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# Geometric Singularities for Hamilton-Jacobi Equation

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## §1. Introduction

In this paper we will consider the Cauchy problem for Hamilton-Jacobi equation :

$$(1) \qquad \frac{\partial y}{\partial t} + H(t, x_1, \ldots, x_n, \frac{\partial y}{\partial x_1}, \ldots, \frac{\partial y}{\partial x_n}) = 0$$

$$(2) \qquad y(0, x_1, \ldots, x_n) = \phi(x_1, \ldots, x_n),$$

where $H$ and $\phi$ are $C^\infty$-functions.

By the method of characteristics, the solution of this problem is explicitly constructed. It is well-known that, even for smooth initial data, the solution becomes multi-valued in finite time. That is, singularities appear.

Recently Tsuji ([6] [7]) and Nakane [5] studied the behavior of solutions near the singular point. They assumed that singularities of the projection to the base space from the multi-valued solution are fold or cusp type singularities. But, other type of singularities may be appeared in generic.

Our purpose is to describe bifurcations of singularities of solutions along the time parameters geometrically. We will study this problem in the framework of the theory of Legendrian unfoldings. In §2 we will introduce the theory of one-parameter Legendrian unfoldings for preparations. In §3 the geometric treatment of Hamilton-Jacobi equation will be given and we will formulate a generalized Cauchy problem associated with the time parameter. Theorem 3.2 is the base of our theory. By this theorem we can apply Arnol'd-Zakalyukin's classifications of one-parameter perestroika of wave front sets and caustics to our situations ([1],[2],[8]).

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All map germs and diffeomorphisms considered here are class $C^\infty$, unless stated otherwise.

## §2. One parameter Legendrian unfoldings

In this section we will briefly introduce the theory of one-parameter Legendrian unfoldings. In [3] we will develop the general theory and its' applications. Thus detailed proof will be appeared in there.

Let $J^1(\mathbb{R}^n, \mathbb{R})$be the 1-jet bundle of functions of $n$-variables. Since we only consider the local situation, the 1-jet bundle $J^1(\mathbb{R}^n, \mathbb{R})$ may be considered as $\mathbb{R}^{2n+1}$ with a natural coordinate system

$$(x_1, \ldots, x_n, y, p_1, \ldots, p_n),$$

where $(x_1, \ldots, x_n)$ is a coordinate system of $\mathbb{R}^n$. We also have natural projections. Namely,

$$\pi : J^1(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}^n \times \mathbb{R} \quad ; \quad \pi(x, y, p) = (x, y)$$

$$\pi' : J^1(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}^n \quad ; \quad \pi'(x, y, p) = x.$$

An immersion germ

$$i : (L, q) \to J^1(\mathbb{R}^n, \mathbb{R})$$

is said to be *a Legendrian immersion germ* if

$$\dim L = n \qquad \text{and} \qquad i^*\theta = 0,$$

where $\theta = dy - \sum_{i=1}^n p_i dx_i$. The image of $\pi \circ i$ is called *a wave front set* of $i$. It is denoted by $W(i)$. We say that $q \in L$ is *a Lagrangian singular point* if

$$\operatorname{rank} d(\pi' \circ i)_q < n.$$

The critical value set of $\pi' \circ i$ is called *a caustics* of $i$. It is denoted by $C(i)$.

Let $i : (L, q) \to J^1(\mathbb{R}^n, \mathbb{R})$ be a Legendrian immersion germ, then $\tilde{\pi} \circ i : (L, q) \to T^*\mathbb{R}^n$ is a Lagrangian immersion germ, where $\tilde{\pi}$ is the canonical projection onto the cotangent bundle $T^*\mathbb{R}^n$. Hence the above definition is reasonable.

We now describe the notion of one-parameter Legendrian unfoldings.

