\}})$$

with $\Psi_{k,l}(\nu)$=ν, where $\mathbf{C}^n - \{k\}$=$\mathbf{C}^n - \{\xi_k = 0\}$.

Remark 1. The $\odot$-product defined on $\mathfrak{a}_{U_{n+1}}$ may *not equal* the usual $\cdot$-product.

Remark 2. By Lemma 3.2 of [10], $\Psi_{k,l}$ are given as the pull back of pre-Weyl diffeomorphisms $\Phi_{k,l}$: $W_{\mathbf{C}^n - \{l\}} \longrightarrow W_{\mathbf{C}^n - \{k\}}$, where $W_{\mathbf{C}^n - \{k\}}$

$=(\mathbf{C}^n - \{k\}) \times \mathbf{W}$. Thus, strictly speaking, we should call the obtained Weyl algebra bundle $W_{P_n(\mathbf{C})}$ a pre-Weyl manifold.

It is, however, possible to correct $W_{P_n(\mathbf{C})}$ to a genuine Weyl manifold defined in [10] by the same procedure discussed in [10, §5]. This proves Theorem 2.5.

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