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Characterization of Images of Radon Transforms

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§0. Introduction

Since F. John [7], the characterization of images of Radon transforms has been one of the main subjects of the theory of Radon transforms. When we recall that the origin of Radon transform was the transform of functions on the 2-sphere by averaging over the great circles, it is rather surprising to find that the characterization of images of Radon transforms on compact symmetric spaces had not been treated until E. Grinberg [4]. There Grinberg showed that the image of Radon transform concerning real or complex Grassmann manifolds can be characterized by an invariant system of differential operators, using the representation theoretical argument. We can see easily that the characterization may also be done by an invariant differential operator of higher order, though Grinberg did not mention it explicitly.

The purpose of this paper is to give another type of characterization, that is, the characterization by an invariant differential operator that takes values in the sections of a vector bundle. The approach by Grinberg used the left action of a group, and ours uses the right action, which lies, in a sense, on the other side with respect to the bi-sided invariant differential operator. We hope our approach will be the first step to fill some vacancy in the theory of invariant differential operators on compact symmetric spaces.

§1. The Radon transform on the sphere

We first consider the case of the standard sphere S^n of radius 1 in the Euclidean space $\mathbf{R}^{n+1}$. A geodesic γ of the sphere S^n is nothing but a great circle, which is determined by a 2-dimensional vector subspace of $\mathbf{R}^{n+1}$. We shall treat the geodesics with their orientation for convenience' sake. The set of oriented geodesics, which we denote by $\mathrm{Geod}\, S^n$, is the oriented real Grassmann manifold $G_{n+1,2}(\mathbf{R})$.

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For a function f on S^n, we define its Radon transform $\mathcal{R}(f)$ to be a function on $\operatorname{Geod} S^n$, the value of which at a point $\gamma \in \operatorname{Geod} S^n$ is given by the average of f over γ. More specifically speaking, we set

$$(\mathcal{R}(f))(\gamma) = \frac{1}{2\pi}\int_0^{2\pi} f(\gamma(s))\,ds,$$

where $\gamma(s)$ is the parametrization of γ by its arclength. We will always concern with smooth functions and denote the space of smooth functions by $\mathcal{F}$. The Radon transform $\mathcal{R}$ is a mapping from the space $\mathcal{F}(S^n)$ to the space $\mathcal{F}(\operatorname{Geod} S^n)$.

The antipodal mapping σ on the sphere is a smooth involution given by $\sigma(x) = -x$ for $x \in S^n \subset \mathbf{R}^{n+1}$. A function f on the sphere is called even when $f \circ \sigma = f$, and odd when $f \circ \sigma = -f$. We denote by $\mathcal{F}_{\text{even}}(S^n)$ and $\mathcal{F}_{\text{odd}}(S^n)$ the spaces of smooth even functions and smooth odd functions, respectively. It is obvious that the space $\mathcal{F}_{\text{odd}}(S^n)$ is included in the kernel of the Radon transform $\mathcal{R}$.

In the case of $n = 2$, $\operatorname{Geod} S^2$ is isomorphic to S^2, for an oriented geodesic has one-to-one correspondence with the *north pole* that makes that geodesic the *equator* with the suitable orientaion of longitude. The Radon transform $\mathcal{R}$ on S^2 is considered to be a mapping from the space $\mathcal{F}(S^2)$ to itself. It is also obvious that the image of $\mathcal{R}$ is included in the space $\mathcal{F}_{\text{even}}(S^2)$.

The following theorem by P. Funk [2] was the starting point of the theory of Radon transform.

Theorem 1.1. *The kernel of the Radon transform $\mathcal{R}$ on S^2 is equal to the space $\mathcal{F}_{\text{odd}}(S^2)$. As the mapping from $\mathcal{F}_{\text{even}}(S^2)$ to itself, the Radon transform $\mathcal{R}$ is an isomorphism.*

We can generalize this theorem to higher dimensions in the same form if we consider not the geodesic, that is, the great circle, but the great sphere of codimension 1. See, for example, S. Helgason [6]. Since the average of a function $f \in \mathcal{F}(S^n)$ over a great sphere of codimension 1 can be calculated by averaging the values of $(\mathcal{R}(f))(\gamma)$ for all the γ included in the great sphere, we can deduce the following theorem.

Theorem 1.2. *The kernel of the Radon transform $\mathcal{R}$ on S^n is equal to the space $\mathcal{F}_{\text{odd}}(S^n)$. The image* $\operatorname{Im}\mathcal{R}$ *is a closed subspace of $\mathcal{F}(\operatorname{Geod} S^n)$ in the C^∞-topology.*

We notice that the latter part of Theorem 1.2 is a consequence of the inversion formula of the Radon transform concerning the great sphere of codimension 1.

Since the dimension of $\operatorname{Geod} S^n$ is greater than n for $n \geq 3$, we cannot expect the Radon transform $\mathcal{R}$ to be surjective. In the next section, we try to find a good characterization of the image of $\mathcal{R}$.

§2. The differential operator $\mathcal{L}$ on $\operatorname{Geod} S^n$

We fix an orthonomal basis $\{e_1, e_2, \dots, e_{n+1}\}$ of $\mathbf{R}^{n+1}$. The special orthogonal group $SO(n+1)$ acts on S^n transitively and isometrically. We set $G = SO(n+1)$, and denote the isotropy group at $e_1 \in S^n$ by $H \cong SO(n)$. The group G acts transitively on the set of all oriented geodesics $\operatorname{Geod} S^n$, too. We take the oriented geodesic γ_0 that passes through e_1 and is pointing e_2 there as the origin of $\operatorname{Geod} S^n$ and denote the isotropy group at γ_0 by $K \cong SO(2) \times SO(n-1)$. We consider $\operatorname{Geod} S^n$ as a symmetric space G/K with the standard invariant metric.