Let $R$ be an $(n+1)$-dimensional smooth manifold and

$$\mu : (R, u_0) \to (\mathbb{R}, t_0)$$

be a submersion germ and

$$\ell : (R, u_0) \to J^1(\mathbb{R}^n, \mathbb{R})$$

be a smooth map germ. We say that the pair $(\mu, \ell)$ is *a Legendrian family* if $\ell_t = \ell|\mu^{-1}(t)$ is a Legendrian immersion germ for any $t \in (\mathbb{R}, t_0)$. Then we have the following simple but very important lemma.

**Lemma 2.1.** *Let $(\mu, \ell)$ be a Legendrian family. Then there exists a unique element $h \in C^\infty_{u_0}(R)$ such that*

$$\ell^*\theta = h \cdot d\mu,$$

*where $C^\infty_{u_0}(R)$ is the ring of smooth function germs at $u_0$.*

We now consider the 1-jet bundle $J^1(\mathbb{R}\times\mathbb{R}^n, \mathbb{R})$ and the canonical 1-form $\Theta$ on the space. Let $(t, x_1, \dots, x_n)$ be canonical coordinate system on $\mathbb{R} \times \mathbb{R}^n$ and

$$(t, x_1, \dots, x_n, y, q, p_1, \dots, p_n)$$

be corresponding coordinate system on $J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$. Then the canonical 1-form is given by

$$\Theta = dy - \sum_{i=1}^{n} p_i \cdot dx_i - q \cdot dt = \theta - q \cdot dt.$$

We define two natural projections. Namely,

$$\Pi : J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}) \to (\mathbb{R} \times \mathbb{R}^n) \times \mathbb{R}$$

by

$$\Pi(t, x, y, q, p) = (t, x, y)$$

and

$$\Pi' : J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}) \to \mathbb{R} \times \mathbb{R}^n$$

by

$$\Pi'(t, x, y) = (t, x).$$

We call the above 1-jet bundle *an unfolded 1-jet bundle.*

Define a map germ

$$\mathcal{L} : (R, u_0) \to J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$$

by

$$\mathcal{L}(u) = (\mu(u), x \circ \ell(u), y \circ \ell(u), h(u), p \circ \ell(u)).$$

Then we can easily show that $\mathcal{L}$ is a Legendrian immersion germ. If we fix 1-forms $\Theta$ and $\theta$, the Legendrian immersion germ $\mathcal{L}$ is uniquely determined by the Legendrian family $(\mu, \ell)$. We call $\mathcal{L}$ *a Legendrian unfolding associated with the Legendrian family* $(\mu, \ell)$.

Since $\mathcal{L}$ is a Legendrian immersion germ, there exists a generating family of $\mathcal{L}$ by the Arnol'd-Zakalyukin's theory ([1],[2],[8]). In this case the generating family is naturally constructed by the one-parameter family of generating families associated with $(\mu, \ell)$.

Let

$$F : ((\mathbb{R} \times \mathbb{R}^n) \times \mathbb{R}^k, 0) \to (\mathbb{R}, 0)$$

be a function germ such that

$$d_2F|0 \times \mathbb{R}^n \times \mathbb{R}^k$$

is non-singular, where

$$d_2F(t, x, q) = (\frac{\partial F}{\partial q_1}(t, x, q), \ldots, \frac{\partial F}{\partial q_k}(t, x, q)).$$

We call $F$ *a generalized phase function germ*. Then $C(F) = d_2F^{-1}(0)$ is a smooth $(n+1)$-manifold germ and

$$\pi_F : (C(F), 0) \to \mathbb{R}$$

is a submersion germ, where

$$\pi_F(t, x, q) = t.$$

Define map germs

$$\tilde{\Phi}_F : (C(F), 0) \to J^1(\mathbb{R}^n, \mathbb{R})$$

by

$$\tilde{\Phi}_F(t, x, q) = (x, F(t, x, q), \frac{\partial F}{\partial x}(t, x, q))$$

and

$$\Phi_F : (C(F), 0) \to J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$$

by

$$\Phi_F(t, x, q) = (t, x, F(t, x, q), \frac{\partial F}{\partial t}(t, x, q), \frac{\partial F}{\partial x}(t, x, q)).$$

Since $\frac{\partial F}{\partial q_i} = 0$ on $C(F)$, we can easily show that

$$(\tilde{\Phi}_F)^*\theta = \frac{\partial F}{\partial t}|C(F) \cdot dt|C(F).$$

By the definition, $\Phi_F$ is a Legendrian unfolding associated with the Legendrian family $(\pi_F, \tilde{\Phi}_F)$. By the same method of the theory of Arnol'd-Zakalyukin ([1],[2],[8]), we can show the following proposition.

**Proposition 2.2.** *Every Legendrian unfolding germs are constructed by the above method.*

## §3. Geometry of Hamilton-Jacobi equation

In this section we will treat Hamilton-Jacobi equation in the framework of the geometric theory of first order partial differential equations [4]. *Hamilton-Jacobi equation* is defined to be a hypersurface

$$E(H) = \{(t, x, y, q, p) \in J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}) | q + H(t, x, p) = 0\}$$

in $J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$. *A geometric solution* of $E(H)$ is a Legendrian submanifold $\mathcal{L}$ lying in $E(H)$.

Since the equation is contact regular at every points (i.e. $\Theta | E(H) \neq 0$), a generalized Cauchy problem (GCP) has a unique solution: It is solved by the method of characteristic equations. In this case the characteristic vector field is given by

$$X_H = -\frac{\partial}{\partial t} - \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\frac{\partial}{\partial x_i} - (\sum_{i=1}^{n} p_i \frac{\partial H}{\partial p_i} - H)\frac{\partial}{\partial y} + \frac{\partial H}{\partial t}\frac{\partial}{\partial q} + \sum_{i=1}^{n} \frac{\partial H}{\partial x_i}\frac{\partial}{\partial p_i}.$$

We say that *a generalized Cauchy problem* (GCP)*is given for an equation* $E(H)$ if there is given an $n$-dimensional submanifold $i : L' \subset E(H)$ such that $i^*\Theta = 0$ and $X_{H,x} \notin T_x(L')$ for any $x \in L'$.

**Theorem 3.1** (Classical existence theorem [4]). *A GCP $i : L' \subset E(H)$ has a unique solution, that is, there is a Legendrian submanifold $\mathcal{L} \subset E(H)$, $L' \subset \mathcal{L}$ and any two such Legendrian submanifolds coincide in a neighbourhood of $L'$.*

But GCP is not enough to serve our purpose. We need a more restricted framework. For any $c \in (\mathbb{R}, 0)$, we set

$$E(H)_c = \{(c, x, y, -H(c, x, p), p) | (x, y, p) \in J^1(\mathbb{R}^n, \mathbb{R})\}.$$

Then it is a $(2n+1)$-dimensional submanifold of $J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ and $\Theta_c = \Theta | E(H)_c = dz - \sum_{i=1}^{n} p_i dx_i$ gives a contact structure on $E(H)_c$.

We define a mapping

$$\iota_c : J^1(\mathbb{R}^n, \mathbb{R}) \to E(H)_c$$

by

$$\iota_c(x, y, p) = (c, x, y, -H(c, x, p), p).$$

Then it is a contact diffeomorphism and the following diagram is commutative:

$$\begin{array}{ccc} J^1(\mathbb{R}^n, \mathbb{R}) & \xrightarrow{\iota_c} & E(H)_c \\ \pi\downarrow & & \downarrow\pi_c \\ \mathbb{R}^n \times \mathbb{R} & = & \mathbb{R}^n \times \mathbb{R}. \end{array}$$

We say that *a generalized Cauchy problem associated with the time parameter*(GCPT) *is given for an equation* $E(H)$ if a GCP $i : L' \subset E(H)$ with $i(L') \subset E(H)_c$ for some $c \in (\mathbb{R}, 0)$ is given.