We take $\{X_{ij}\}_{1 \leq j < i \leq n+1}$ as a basis of the Lie algebra $\mathfrak{g}$ of G, where X_{ij} is a matrix whose (k,l)-element is given by $\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$. As usual, the orthogonal complement of the Lie algebra $\mathfrak{k}$ of K in $\mathfrak{g}$ is denoted by $\mathfrak{m}$.

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}, \qquad \mathfrak{m} = \bigoplus_{\substack{3 \leq a \leq n+1 \\ l=1,2}} \mathbf{R} X_{a,l}.$$

We always consider the action of G on the functions of a G-space to be the contravariant action of the left action.

$$(g \cdot F)(x) = F(g^{-1}x), \quad \text{for } g \in G,\ x \text{ a point of a } G\text{-space}.$$

We consider the group G to be a G-space by multiplication from the left. The element X_{ij} is considered to be an invariant differential operator acting on the space $\mathcal{F}(G)$ as follows.

$$(X_{ij}F)(g) = \frac{d}{dt} F(g \exp tX_{ij})|_{t=0}, \quad \text{for } F \in \mathcal{F}(G),\ g \in G.$$

For each pair of integers a and b satisfying $3 \leq a < b \leq n+1$, we define a second order differential operator L_{ab} acting on $\mathcal{F}(G)$ by

$$(L_{ab}F)(g) = (X_{a1}(X_{b2}F))(g) - (X_{a2}(X_{b1}F))(g).$$

The commutation relations $[X_{a1}, X_{b2}] = [X_{a2}, X_{b1}] = 0$ enable us to rewrite it as

$$(L_{ab}F)(g) = \left(\frac{\partial^2}{\partial t_{a1} \partial t_{b2}} - \frac{\partial^2}{\partial t_{a2} \partial t_{b1}} \right) F(g \exp X(t)) \bigg|_{t=0},$$

where an element $X(t)$ of $\mathfrak{m}$ depending on $t = \{t_{al}\}_{\substack{3\le a\le n+1\\ l=1,2}}$ is given by $X(t) = \sum_{\substack{3\le a\le n+1\\ l=1,2}} t_{al}X_{al}$.

The space $\mathcal{F}(\mathrm{Geod}\, S^n)$ is regarded as a subspace of $\mathcal{F}(G)$ consisting of the elements F that satisfy $F(gk) = F(g)$ for all $k \in K$, $g \in G$. For these elements F, we have

$$\begin{aligned}(L_{ab}F)(gk) &= \left(\frac{\partial^2}{\partial t_{a1}\partial t_{b2}} - \frac{\partial^2}{\partial t_{a2}\partial t_{b1}}\right) F(gk\exp X(t)k^{-1}k)\Big|_{t=0}\\ &= \left(\frac{\partial^2}{\partial t_{a1}\partial t_{b2}} - \frac{\partial^2}{\partial t_{a2}\partial t_{b1}}\right) F(g\exp \mathrm{Ad}(k)X(t))\Big|_{t=0}.\end{aligned}$$

Notice that $\mathrm{Ad}(k)X(t)$ is written as $X(t')$, where t' is a linear combination of t determined by k. For an element $k \in K$ of the form

$$k = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix} \times (k_{cd})_{3\le c,d\le n+1} \qquad ((k_{cd}) \in SO(n-1)),$$

an easy calculation gives

$$(L_{ab}F)(gk) = \sum_{3\le c<d\le n+1} (k_{ca}k_{db} - k_{da}k_{cb})(L_{cd}F)(g).$$

Now we consider a vector space V of dimension $(n-1)(n-2)/2$, with a fixed basis $\{u_a \wedge u_b\}$ $(3 \le a < b \le n+1)$ and an action ρ of K given by

$$\rho(k)(u_a \wedge u_b) = \sum_{3\le c<d\le n+1} (k_{ca}k_{db} - k_{da}k_{cb})\, u_c \wedge u_d,$$

and define a V-valued function $\mathcal{L}F$ on G by

$$\mathcal{L}F = \sum_{3\le a<b\le n+1} (L_{ab}F)\, u_a \wedge u_b.$$

Then we have

$$\begin{aligned}(\mathcal{L}F)(gk) &= \sum_{3\le a<b\le n+1} (L_{ab}F)(gk)\, u_a \wedge u_b\\ &= \rho(k^{-1})(\mathcal{L}F)(g),\end{aligned}$$

which means that $\mathcal{L}F$ is a section of the vector bundle $E = G \times_K V$ on $G/K \cong \mathrm{Geod}\, S^n$ of rank $(n-1)(n-2)/2$, associated with the principal bundle $G \to G/K$ under the representation ρ.

We take the formal adjoint operator $\mathcal{L}^*$ of $\mathcal{L}$ with respect to the invariant inner products on $\mathcal{F}(\mathrm{Geod})$ and $C^\infty(E)$ induced by the invariant measure on G, and set $\mathcal{D} = \mathcal{L}^*\mathcal{L}$. In fact, the differential operator $\mathcal{D}$ is given by

$$(\mathcal{D}F)(g) = \sum_{3 \leq a < b \leq n+1} (L_{ab}(L_{ab}F))(g).$$

By construction, it is obvious that $\mathcal{L}$ and $\mathcal{D}$ are invariant differential operators.

Proposition 2.1. *The image of the Radon transform $\mathcal{R}$ is included in the kernel of the differential operator $\mathcal{D}$.*

Proof. Since $\operatorname{Ker}\mathcal{D}$ is equal to $\operatorname{Ker}\mathcal{L}$, it is enough to show that $\mathcal{L}(\mathcal{R}(f))$ vanishes for any function f on S^n. Since $\mathcal{R}$ and $\mathcal{L}$ are invariant operators, it is enough to show $\mathcal{L}(\mathcal{R}(f))(\gamma_0) = 0$ for any $f \in \mathcal{F}(S^n)$. (Notice that, for any $\gamma \in \mathrm{Geod}\, S^n$, there exists an element $g \in G$ that satisfies $\gamma = g\gamma_0$, and that we have $\mathcal{L}(\mathcal{R}(f))(\gamma) = \mathcal{L}(\mathcal{R}(g^{-1} \cdot f))(\gamma_0)$.)