*Remark.* The Cauchy problem $y(0, x_1, \ldots, x_n) = \phi(x_1, \ldots, x_n)$ is a GCPT. The initial submanifold is given by

$$L_{\phi,0} = \left\{(0, x, \phi(x), -H(0, x, \frac{\partial\phi}{\partial x}), \frac{\partial\phi}{\partial x})|x \in \mathbb{R}^n\right\} \subset E(H)_0.$$

Our purpose is formulated as follows:

**Problem.** Classify the generic bifurcations of singularities of

$$\pi_t| : \mathcal{L} \cap E(H)_t \to \mathbb{R}^n \times \mathbb{R}$$

and

$$\pi'_t| : \mathcal{L} \cap E(H)_t \to \mathbb{R}^n$$

with respect to the parameter t.

*Remark.* In their papers Tsuji ([6],[7]) and Nakane [5] assumed that singularities of $\pi'_t|$ are fold or cusp type singularities. But these singularities are stable in the sense of Thom, then these do not bifurcate along the time parameter. Since the initial condition of the Cauchy problem is non-singular, then other types of singularities must be appeared in generic.

Let $i : L' \subset E(H)_0 \subset E(H)$ be a GCPT and $\mathcal{L}$ be the unique solution of $L'$. Since $X_{H,x} \notin T_xE(H)_c$, then $\mathcal{L}$ is transverse to $E(H)_c$ in $E(H)$ for any $c \in (\mathbb{R}, 0)$. It follows that $\mathcal{L}_c = \mathcal{L} \cap E(H)_c$ is an $n$-dimensional submanifold of $E(H)_c$ and it satisfies $\Theta_c|\mathcal{L}_c = 0$ (i.e. $\mathcal{L}_c$ is a Legendrian submanifold of $E(H)_c$). If we consider the local parametrization of $\mathcal{L}$, we may assume that $\mathcal{L}$ is a image of an immersion germ

$$\mathcal{L} : (\mathbb{R} \times \mathbb{R}^n, 0) \to E(H)$$

such that $\mathcal{L}|(c \times \mathbb{R}^n)$ is a Legendrian immersion germ of $E(H)_c$. Hence the coordinate representation of $\mathcal{L}$ is given by

$$\mathcal{L}(t,u) = (t, x(t,u), y(t,u), -H(t, x(t,u), p(t,u)), p(t,u)).$$

We now define the projection

$$\tilde{\pi} : J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}) \to J^1(\mathbb{R}^n, \mathbb{R})$$

by

$$\tilde{\pi}(t, x, y, q, p) = (x, y, p).$$

It follows from the above arguments that $(\pi_1, \tilde{\pi} \circ \mathcal{L})$ is a Legendrian family, where

$$\pi_1 : (\mathbb{R} \times \mathbb{R}^n, 0) \to (\mathbb{R}, 0)$$

is the canonical projection. Hence $\mathcal{L}$ is a Legendrian unfolding associated with $(\pi_1, \tilde{\pi} \circ \mathcal{L})$.

The following theorem is fundamental in our theory.

**Theorem 3.2.** (1) *The local solution of the generalized Cauchy problem associated with the time parameter for Hamilton-Jacobi equation*

$$q + H(t, x, p) = 0$$

*is a Legendrian unfolding*

$$\mathcal{L} : (\mathbb{R} \times \mathbb{R}^n, 0) \to J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}).$$

(2) *Let* $\mathcal{L} : (\mathbb{R} \times \mathbb{R}^n, 0) \to J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ *be a Legendrian unfolding associated with* $(\pi_1, \ell)$. *Then there exists a* $C^\infty$*-function germ* $H(t, x_1, \dots, x_n, p_1, \dots, p_n)$ *such that* $\mathcal{L}$ *is a local solution of the generalized Cauchy problem associated with the time parameter for Hamilton-Jacobi equation*

$$q + H(t, x, p) = 0,$$

*where the initial condition is given by* $\ell(0, u)$.

*Proof.* The assertion (1) is already proved by the above arguments. We now prove the assertion (2). Taking a coordinate representation of $\ell$, we have

$$\ell(t,u) = (x(t,u), y(t,u), p(t,u)).$$

Since $(\pi_1, \ell)$ is a Legendrian family, we have

$$\langle dt \rangle_{\mathcal{E}_{n+1}} \supset \langle \ell^* \theta \rangle_{\mathcal{E}_{n+1}}.$$

Hence, there exists a $C^\infty$-function germ $h(t,u)$ such that

$$dy(t,u) - \sum_{i=1}^{n} p_i(t,u)dx_i(t,u) = h(t,u)dt.$$

By the definition of the Legendrian unfolding, we have

$$\mathcal{L}(t,u) = (t, x(t,u), y(t,u), h(t,u), p(t,u)).$$

We now define a $C^\infty$-map germ

$$\tilde{\ell} : (\mathbb{R} \times \mathbb{R}^n, 0) \to T^*\mathbb{R}^n$$

by

$$\tilde{\ell}(t,u) = (x(t,u), p(t,u)).$$

Since $(\pi_1, \ell)$ is a Legendrian family, $\tilde{\ell}_t$ is a Lagrangian immersion germ with respect to the canonical symplectic structure on $T^*\mathbb{R}^n$ for any $t \in (\mathbb{R}, 0)$.

We also define a $C^\infty$-map germ

$$\ell' : (\mathbb{R} \times \mathbb{R}^n, 0) \to \mathbb{R} \times T^*\mathbb{R}^n$$

by

$$\ell'(t,u) = (t, x(t,u), p(t,u)).$$

By the above argument, $\ell'$ is an immersion germ. Then

$$\ell'^* : C^\infty_{\ell'(0)}(\mathbb{R} \times T^*\mathbb{R}^n) \to C^\infty_0(\mathbb{R} \times \mathbb{R}^n)$$

is an epimorphism. Since $h \in C^\infty_0(\mathbb{R} \times \mathbb{R}^n)$, there exists $H \in C^\infty_{\ell'(0)}(\mathbb{R} \times T^*\mathbb{R}^n)$ such that $\ell'^*(H) = -h$. That is,

$$-H(t, x(t,u), p(t,u)) = h(t,u).$$

Thus the Legendrian unfolding

$$\mathcal{L} : (\mathbb{R} \times \mathbb{R}^n, 0) \to J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$$

is a geometric solution of the Hamilton-Jacobi equation

$$q + H(t,x,p) = 0.$$

By the uniqueness of the solution, it is also a local solution of GCPT of the Hamilton-Jacobi equation whose initial condition is $\ell(0,u)$.

## §4. Classifications

We now give generic classifications for bifurcations of singularities of solutions along the time parameter. By Theorem 3.2, we can apply the classification theory of bifurcations of singularities of one-parameter Legendrian unfoldings. This section depends heavily on Arnol'd and Zakalyukin's results ([1],[2],[8]). Let

$$\mathcal{L} : (R, u_0) \to (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), z_0)$$

and

$$\mathcal{L}' : (R, u_1) \to (J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}), z_1)$$

be Legendrian unfoldings. We say that *two wave front sets $W(\mathcal{L})$ and $W(\mathcal{L}')$ have diffeomorphic bifurcations* if there exists a diffeomorphism germ

$$\Phi : ((\mathbb{R} \times \mathbb{R}^n) \times \mathbb{R}, \Pi(z_0)) \to ((\mathbb{R} \times \mathbb{R}^n) \times \mathbb{R}, \Pi(z_1))$$

of the form $\Phi(t, x, y) = (\phi(t), \Phi_2(t, x, y))$ such that

$$\Phi(W(\mathcal{L})) = W(\mathcal{L}').$$

We also say that *two caustics $C(\mathcal{L})$ and $C(\mathcal{L}')$ have diffeomorphic bifurcations* if there exists a diffeomorphism germ