Let us fix the indices a and b and show that $L_{ab}(\mathcal{R}(f))(e) = 0$ for any $f \in \mathcal{F}(S^n)$. We recall that, for $f \in \mathcal{F}(G/H)$, our definition of the Radon transform $\mathcal{R}$ is rewritten as

$$\mathcal{R}(f)(g) = \frac{1}{2\pi} \int_0^{2\pi} f(gk(\theta))\, d\theta,$$

where

$$k(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \times \mathrm{Id}.$$

Therefore we have

$$\begin{aligned}
&L_{ab}(\mathcal{R}(f))(e) \\
&= \frac{1}{2\pi} \left(\frac{\partial^2}{\partial t_{a1}\partial t_{b2}} - \frac{\partial^2}{\partial t_{a2}\partial t_{b1}} \right) \int_0^{2\pi} f(\exp X(t)k(\theta))\, d\theta \bigg|_{t=0} \\
&= \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial^2}{\partial t_{a1}\partial t_{b2}} f(\exp X(t)k(\theta)) \bigg|_{t=0} d\theta \\
&\quad - \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial^2}{\partial t_{a2}\partial t_{b1}} f(\exp X(t)k(\theta)) \bigg|_{t=0} d\theta.
\end{aligned}$$

The value of $f \in \mathcal{F}(G)$ at the point $\exp X(t)k(\theta) \in G$ where the components of t vanish except for $t_{a1} = r$ and $t_{b2} = s$ is given by the value of $f \in \mathcal{F}(S^n)$ at the point $\cos\theta(\cos r\, e_1 + \sin r\, e_a) + \sin\theta(\cos s\, e_2 +$

$\sin s\, e_b) \in S^n$. Therefore the former integral in the last expression is equal to

$$\frac{1}{2\pi}\int_0^{2\pi} \cos\theta \sin\theta (\nabla_{e_a}\nabla_{e_b} f)(\cos\theta\, e_1 + \sin\theta\, e_2)\, d\theta.$$

Since the point $\exp X(t)k(\theta)$ where the components of t vanish except for $t_{a2} = r$ and $t_{b1} = s$ corresponds to the point $\cos\theta(\cos s\, e_1 + \sin s\, e_b) + \sin\theta(\cos r\, e_2 + \sin r\, e_a)$, the latter integral in the last expression has the same value, and hence $L_{ab}(\mathcal{R}(f))(e)$ vanishes. Q.E.D.

Remark 2.2. The vanishing of $L_{ab}(\mathcal{R}(f))(e)$ is deduced from the geometric observation related to two 2-parameter families of geodesics, which is the same argument as given in F. John [7].

Remark 2.3. The ring of invariant differential operators on the rank 2 symmetric space $\mathrm{Geod}\, S^n$ ($n > 3$) is generated by the Laplace operator Δ and the 4-th order differential operator $\mathcal{D}$. For the case $n = 3$, see the next section.

§3. The case $n = 3$

Let us recall the elementary facts on $\mathrm{Geod}\, S^3$. An oriented great circle on S^3 is specified by its point e_1 and its unit tangent vector e_2 at e_1, and corresponds one-to-one to the exterior product $\omega = e_1 \wedge e_2$ with unit norm. A 2-vector $\omega \in \bigwedge^2 \mathbf{R}^4$ with unit norm corresponds to a great circle if and only if it is decomposable, that is, $\omega \wedge \omega$ vanishes. In view of the Hodge star operator $*$ on $\bigwedge^2 \mathbf{R}^4$, the latter condition is the same as saying the norm of the self-dual part $\omega_+ = (\omega + *\omega)/2$ is equal to the norm of th anti-self-dual part $\omega_- = (\omega - *\omega)/2$. Since the self-dual 2-vectors and the anti-self-dual 2-vectors both form the 3-dimensional vector spaces V_+ and V_-, a decomposable 2-vector with unit norm has one-to-one correspondence with the product of two 2-spheres, $S^2_+ \subset V_+$ and $S^2_- \subset V_-$, with radius $1/\sqrt{2}$. We thus have the isomorphism $\mathrm{Geod}\, S^3 \cong S^2_+ \times S^2_-$.

In the case $n = 3$, since the representation ρ in the last section is trivial, the vector bundle E of rank 1 is also trivial. We have only to consider the invariant differential operator $\mathcal{L} = L_{34}$. In view of the above isomorphism, $\mathcal{L}$ is shown to be the differential operator $\Delta_+ - \Delta_-$, where $\Delta_\pm$ is the Laplace operator on $S^2_\pm$.

Notice that the ring of invariant differential operators on the rank 2 (but not irreducible) symmetric space $\mathrm{Geod}\, S^3$ is generated by the Laplacian $\Delta = \Delta_+ + \Delta_-$ and the second order differential operator $\mathcal{L}$.

We shall show the main theorem for S^3 by means of the representation theory of $SO(4)$.

We denote by E^n_k the space of fucntions on S^n that are the restrictions of the harmonic polynomials on $\mathbf{R}^{n+1}$ of degree k. The following decompositions of the function spaces are well-known.

Lemma 3.1. *We have the direct sum decompositions*

$$\mathcal{F}(S^3) \approx \sum_{k=0}^{\infty} E^3_k,$$

$$\mathcal{F}_{\text{even}}(S^3) \approx \sum_{k=0}^{\infty} E^3_{2k},$$

$$\mathcal{F}(S^2_+ \times S^2_-) \approx \sum_{k,l=0}^{\infty} E^2_k \boxtimes E^2_l,$$

where the symbol $\approx$ means that the right-hand side is densely included in the left-hand side.

The above decompositions are in fact the decompositions as $SO(4)$-modules. We fix the Lie algebra $\mathfrak{t} \subset \mathfrak{g}$ corresponding to $SO(2)\times SO(2) \subset SO(4)$, and the basis $\{\lambda_1, \lambda_2\}$ of the complexified dual space $\mathfrak{t}^*_{\mathbf{C}}$ of $\mathfrak{t}$ as follows.

$$\lambda_1\left(\begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}, \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix}\right) = \sqrt{-1}a,$$

$$\lambda_2\left(\begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}, \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix}\right) = \sqrt{-1}b.$$

We order them as $\lambda_1 > \lambda_2$. The following lemma is easy to verify.

Lemma 3.2. *The space E^3_k is an irreducible $SO(4)$-module with the highest weight $k\lambda_1$. The space $E^2_k \boxtimes E^2_l$ is an irreducible $SO(4)$-module with the highest weight $(k+l)\lambda_1 + (k-l)\lambda_2$.*

Theorem 3.3. *The image of the Radon transform $\mathcal{R}$ is equal to the kernel of the differential operator $\mathcal{L}$.*

Proof. Since the operator $\mathcal{R}$ is injective on $\mathcal{F}_{\text{even}}(S^3)$ and commutes with the action of $SO(4)$, it isomorphically maps the space E^3_{2k} with the highest weight $2k\lambda_1$ to the space of the same highest weight, which must be the space $E^2_k \boxtimes E^2_k$. Therefore we have $\operatorname{Im}\mathcal{R} \approx \sum_k E^2_k \boxtimes E^2_k$.

On the other hand, since Δ_+ acts on $E_k^2 \boxtimes E_l^2$ as $k(k+1)\mathrm{Id}$ and Δ_- acts on it as $l(l+1)\mathrm{Id}$,

$$\mathrm{Ker}(\Delta_+ - \Delta_-) \approx \sum_{\substack{k,l \\ k(k+1)=l(l+1)}} E_k^2 \boxtimes E_l^2 = \sum_k E_k^2 \boxtimes E_k^2.$$

Since we have $\mathrm{Im}\,\mathcal{R} \subset \mathrm{Ker}\,\mathcal{L}$ and these closed subspaces include the same dense subspace in common, they must coincide. Q.E.D.

§4. Reduction to the case $n = 3$

We shall prove the main theorem for a general case by reducing it to the case $n = 3$.

We denote by S_0^3 the totally geodesic 3-sphere in S^n that is included in the subspace spanned by e_1, e_2, e_3, and e_4. All the other totally geodesic 3-sphere in S^n is written as gS_0^3 for some element $g \in G = SO(n+1)$. The manifold $N = \mathrm{Geod}\,S_0^3$ of the oriented great circles included in S_0^3 is a homogeneous manifold G'/K', where G' is $SO(4)$ considered as a subgroup of G and K' is $G' \cap K \cong SO(2) \times SO(2)$.

Now let us consider what happens when the vector bundle E is restricted to N. Since E is an associated vector bundle $G \underset{K}{\times} V$ and N is a homogeneous manifold G'/K', we have $E|_N = G' \underset{K'}{\times} V$, where the action of K' on V is that of K restricted. When the representation ρ of K in V is restricted to the subgroup K', it decomposes to a sum of irreducible components and has the subspace spanned by the vector $u_3 \wedge u_4$ as its irreducible component with trivial action. Therefore the vector bundle $E|_N$ splits to a sum of subbundles, one of which is the trivial subbundle of rank 1 corresponding to $u_3 \wedge u_4$.

When a section $\mathcal{L}(F)$ of E for $F \in \mathcal{F}(\mathrm{Geod}\,S^n)$ is restricted to N, its $u_3 \wedge u_4$-component is just $L_{34}(F)$, and, by construction, is equal to $\mathcal{L}(F|_N)$. The vanishing of $\mathcal{L}(F)$ implies the vanishing of $\mathcal{L}(F|_N)$, and $F|_N$ is in the image of the Radon transform on S_0^3 by Theorem 2.1. Taking account of the equivariance of our construction, we have the following lemma.

Lemma 4.1. *If $F \in \mathcal{F}(\mathrm{Geod}\,S^n)$ is in the kernel of $\mathcal{L}$, its restriction to the submanifold $gN = \mathrm{Geod}(gS_0^3)$ is in the image of the Radon transform on $S^3 = gS_0^3$ for every $g \in G$.*

We notice that this lemma implys that $F \in \mathrm{Ker}\,\mathcal{L}$ is an even function in the sense that, for any totally geodesic $S^2 \subset S^n$, the restriction of

F to $\operatorname{Geod} S^2$ is an even function; there exists a totally geodesic S^3 statisfying $S^2 \subset S^3 \subset S^n$ and the restriction of F to $\operatorname{Geod} S^3$, and hence to $\operatorname{Geod} S^2$, is in the image of the Radon transform.

Theorem 4.2. *For $n \geq 3$, if $F \in \mathcal{F}(\operatorname{Geod} S^n)$ is in the kernel of $\mathcal{L}$, it is in the image of the Radon transform $\mathcal{R}$ on S^n. Therefore we have* $\operatorname{Ker} \mathcal{D} = \operatorname{Ker} \mathcal{L} = \operatorname{Im} \mathcal{R}$.

Proof. We take a point $x \in S^n$ and shall fix a value $f(x)$ of a function $f \in \mathcal{F}(S^n)$ for which we should have $\mathcal{R}(f) = F$.

If we choose a totally geodesic 2-sphere S^2 containing x, we can uniquely determine an even function f on S^2 with the property that the image of the Radon transform on S^2 of f is equal to the restriction of F, since the Radon transform on S^2 is an isomorphism on the even functions. We claim that the value $f(x)$ does not depend on the totally geodesic S^2 chosen.