$$\Psi : (\mathbb{R} \times \mathbb{R}^n, \Pi'(z_0)) \to (\mathbb{R} \times \mathbb{R}^n, \Pi'(z_1))$$

of the form $\Psi(t, x) = (\psi(t), \Psi_2(t, x))$ such that

$$\Psi(C(\mathcal{L})) = C(\mathcal{L}').$$

In their papers ([1],[2],[8]) Arnol'd and Zakalyukin gave generic classifications of one-parameter perestroika of wave front sets and caustics in the case $n \leq 4$. As an application of their classifications, we have classifications as follows.

**Theorem 4.1.** (1) *Bifurcations of wave front sets in generic one-parameter Legendrian unfoldings for $n \leq 2$ are diffeomorphic to one of the bifurcation of wave front sets defined by generalized phase function germs from the following list :*
$n = 1$ :
${}^0A_1$ : $q_1^2$ ;
${}^0A_2$ : $q_1^3 + x_1 q_1$ ;
${}^1A_3$ : $q_1^4 + q_1^2 t + x_1 q_1$.
$n = 2$ :
${}^0A_1$ : $q_1^2$ ;

${}^0A_2$ : $q_1^3 + x_1q_1$ ;
${}^0A_3$ : $q_1^4 + x_1q_1 + x_2q_1^2$ ;
${}^1A_3$ : $q_1^4 + q_1^2(t \pm x_2^2) + x_1q_1$ ;
${}^1A_4$ : $q_1^5 + q_1^3t + x_1q_1 + x_2q_1^2$ ;
${}^1D_4$ : $q_2q_1^2 \pm q_2^3 + q_2^2t + x_2q_2 + x_1q_1$.

(2) *Bifurcations of caustics in generic one-parameter Legendrian unfoldings for $n \leq 2$ are diffeomorphic to one of the bifurcation of caustics defined by generalized phase function germs from the following list :*

$n = 1$ :
$A_2$ : $q_1^3 + x_1q_1$ ;
$A_3$ : $q_1^4 + tq_1^2 + x_1q_1$ ;
$n = 2$ :
$A_2$ : $q_1^3 + x_1q_1$ ;
$A_3$ : $q_1^4 + x_2q_1^2 + x_1q_1$ ;
$A_3^1$ : $q_1^4 + (t \pm x_2^2)q_1^2 + x_1q_1$ ;
$A_4$ : $q_1^5 + tq_1^3 + x_2q_1^2 + x_1q_1$ ;
$D_4^\pm$ : $q_2q_1^2 \pm q_2^3 + q_2^2(x_1 + ax_2 \pm t) + x_1q_1 + x_2q_2, a \in \mathbb{R}$, *where $a$ is a moduli parameter.*

In their classifications ([1],[2],[8]), ${}^1A_1$-type (i.e., $q_1^2 + t^2 \pm x_1^2 \pm y^2$, $q_1^2 + t \pm x_1^2 \pm x_2^2 \pm y^2$) and ${}^1A_2$-type (i.e., $q_1^3 + q_1(t \pm x_1^2)$, $q_1^3 + q_1(t \pm x_1^2 \pm x_2^2)$) bifurcations are contained in the list of perestroikas of wave front sets. Because the notion of extended Legendrian manifolds in [8] is slightly different from our notion of Legendrian unfoldings, thus ${}^1A_1$-type and ${}^1A_2$-type bifurcations do not appear in our list. We can also list up normal forms of bifurcations in the case when $n = 3$ or $n = 4$. But it is too complicated to explain here.