For any two totally geodesic 2-spheres S_a^2 and S_b^2 containing x, there exist the third totally geodesic 2-sphere S_c^2 containing x and two totally geodesic 3-spheres S_{ac}^3 and S_{bc}^3 that satisfy S_a^2, $S_c^2 \subset S_{ac}^3$ and S_b^2, $S_c^2 \subset S_{bc}^3$. We denote by f_r ($r = a$, b, or c) the even functions on S_r^2 with the property that the image of the Radon transform on S_r^2 of f_r is equal to the restriction of F to $\operatorname{Geod} S_r^2$. By the last lemma, the restriction of F to $\operatorname{Geod} S_{ac}^3$ is in the image of the Radon transform on S_{ac}^3 of a function on S_{ac}^3, say, f_{ac}. Taking the even part of f_{ac} if needed, we may assume that f_{ac} is an even function. Since the Radon transform is injective on the even functions, the restriction of f_{ac} to S_a^2 is equal to f_a and that to S_c^2 is equal to f_c. Therefore we have $f_a(x) = f_c(x)$ and, by the same reasoning, $f_c(x) = f_b(x)$, which assures our claim.

We see easily that the function f on S^n thus constructed is continuous and has the property $\mathcal{R}(f) = F$. By the inversion formula, f is shown to be smooth. Q.E.D.

§5. The case of the complex projective space

In the case of the complex projective space $P^n(\mathbf{C})$, we consider the projective line as its counter part of the oriented geodesic in the sphere. Since a projective line $C \subset P^n(\mathbf{C})$ corresponds to a 2-dimentional complex vector subspace of $\mathbf{C}^{n+1}$, the set of projective lines is the complex Grassmann manifold $G_{n+1,2}(\mathbf{C})$. We define the Radon transform $\mathcal{R}(f)$ of a function f on the complex projective space $P^n(\mathbf{C})$ by assigning to each point C of $G_{n+1,2}(\mathbf{C})$ the averaged value of f over C.

By the same argument as in Theorem 1.2, we have the following theorem.

Theorem 5.1. *The Radon transform $\mathcal{R}$ on $P^n(\mathbf{C})$ is an injective mapping from $\mathcal{F}(P^n(\mathbf{C}))$ to $\mathcal{F}(G_{n+1,2}(\mathbf{C}))$. The image* $\operatorname{Im}\mathcal{R}$ *is closed in the C^∞-topology.*

For $n=2$, the complex Grassmann manifold $G_{3,2}(\mathbf{C})$ is isomorphic to the complex projective space $P^2(\mathbf{C})$ and the Radon transform on $P^2(\mathbf{C})$ is an isomorphism. For $n \geq 3$, the dimension of $G_{n+1,2}(\mathbf{C})$ is greater than that of $P^n(\mathbf{C})$ and the Radon transform is not surjective.

We fix an orthonomal basis $\{e_1, e_2, \dots, e_{n+1}\}$ of $\mathbf{C}^{n+1}$. The unitary group $U(n+1)$ acts on $P^n(\mathbf{C})$ transitively and isometrically. We here set $G = U(n+1)$, and denote the isotropy group at $[e_1] \in P^n(\mathbf{C})$ by $H \cong U(1) \times U(n)$. The group G acts transitively on the complex Grassmann manifold $G_{n+1,2}(\mathbf{C})$, too. We take the vector subspace spanned by e_1 and e_2 as the origin C_0 of $G_{n+1,2}(\mathbf{C})$ and denote the isotropy group at C_0 by $K = U(2) \times U(n-1)$. We consider $G_{n+1,2}(\mathbf{C})$ as a symmetric space G/K with the standard invariant metric.

We denote by $\mathfrak{g}$, $\mathfrak{k}$, and $\mathfrak{h}$ the Lie algebras of G, K, and H, respectively. The orthogonal complement $\mathfrak{m}$ of $\mathfrak{k}$ in $\mathfrak{g}$ is the subspace given by

$$\mathfrak{m} = \{\, Z(z_1, z_2) \in M(n+1; \mathbf{C}) \mid z_1, z_2 \in \mathbf{C}^{n-1} \,\},$$

where, for each two elements $z_l = (z_{al}) \in \mathbf{C}^{n-1}$ ($3 \leq a \leq n+1$, $l = 1$ or 2), the (i,j)-element $(Z)_{ij}$ of an $(n+1) \times (n+1)$-matrix $Z = Z(z_1, z_2)$ is given by

$$(Z)_{ij} = \begin{cases} z_{ij}, & \text{for } 3 \leq i \leq n+1,\ j = 1 \text{ or } 2, \\ -\overline{z_{ji}}, & \text{for } i = 1 \text{ or } 2,\ 3 \leq j \leq n+1, \\ 0, & \text{otherwise.} \end{cases}$$

In the following we always treat the $\mathbf{C}$-valued functions and denote by $\mathcal{F}(G)$ the space of $\mathbf{C}$-valued smooth functions on G. For each pair (a,l) of indices with $3 \leq a \leq n+1$ and $l = 1$ or 2, we define differential operators Z_{al} and $\overline{Z}_{al}$ on $\mathcal{F}(G)$ by

$$(Z_{al}F)(g) = \left.\frac{\partial}{\partial z_{al}} F(g \exp Z)\right|_{z_1 = z_2 = 0},$$
$$(\overline{Z}_{al}F)(g) = \left.\frac{\partial}{\partial \bar{z}_{al}} F(g \exp Z)\right|_{z_1 = z_2 = 0},$$
$$(F \in \mathcal{F}(G),\ g \in G).$$

The formal adjoint operator $(Z_{al})^*$, with respect to the invariant hermitian inner product on $\mathcal{F}(G)$ induced by the invariant measure on G, is equal to $-\overline{Z}_{al}$. For $3 \leq a < b \leq n+1$, we define differential operators L_{ab} and L^*_{ab} on $\mathcal{F}(G)$ by

$$\begin{aligned}(L_{ab}F)(g) &= \left(\frac{\partial^2}{\partial z_{a1}\partial z_{b2}} - \frac{\partial^2}{\partial z_{a2}\partial z_{b1}}\right) F(g\exp Z)\bigg|_{z_1=z_2=0} \\ &= (Z_{a1}(Z_{b2}F))(g) - (Z_{a2}(Z_{b1}F))(g), \\ (L^*_{ab}F)(g) &= \left(\frac{\partial^2}{\partial \bar{z}_{a1}\partial \bar{z}_{b2}} - \frac{\partial^2}{\partial \bar{z}_{a2}\partial \bar{z}_{b1}}\right) F(g\exp Z)\bigg|_{z_1=z_2=0} \\ &= (\overline{Z}_{a1}(\overline{Z}_{b2}F))(g) - (\overline{Z}_{a2}(\overline{Z}_{b1}F))(g), \\ &\qquad\qquad (F \in \mathcal{F}(G),\ g \in G).\end{aligned}$$

Let F be a smooth function on G/K, that is, a function $F \in \mathcal{F}(G)$ statisfying $F(gk) = F(g)$ $(k \in K)$. For an element $k = (k_{ij})$ of K, we have

$$(L_{ab}F)(gk) = \overline{(k_{11}k_{22} - k_{12}k_{21})} \sum_{3\leq c<d\leq n+1} (k_{ca}k_{db} - k_{da}k_{cb})(L_{cd}F)(g),$$
$$(L^*_{ab}F)(gk) = (k_{11}k_{22} - k_{12}k_{21}) \sum_{3\leq c<d\leq n+1} \overline{(k_{ca}k_{db} - k_{da}k_{cb})}(L^*_{cd}F)(g).$$

Now we consider a complex vector space V of dimension $(n-1)(n-2)/2$, with a fixed basis $\{u_a \wedge u_b\}$ $(3 \leq a < b \leq n+1)$ and an action ρ of K given by

$$\rho(k)(u_a \wedge u_b) = (k_{11}k_{22} - k_{12}k_{21}) \sum_{3\leq c<d\leq n+1} \overline{(k_{ca}k_{db} - k_{da}k_{cb})}\, u_c \wedge u_d.$$

For a function $F \in \mathcal{F}(G/K)$, a V-valued function $\mathcal{L}F$ on G defined by $\mathcal{L}F = \sum_{3\leq a<b\leq n+1}(L_{ab}F)\, u_a \wedge u_b$ satisfies

$$\begin{aligned}&(\mathcal{L}F)(gk) \\ &\quad = \sum (L_{ab}F)(gk)\, u_a \wedge u_b \\ &\quad = \overline{(k_{11}k_{22} - k_{12}k_{21})} \sum (k_{ca}k_{db} - k_{da}k_{cb})(L_{cd}F)(g)\, u_a \wedge u_b \\ &\quad = (k^{-1}_{11}k^{-1}_{22} - k^{-1}_{21}k^{-1}_{12}) \sum \overline{(k^{-1}_{ac}k^{-1}_{bd} - k^{-1}_{ad}k^{-1}_{bc})}(L_{cd}F)(g)\, u_a \wedge u_b \\ &\quad = \sum (L_{cd}F)(g)\, \rho(k^{-1})(u_c \wedge u_d), \\ &\quad = \rho(k^{-1})((\mathcal{L}F)(g)).\end{aligned}$$

It means that $\mathcal{L}F$ can be considered as a section of the vector bundle $E = G \times_K V$ over G/K.

We define a differential operator $\mathcal{D}$ on $\mathcal{F}(G/K)$ by $\mathcal{D} = \mathcal{L}^*\mathcal{L}$, where $\mathcal{L}^*$ is the formal adjoint operator of $\mathcal{L}$. It can be explicitly written as follows.

$$(\mathcal{D}F)(g) = \sum_{3\leq a<b\leq n+1} (L_{ab}^*(L_{ab}F))(g).$$

By construction, it is obvious that both $\mathcal{L}$ and $\mathcal{D}$ are invariant differential operators. In fact, it can be shown that the ring of invariant differential operators on the rank 2 symmetric space G/K is generated by the differential operator $\mathcal{D}$ and the Laplacian Δ.

Theorem 5.2. *The image of the Radon transform $\mathcal{R}$ on the complex projective space $P^n(\mathbf{C})$ is equal to the kernel of the differential operator $\mathcal{D}$ on the complex Grassmann manifold $G_{n+1,2}(\mathbf{C})$.*

We prove this theorem in the next section following the same steps as the sphere case.

§6. The proof of Theorem 5.2

We first fix a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{g}$, which is included in both $\mathfrak{k}$ and $\mathfrak{h}$, by

$$\mathfrak{t} = \{\,\mathrm{diag}(\mu_1,\dots,\mu_{n+1}) \mid \mu_i \in \sqrt{-1}\mathbf{R}, \quad \text{for } 1\leq i\leq n+1\,\},$$

where $\mathrm{diag}(\mu_1,\dots,\mu_{n+1})$ is a diagonal matrix with the diagonal elements $\mu_1, \dots, \mu_{n+1}$. We take as the basis of the complexified dual vector space $\mathfrak{t}_{\mathbf{C}}^*$ of $\mathfrak{t}$ the following elements $\lambda_1, \dots, \lambda_{n+1}$.

$$\lambda_i(\mathrm{diag}(\mu_1,\dots,\mu_{n+1})) = \mu_i, \qquad \text{for } 1\leq i\leq n+1.$$

We introduce an order on the real vector subspace of $\mathfrak{t}_{\mathbf{C}}^*$ spanned by $\lambda_1, \dots, \lambda_{n+1}$ such as $\lambda_1 > \cdots > \lambda_{n+1}$.