We now represent the list of bifurcations in the above theorem by using the coordinate representation of map germs. Let

$$f, g : (\mathbb{R} \times \mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$$

be smooth map germs. We define map germs

$$F, G : (\mathbb{R} \times \mathbb{R}^n, 0) \to (\mathbb{R} \times \mathbb{R}^p, 0)$$

by

$$F(t, x) = (t, f(t, x)) \qquad \text{and} \qquad G(t, x) = (t, g(t, x)).$$

We say that *two images* Image($F$) *and* Image($G$) *have diffeomorphic bifurcations* if there exists a diffeomorphism germ

$$\Phi : (\mathbb{R} \times \mathbb{R}^p, 0) \to (\mathbb{R} \times \mathbb{R}^p, 0)$$

of the form $\Phi(t,y) = (\phi(t), \Phi_2(t,y))$ such that

$$\Phi(\text{Image}(F)) = \text{Image}(G).$$

We also say that *two critical value sets $C(F)$ and $C(G)$ have diffeomorphic bifurcations* if there exists diffeomorphism germ

$$\Psi : (\mathbb{R} \times \mathbb{R}^p, 0) \to (\mathbb{R} \times \mathbb{R}^p, 0)$$

of the form $\Psi(t,y) = (\psi(t), \Psi_2(t,y))$ such that

$$\Psi(C(F)) = C(G).$$

Let $\mathcal{L} : (\mathbb{R} \times \mathbb{R}^n, 0) \to J^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ be Legendrian unfolding associated with $(\pi_1, \ell)$. Then the wave front set $W(\mathcal{L})$ is the image of $\Pi \circ \mathcal{L}$ and the caustics $C(\mathcal{L})$ is the critical value set of $\Pi' \circ \mathcal{L}$. Thus we have the following corollary of Theorem 4.1.

**Corollary 4.2.** (1) *Bifurcations of wave front sets in generic one-parameter Legendrian unfoldings for $n \leq 2$ are diffeomorphic to one of the bifurcation of images defined by map germs from the following list :*

$n = 1$ :
${}^0A_1$ : $(x, 0)$ ;
${}^0A_2$ : $(x^2, x^3)$ ;
${}^1A_3$ : $(2x^3 + tx, 3x^4 + tx^2)$.
$n = 2$ :
${}^0A_1$ : $(x_1, x_2, 0)$ ;
${}^0A_2$ : $(x_1, x_2^2, x_2^3)$ ;
${}^0A_3$ : $(x_1, x_2^3 + tx_2, 3x_2^4 + x_1x_2^2)$ ;
${}^1A_3$ : $(x_1, 2x_2^3 + x_2(t \pm x_1^2), 3x_2^4 + x_2^2(t \pm x_1^2))$ ;
${}^1A_4$ : $(x_1, 5x_2^4 + 3x_2^2t + 2x_2x_1, 4x_2^5 + 2x_2^3t + x_2^2x_1)$ ;
${}^1D_4$ : $(x_1x_2, (x_1^2 \pm 3x_2^2) + 2x_2t, 2(x_1^2x_2 \pm x_2^3) + x_2^2t)$.

(2) *Bifurcations of caustics in generic one-parameter Legendrian unfoldings for $n \leq 2$ are diffeomorphic to one of the bifurcation of critical value sets defined by map germs from the following list :*

$n = 1$ :
$A_2$ : $x^2$ ;
$A_3$ : $x^3 + tx$.
$n = 2$ :
$A_2$ : $(x_1, x_2^2)$ ;
$A_3$ : $(x_1, x_2^3 + x_1x_2)$ ;
$A_3^1$ : $(x_1, x_2^3 + x_2(t \pm x_1^2))$ ;
$A_4$ : $(x_1, x_2^4 + x_2^2t + x_2x_1)$ ;

$D_4^\pm$ : $(x_1x_2, \frac{1}{1+2ax_2}(4x_1x_2^2 \mp 2x_2t - x_1^2 \mp 3x_2^2)), a \in \mathbb{R}$, *where $a$ is a moduli parameter.*

*Remark.* In the above lists, bifurcations of caustics given by $A_3$ for $n=1$ and $A_3^1$ for $n=2$ describe the process of birth of the caustics from the empty. If the initial condition of the Cauchy problem is smooth, these process must exist for a neighbourhood of some $t_0$.

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