An irreducible G-module is specified by the highest weight, which has the form $l_1\lambda_1 + \cdots + l_{n+1}\lambda_{n+1}$, where $l_1, \dots, l_{n+1}$ are integers satisfying $l_1 \geq \cdots \geq l_{n+1}$. The same is true for an irreducible H-module or an irreducible K-module, and its highest weight has the form $h_1\lambda_1 + \cdots + h_{n+1}\lambda_{n+1}$, where $h_1, \dots, h_{n+1}$ are integers satisfying $h_2 \geq \cdots \geq h_{n+1}$, for the former, or the form $k_1\lambda_1 + \cdots + h_{n+1}\lambda_{n+1}$, where $k_1, \dots, k_{n+1}$ are integers satisfying $k_1 \geq k_2$ and $k_3 \geq \cdots \geq k_{n+1}$, for the latter. We shall denote an irreducible module with the highest weight Λ by $V(\Lambda)$.

When an irreducible G-module is considered as an H-module (resp. a K-module) by restricting the action, it decomposes into the sum of irreducible H-modules (resp. K-modules). The following two branching laws specify which irreducible module appears in the decomposition.

Theorem 6.1. *In the decomposition of the irreducible G-module with the highest weight $l_1\lambda_1 + \cdots + l_{n+1}\lambda_{n+1}$, an irreducible H-module with the highest weight $h_1\lambda_1 + \cdots + h_{n+1}\lambda_{n+1}$ appears if and only if $l_1 \geq h_2 \geq l_2 \geq \cdots \geq h_{n+1} \geq l_{n+1}$ and $\sum_{i=1}^{n+1} l_i = \sum_{i=1}^{n+1} h_i$. And then it appears only once.*

Theorem 6.2. *In the decomposition of the irreducible G-module with the highest weight $l_1\lambda_1 + \cdots + l_{n+1}\lambda_{n+1}$, an irreducible K-module with the highest weight $k_1\lambda_1 + \cdots + k_{n+1}\lambda_{n+1}$ appears if and only if $l_i \geq k_{i+2} \geq l_{i+2}$ for $1 \leq i \leq n-1$, $\sum_{i=1}^{n+1} l_i = \sum_{i=1}^{n+1} k_i$ $(= p)$, and the following condition is satisfied: Let the integers $m_1, \ldots, m_{2n}$ be the descending reordering of $l_1, \ldots, l_{n+1}$ and $k_3, \ldots, k_{n+1}$. The irreducible $U(2)$-module $V((p-k_2)\lambda_1 + (p-k_1)\lambda_2)$ appears in the decomposition of the tensor product of irreducible $U(2)$-modules $V(m_1\lambda_1 + m_2\lambda_2) \otimes \cdots \otimes V(m_{2n-1}\lambda_1 + m_{2n}\lambda_2)$.*

An irreducible K-module appears in the same times as the corresponding irreducible $U(2)$-module.

For their proofs, see H. Boerner [1] and J. Mikelsson [7].

Frobenius' reciprocity law enables us to determine the irreducible decomposition of the spaces $\mathcal{F}(G/H)$, $\mathcal{F}(G/K)$, and $C^\infty(E)$ as G-modules. For example, a G-module appears in the decomposition of $\mathcal{F}(G/H)$ if and only if its irreducible decomposition as an H-module includes a trivial H-module. An easy calculation shows the following proposition.

Proposition 6.3. *For $n \geq 3$, we have the direct sum decompositions*

$$\mathcal{F}(G/H) \approx \sum_{l=0}^{\infty} V(l\lambda_1 - l\lambda_{n+1}),$$

$$\mathcal{F}(G/K) \approx \sum_{l,m=0}^{\infty} V((l+m)\lambda_1 + m\lambda_2 - m\lambda_n - (l+m)\lambda_{n+1}).$$

In the same way, we can compute the decomposition of $C^\infty(E)$. A G-module appears in the decomposition of $C^\infty(E)$ if and only if its irreducible decomposition as a K-module includes a K-module isomorphic to (V, ρ), which is an irreducible K-module with the highest weight $\lambda_1 + \lambda_2 - \lambda_n - \lambda_{n+1}$. The result varies depending on n and is somewhat cumbersome. Anyway, what we need is the following proposition, which can be shown easily.

Proposition 6.4. *An irreducible G-module with the highest weight $l\lambda_1 - l\lambda_{n+1}$ $(l \geq 0)$ never appears in the decomposition of $C^\infty(E)$.*

Theorem 6.5. *The image of the Radon transform $\mathcal{R}$ on $P^n(\mathbf{C})$ is included in the kernel of the differential operator $\mathcal{L}$.*

Proof. Let us denote by W_l the irreducible G-submodule of $\mathcal{F}(P^n(\mathbf{C})) = \mathcal{F}(G/H)$ with the highest weight $l\lambda_1 - l\lambda_{n+1}$ $(l \geq 0)$. Since the Radon transform $\mathcal{R}$ is an injective G-homomorphism, $\mathcal{R}(W_l)$ is an irreducible G-submodule of $\mathcal{F}(G/K)$ with the same highest weight, by Schur's lemma. The differential operator $\mathcal{L}$ is also an G-homomorphism, and therefore $\mathcal{L}(\mathcal{R}(W_l))$ is an irreducible G-submodule of $C^\infty(E)$ with the same highest weight or vanishes totally. But, by Proposition 6.4, an irreducible G-module with the highest weight $l\lambda_1 - l\lambda_{n+1}$ $(l \geq 0)$ cannot be a G-submodule of $C^\infty(E)$. Thus we have $\mathcal{L}(\mathcal{R}(W_l)) = \{0\}$.

Since the direct sum $\sum \mathcal{R}(W_l)$ is dense in $\operatorname{Im}\mathcal{R}$, the image $\operatorname{Im}\mathcal{R}$ itself is included in the kernel $\operatorname{Ker}\mathcal{L}$. Q.E.D.

To prove the other inclusion, it is enough to show $\operatorname{Im}\mathcal{R} = \operatorname{Ker}\mathcal{D}$ for $n = 3$, because the same argument as in §4 holds for $P^n(\mathbf{C})$. In the case $n = 3$, we can explicitly compute how $\mathcal{D}$ acts on each irreducible G-sumodule of $\mathcal{F}(G/K)$. $(G = U(4),\ K = U(2) \times U(2).)$

A G-module U_{lm} with the highest weight $(l+m)\lambda_1 + m\lambda_2 - m\lambda_3 - (l+m)\lambda_4$ can be endowed with an invariant hermitian inner product, which is unique up to a constant factor. We fix one and denote it by $\langle\ ,\ \rangle$. By Theorem 6.2, the K-invariant elements in U_{lm} forms a 1-dimensional subspace, and we fix a K-invariant element v_K with unit norm. A G-isomorphism from U_{lm} into $\mathcal{F}(G/K)$ is given by

$$U_{lm} \ni v \mapsto f_v(g) = \langle \rho(g)v_K, v\rangle \in \mathcal{F}(G/K),$$

where ρ denotes the action of G on U_{lm}.

The computation can be simplified by studying the relations in the universal enveloping algebra $\mathcal{U}(\mathfrak{g}^{\mathbf{C}})$ of the complexification $\mathfrak{g}^{\mathbf{C}} = M(4, \mathbf{C})$ of the Lie algebra $\mathfrak{g}$. Notice that the action ρ of G can be extended to the action of $\mathcal{U}(\mathfrak{g}^{\mathbf{C}})$, denoted by the same letter ρ.

The differential operator $\mathcal{D}$ corresponds to an element D in $\mathcal{U}(\mathfrak{g}^{\mathbf{C}})$ by the following formula.

$$(\mathcal{D}f_v)(g) = \langle \rho(g)\rho(D)v_K, v\rangle.$$

We denote by E_{ij} a matrix whose (k,l)-element is given by $\delta_{ik}\delta_{jl}$. Then the element D in $\mathcal{U}(\mathfrak{g}^{\mathbf{C}})$ is written explicitly as

$$D = (E_{13}E_{24} - E_{14}E_{23})(E_{31}E_{42} - E_{32}E_{41}).$$

This element D commutes with the elements of $\mathfrak{k}^{\mathbf{C}}$ in $\mathcal{U}(\mathfrak{g}^{\mathbf{C}})$, but it does not belong to the center of $\mathcal{U}(\mathfrak{g}^{\mathbf{C}})$.

We introduce two elements D_1 and D_2 in the center of $\mathcal{U}(\mathfrak{g}^{\mathbf{C}})$.

$$\begin{aligned} D_1 &= \sum_{i,j=1}^{4} E_{ij}E_{ji}, \\ D_2 &= \sum_{\sigma,\tau\in\mathfrak{S}_4} \operatorname{sgn}(\sigma\tau)E_{\sigma(1)\tau(1)}E_{\sigma(2)\tau(2)}E_{\sigma(3)\tau(3)}E_{\sigma(4)\tau(4)}. \end{aligned}$$

Then a straightforward computation yields

$$24D \equiv D_2 + 2D_1 \bmod \mathcal{U}(\mathfrak{g}^{\mathbf{C}})\mathfrak{k}^{\mathbf{C}}.$$

Therefore we have $\langle \rho(g)\rho(D)v_K, v\rangle = (1/24)\langle \rho(g)\rho(D_2+2D_1)v_K, v\rangle$.

Since D_2+2D_1 is in the center of $\mathcal{U}(\mathfrak{g}^{\mathbf{C}})$, its action on the irreducible G-module U_{lm} is nothing but multiplication by a constant. The constant can be computed by its action on the maximal vector v_Λ, i.e., the vector of the highest weight. Let us denote by $\mathfrak{b}^+$ the subalgebra of $\mathfrak{g}^{\mathbf{C}}$ spanned by $\{E_{ij}\}_{i<j}$. Then a straightforward computation yields

$$\begin{aligned} D_1 \equiv{}& E_{11}^2 + E_{22}^2 + E_{33}^2 + E_{44}^2 \\ &+ 3(E_{11} - E_{44}) + E_{22} - E_{33} \bmod \mathcal{U}(\mathfrak{g}^{\mathbf{C}})\mathfrak{b}^+, \\ D_2 \equiv{}& 24E_{11}E_{22}E_{33}E_{44} \\ &- 36E_{22}E_{33}(E_{11} - E_{44}) - 12E_{11}E_{44}(E_{22} - E_{33}) \\ &+ 28(E_{11}E_{22} + E_{33}E_{44}) - 8(E_{11}E_{33} + E_{22}E_{44}) \\ &+ 4E_{11}E_{44} - 44E_{22}E_{33} \\ &- 6(E_{11} - E_{44}) + 22(E_{22} - E_{33}) \bmod \mathcal{U}(\mathfrak{g}^{\mathbf{C}})\mathfrak{b}^+. \end{aligned}$$

Since we have $\rho(E_{ij})v_\Lambda = 0$ for $i < j$ and $\rho(E_{ii})v_\Lambda = \Lambda(E_{ii})v_\Lambda$, the following proposition can be easily deduced.

Proposition 6.6. *The action of the differential operator $\mathcal{D}$ on the irreducible G-submodule of $\mathcal{F}(G/K)$ isomorphic to U_{lm} is multiplication by the constant $m(m+1)(l+m+1)(l+m+2)$.*

Therefore the irreducible G-submodule of $\mathcal{F}(G/K)$ isomorphic to U_{lm} is in the kernel of $\mathcal{D}$ if and only if m vanishes. Since the irreducible G-submodule of $\mathcal{F}(G/K)$ with the highest weight $l\lambda_1 - l\lambda_4$ is unique, the module then coincides with the image of W_l by $\mathcal{R}$. By the same argument as in §3, we can prove $\operatorname{Ker}\mathcal{D} = \operatorname{Im}\mathcal{R}$, and thus our proof of Theorem 5.2 is completed.

Remark 6.7. The eigenvalue of the differential operator $\mathcal{D}$ can be computed also by using the formula that gives the radial part of $\mathcal{D}$. The first author has exploited this approach and the further results will be shown in the forthcoming papers.

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